Fundamentals of Logic

No.5 Soundness and Completeness

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So far

- Propositional Logic
  - Logical Connectives
  - Truth Table
  - Tautology
- Normal Form
  - Disjunctive Normal Form
  - Conjunctive Normal Form
  - Restricting Logical Connectives
- Proof
  - Axiom and Theorem
  - LK Framework

Extending Tautology

- Extending the notion of tautology to sequent
- Let Γ be a sequence of formula $A_1, \ldots, A_m$

$$\Gamma^* = \begin{cases} A_1 \vee \cdots \vee A_m & \text{when } m > 0 \\ \bot & \text{when } m = 0 \end{cases}$$

$$\Gamma_* = \begin{cases} A_1 \wedge \cdots \wedge A_m & \text{when } m > 0 \\ \top & \text{when } m = 0 \end{cases}$$

- Sequent $\Gamma \to \Delta$ is a *tautology* $\iff$
  - $\Gamma_* \Rightarrow \Delta^*$ is a tautology

Soundness and Completeness

- $\Gamma \to \Delta$ is a tautology:
  - For any assignment v, $v(\Gamma_* \Rightarrow \Delta^*) = \mathrm{t}$
  - For any assignment of truth values to propositional variables, we can calculate the truth value of $\Gamma_* \Rightarrow \Delta^*$ using the truth tables and check it is always true.
- $\Gamma \to \Delta$ is provable in LK:
  - There is a proof figure of which end sequent is $\Gamma \to \Delta$.
  - Starting from initial sequents, we can construct a proof figure of $\Gamma \to \Delta$.
- *Soundness* of LK:
  - Any sequent which is provable is a tautology.
- *Completeness* of LK:
  - Any tautology is provable in LK.

LK Soundness Theorem (1)

- **Theorem:** For any sequent $\Gamma \to \Delta$, if $\Gamma \to \Delta$ is provable in LK, it is a tautology.

- **Proof:** We need to show the following two things:
  (1) An initial sequent is a tautology.
  (2) For each inference rule, if all the sequents above are tautologies, the the bottom sequent is also a tautology.

  For (1), an initial sequent $A \to A$ is $A \Rightarrow A$ and is a tautology.
  For (2), we need to check for each inference rule. For example,

$$(\Rightarrow\text{R}) \qquad \frac{\Gamma \to \Delta, A \quad B, \Pi \to \Sigma}{A \Rightarrow B, \Gamma, \Pi \to \Delta, \Sigma}$$

  we need to show when $\Gamma_* \Rightarrow (\Delta^* \vee A)$ and $B \wedge \Pi_* \Rightarrow \Sigma^*$ are tautologies, $((A \Rightarrow B) \wedge \Gamma_* \wedge \Pi_*) \Rightarrow (\Delta^* \vee \Sigma^*)$ is also a tautology.

LK Soundness Theorem (2)

- When $v((A \Rightarrow B) \wedge \Gamma_* \wedge \Pi_*) = \mathrm{t}$, $v(A \Rightarrow B) = \mathrm{t}$, $v(\Gamma_*) = \mathrm{t}$ and $v(\Pi_*) = \mathrm{t}$.
  (1) If $v(A) = \mathrm{f}$, from $v(\Gamma_* \Rightarrow (\Delta^* \vee A)) = \mathrm{t}$ and $v(\Gamma_*) = \mathrm{t}$, $v(\Delta^*) = \mathrm{t}$.
  (2) If $v(A) = \mathrm{t}$, from $v(A \Rightarrow B) = \mathrm{t}$, $v(B) = \mathrm{t}$. Since $v(B \wedge \Pi_* \Rightarrow \Sigma^*) = \mathrm{t}$ and $v(\Pi_*) = \mathrm{t}$, $v(\Sigma^*) = \mathrm{t}$.

  In both cases $v(\Delta^* \vee \Sigma^*) = \mathrm{t}$. Therefore, $((A \Rightarrow B) \wedge \Gamma_* \wedge \Pi_*) \Rightarrow (\Delta^* \vee \Sigma^*)$ is a tautology.

- For other rules, we can show the bottom sequent is a tautology when the above sequents are tautologies.

- We have shown that LK framework is sound.

LK Completeness Theorem (1)

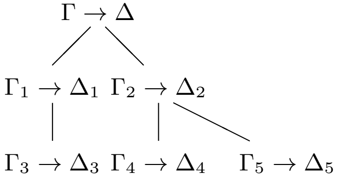
- **Theorem:** If $\Gamma \to \Delta$ is a tautology, it is provable in LK without using cut inference rule.
- **Proof:** First, define the decomposition of a sequent as follows:

$$\begin{array}{lcl}
\Gamma \to \Delta_1, A \wedge B, \Delta_2 & \Rightarrow & \Gamma \to \Delta_1, A, \Delta_2 \quad \text{and} \quad \Gamma \to \Delta_1, B, \Delta_2 \\
\Gamma_1, A \wedge B, \Gamma_2 \to \Delta & \Rightarrow & \Gamma_1, A, B, \Gamma_2 \to \Delta \\
\Gamma \to \Delta_1, A \vee B, \Delta_2 & \Rightarrow & \Gamma \to \Delta_1, A, B, \Delta_2 \\
\Gamma_1, A \vee B, \Gamma_2 \to \Delta & \Rightarrow & \Gamma_1, A, \Gamma_2 \to \Delta \quad \text{and} \quad \Gamma_1, B, \Gamma_2 \to \Delta \\
\Gamma \to \Delta_1, A \Rightarrow B, \Delta_2 & \Rightarrow & A, \Gamma \to \Delta_1, B, \Delta_2 \\
\Gamma_1, A \Rightarrow B, \Gamma_2 \to \Delta & \Rightarrow & \Gamma_1, \Gamma_2 \to \Delta, A \quad \text{and} \quad \Gamma_1, B, \Gamma_2 \to \Delta \\
\Gamma \to \Delta_1, \neg A, \Delta_2 & \Rightarrow & A, \Gamma \to \Delta_1, \Delta_2 \\
\Gamma_1, \neg A, \Gamma_2 \to \Delta & \Rightarrow & \Gamma_1, \Gamma_2 \to \Delta, A
\end{array}$$

A sequent is decomposed into one or two sequents.

LK Completeness Theorem (2)

- Two facts about decomposition:
  - The number of logical connectives are less in the decomposed sequents.
  - If a sequent is a tautology, its decompositions are tautologies.
- Apply decomposition repeatedly:



- Sequents at the bottom row do not contain logical connectives.
  - *Complete decomposition tree*
  - If $\Gamma \to \Delta$ is a tautology, the bottom row of the complete decomposition contains tautology sequents containing propositional variables (no logical connectives).

LK Completeness Theorem (3)

- A sequent $p_1, \ldots, p_m \to q_1, \ldots, q_n$ of propositional variables is a tautology if and only if one of p_i is equal to one of q_j.
- $\Gamma_1, A, \Gamma_2 \to \Delta_1, A, \Delta_2$ is provable without cut rule.
  - Apply weakening and exchange to initial sequent $A \to A$.
- When $\Gamma \to \Delta$ is decomposed into $\Gamma_1 \to \Delta_1$ and $\Gamma_2 \to \Delta_2$, if $\Gamma_1 \to \Delta_1$ and $\Gamma_2 \to \Delta_2$ are provable without cut, $\Gamma \to \Delta$ is also provable without cut.
  - For each decomposition, use rules for logical connectives, exchange and contraction.
- Therefore, if a sequent is a tautology, using its complete decomposition it is provable.
- LK framework is complete.
- This proof provides an algorithm to check whether a sequent is provable or not.

Example

- Find the complete decomposition of $\rightarrow ((p \Rightarrow q) \Rightarrow p) \Rightarrow p$ and find its proof figure without cut.

Cut Elimination Theorem

- **Theorem:** If $\Gamma \to \Delta$ is provable in LK, there exists a proof figure of $\Gamma \to \Delta$ without cut.
  - **Proof:** Since a provable sequent is a tautology, it is provable without cut.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}}{\to A, A \lor \neg A}}{\to A \lor \neg A, A}}{\to A \lor \neg A, A \lor \neg A}}{\to A \lor \neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \to A}{B, A \to A}}{A \to B \Rightarrow A}}{A \to (A \Rightarrow B) \lor (B \Rightarrow A)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}}{\neg A, A \to B}}{\neg A \to A \Rightarrow B}}{\neg A \to (A \Rightarrow B) \lor (B \Rightarrow A)}}{A \lor \neg A \to (A \Rightarrow B) \lor (B \Rightarrow A)}}{\to (A \Rightarrow B) \lor (B \Rightarrow A)}$$

- **Theorem:** Any formula in a cut free proof figure of $\Gamma \to \Delta$ is a subformula of $\Gamma \to \Delta$.

Cut Elimination Theorem

- Find a proof figure of $\rightarrow (A \Rightarrow B) \vee (B \Rightarrow A)$ without cut.

Duality

- ($\neg$L) and ($\neg$R) is symmetrical by exchanging left and right of $\Gamma \to \Delta$.

$$(\neg\text{L})\quad \frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta} \qquad (\neg\text{R})\quad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A}$$

- Inference rules for $\vee$ and $\wedge$ are symmetrical by replacing left and right and $\vee$ and $\wedge$.

$$(\wedge\text{R})\quad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B} \qquad (\vee\text{L})\quad \frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}$$

$$(\wedge\text{L1})\quad \frac{A, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta} \qquad (\vee\text{R1})\quad \frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, A \vee B}$$

$$(\wedge\text{L2})\quad \frac{B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta} \qquad (\vee\text{R2})\quad \frac{\Gamma \to \Delta, B}{\Gamma \to \Delta, A \vee B}$$

- **Duality Theorem:** Let A and B be formulae without $\Rightarrow$. Let $\tilde{A}$ and $\tilde{B}$ be formulae A and B replacing $\vee$ and $\wedge$. If $A \to B$ is provable, $\tilde{B} \to \tilde{A}$ is also provable.

Hilbert Framework

- In LK framework, there is only one axiom (for initial sequent) and a lot of inference rules.
- In Hilbert framework, there are several axioms, but only one inference rule (modus ponens).
  - $A \Rightarrow (B \Rightarrow A)$
  - $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
  - $A \Rightarrow (A \vee B)$
  - $B \Rightarrow (A \vee B)$
  - $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C))$
  - $(A \wedge B) \Rightarrow A$
  - $(A \wedge B) \Rightarrow B$
  - $(C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \wedge B)))$
  - $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$
  - $A \Rightarrow (\neg A \Rightarrow B)$
  - $A \vee \neg A$

Summary

- Soundness of LK framework
  - Any sequent which is provable in LK is a tautology.
- Completeness of LK framework
  - A tautology is provable in LK.
  - There is an algorithm of checking whether a formula is provable or not.
- Properties of LK framework
  - cut elimination theorem
  - duality theorem
- Hilbert framework