# Distributed Quantum Error Correction

Rodney Van Meter
*Environment and Information Studies, Keio University, and National Institute of Informatics, Japan* *

Jacob M. Taylor
*Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

Liang Jiang
*Department of Physics, Harvard University, Cambridge, MA 02138, USA*
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The recent discovery of the use of operator subspaces as a generalization of stabilizers for quantum error correction (QEC) has provided a more flexible framework for the implementation of QEC. We have analyzed implementation of one such code, the Bacon-Shor code, in a two-dimensional memory layout where operations on one axis are substantially more expensive and error-prone that on the other, such as node-local versus inter-node operations in a distributed system. By favoring stabilizers that use local operations, rather than long-distance operations, we solve a key problem in the implementation of a quantum multicomputer with code words spanning multiple nodes. We also show efficient distributed creation of the cat states necessary for syndrome calculation. This work improves the prospects for the use of very small nodes, holding only a few physical qubits each.

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Hardware systems that can cleanly scale the number of physical qubits in a single system are hard to build [1]. Thus, we have been investigating *quantum multicomputer* architectures, in which a set of quantum computers are connected via quantum channels into a system that can solve problems that are beyond the reach of any individual quantum computer [2].

In prior work on fault tolerance in multicomputer architectures, Van Meter *et al.* showed that Steane's method of measuring syndromes, which uses logical zero states, is not well suited to quantum error correction on distributed code words, and argued that individual multicomputer nodes therefore should contain complete logical qubits [3]. Logical gates are performed by teleporting complete logical qubits, then performing the gates locally. Jiang *et al.* explored minimum-size nodes, using purification to insure the quality of distributed Bell pairs; each node effectively holds a single data qubit and a few ancillae [4]. The inter-node interconnect is treated as the first-level qubit-to-qubit interconnect, for purposes of calculating an error threshold and planning logical operations. The system proposed by Oi *et al.* falls in between, holding a single, first-level logical qubit in each node [5]. The logical qubits do not move, and distributed zeros built on ancillae are used to execute two-qubit gates.

Operator subspaces relax some of the constraints on quantum error correction codes by recognizing that many errors that can occur do not impact our ability to continue to use a particular state for quantum computation [6, 7]. Aliferis and Cross have shown that a qubit encoded in the Shor nine-bit code can be protected using only nearest-neighbor interactions on a 2D lattice and gauge bits along the edges of the lattice [8].

However, when using a multicomputer architecture, the qubit-to-qubit interconnect is not fully symmetric: interactions within a node are generally direct qubit-to-qubit actions, governed by the node's internal topology and technology, but interactions between nodes are very difficult, typically mediated by Bell pairs and teleportation operations. Our question then becomes how to lay out the qubits to minimize the number of high-fidelity Bell pairs that must be created using the inter-node interconnect.

We find that the standard stabilizer set for a generalized Bacon-Shor code, of the form $[[n^2, 1, n]]$, works well in a multicomputer, when cat states and Shor's fault tolerance method are used instead of logical zeros and Steane's method. A particularly elegant approach becomes apparent for $n$ multicomputer nodes holding a single logical qubit. Then, only $O(n)$ distributed Bell pairs are necessary for fault tolerant syndrome extraction and error correction.

Writing stabilizers as a matrix with the $j$th column corresponding to qubits in the $j$th node of the multicomputer, we use

$$S^x_{ij} = X_{i,j}X_{i+1,j} \quad S^z_j = \prod_{i=1}^{n} Z_{i,j}Z_{i,j+1} \tag{1}$$

That is, paired pauli $X$ operators within a given node, and two columns of $Z$ operators between adjacent nodes, as illustrated in Fig. 1a. Measurement of $S^x_{ij}$ is trivial with good internal node operations. Our task is then reduced to efficient measurement of the $n-1$ paired-column stabilizers, $S^z_j$.

For this Pauli operator on $2n$ qubits, we will use Shor-type syndrome extraction. This operation requires

*Electronic address: rdv@sfc.wide.ad.jp

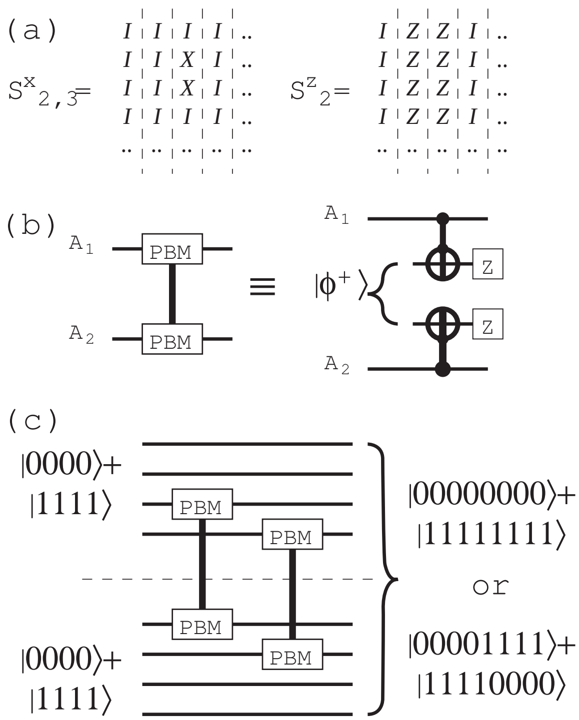


FIG. 1: (a) Stabilizers for a $[[n^2, 1, n]]$ Bacon-Shor code, distributed between different multicomputer nodes (separation indicated by the dashed lines). (b) A partial Bell measurement (PBM) circuit using one entangled Bell pair, with result $\pm 1$ given by the product of the $Z$ measurement results. (c) Fault tolerant "patching" of two 4-qubit cat states. The result is kept only when both PBM measurements give the same result. For PBM results of 1,1, we get the top GHZ state, while -1,-1, gives the bottom.

preparing a $2n$ GHZ state of an ancillary system, followed by pairwise controlled-not operation and measurement of the ancillae. In a given node, we need $n$ qubits for the code word, $n$ qubits for the cat state, and one additional qubit for verification and entanglement generation.

Fault tolerance is achieved if this procedure does not amplify errors: no single bit error, occurring with probability $p$, can produce a two-bit errors on the code-word qubits. We choose a recursive approach to proving fault tolerance, using ancilla-mediated partial Bell measurements to verify the cat state against errors. In essence, we can fault tolerantly "patch" two $n$ cat states together to a single cat state of size $2n$. This operation requires two partial Bell measurements (sequential or simultaneous), necessitating two Bell pairs be consumed for each $2n$ cat state needed.

Let us define the partial Bell measurement (PBM), which will account for the entangling operations necessary to "patch" two cat states together. The idea of PBM is to projectively measure two qubits (one in each smaller cat) in two orthogonal subspaces: one subspace is spanned by the Bell states $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$, and the other is spanned by the Bell states $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$.

We can achieve PBM by entanglement generation and local operations. For distributed nodes (registers), we implement PBM via the following procedure (shown in Fig. 1b): (1) Prepare a high fidelity EPR pair shared by the communication qubits from the two nodes, say $|\Phi^+\rangle_{C_1,C_2}$. (2) Then apply local CNOT gate. (3) The two storage qubits ($S_1$ and $S_2$) are projected to the subspace spanned by the Bell states $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ if the measurement outcomes are the same, or they are projected to the subspace spanned by the Bell states $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$ if the measurement outcomes are different. (4) (Optional) For the latter case, we may further flip one of the storage qubits, so that they are projected to the subspace spanned by $|\Phi^\pm\rangle$.

More specifically, we assume we have fault tolerantly prepared two cat states of size $n/2$, spanned by $|\bar{0}\rangle \equiv |000\cdots\rangle$ and $|\bar{1}\rangle \equiv |111\cdots\rangle$. Suppose we start with two such blocks, prepared in $|\bar{0}\rangle_1 + |\bar{1}\rangle_1$ and $|\bar{0}\rangle_2 + |\bar{1}\rangle_2$ states, respectively (i.e., two small GHZ/cat states). Each block is associated with a communication qubit for entanglement generation. We want to prepare a large GHZ state $|\bar{0}\rangle_1 |\bar{0}\rangle_2 + |\bar{1}\rangle_1 |\bar{1}\rangle_2$ fault tolerantly. We may achieve this task by applying PBM twices as shown in Fig. 1c. The fault-tolerance comes from the second (redundant) PBM. Each PBM will indicate whether the upper qubits have the same value as the lower ones. The indications from two PBMs should be the same; otherwise there must be something wrong. There are several sources that can lead to the inconsistency: (1) There is a bit error in one of the physical qubits connect to the PBM. (2) The EPR pair $|\Phi^+\rangle_{C_1,C_2}$ for one PBM has a bit-flip error. (3) The measurement of the communication qubit within the PBM has a mistake. If *one* of the above error occurs, it will be detected by our double-PBM operation (Fig. 1c) [9]. The possibility for these errors to induce correlated errors in the large GHZ state is suppressed to the second order by the circuit.

This specific example illustrates our approach for minimizing the number of Bell pairs necessary between separated nodes in a quantum multicomputer. Generalizations of this approach to non-degenerate quantum codes and concatenated codes remains an outstanding problem.

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[1] Rodney Van Meter and Mark Oskin. Architectural implications of quantum computing technologies. *ACM Journal of Emerging Technologies in Computing Systems*, 2(1):31–

63, January 2006.

[2] Rodney Van Meter, W. J. Munro, Kae Nemoto, and Kohei M. Itoh. Arithmetic on a distributed-memory quantum multicomputer. *ACM Journal of Emerging Technologies in Computing Systems*, 2007. to appear; available as arXiv:quant-ph/0607160v2.

[3] Rodney Van Meter, Kae Nemoto, and W. J. Munro. Communication links for distributed quantum computation. *IEEE Transactions on Computers*, 2007. to appear; available as http://arxiv.org/quant-ph/0701043.

[4] L. Jiang, J. M. Taylor, A. S. Sorensen, and M. D. Lukin. Scalable quantum networks based on few-qubit registers. quant-ph/0703029, 2007.

[5] Daniel K. L. Oi, Simon J. Devitt, and Lloyd C. L. Hollenberg. Scalable error correction in distributed ion trap computers. *Physical Review A*, 74:052313, 2006.

[6] D. Bacon. Operator quantum error-correcting subsystems for self-correcting quantum memories. *Physical Review A*, 73(1):12340, 2006.

[7] Dave Bacon and Andrea Casaccino. Quantum error correcting subsystem codes from two classical linear codes. quant-ph/0610088, October 2006.

[8] Panos Aliferis and Andrew W. Cross. Subsystem fault tolerance with the Bacon-Shor code. *Physical Review Letters*, 98(22):220502, May 2007. arXiv version quant-ph/0610063 includes extensive, helpful appendix.

[9] We apply the double-PBM operation on different physical qubits, rather than repeating the PBM twice with the same physical qubits. This helps to suppress the type (1) error to the second order.