Fundamentals of Logic

No.7 Predicate Logic

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Limitation of Propositional Logic

- Propositional Logic
  - Each proposition is either true or false.
  - The truth value does not change.
  - The truth value does not depend of objects which are referred in the proposition.
- Socrates problem:
  - Socrates is a man.
  - All men are mortal.
  - Therefore, Socrates is mortal.
- In propositional logic:
  - $p =$ "Socrates is a man"
  - $q =$ "Socrates is mortal"
  - $p \Rightarrow q$?

Propositional Logic to Predicate Logic

- Extend logic to handle objects and express properties and relations of objects.
- Set of objects
  - Integer
  - Human
- Variable over a set of objects
  - *object variable*
  - $x, y, z, \ldots$
- Name of object
  - *object constant*
  - Socrates, Pythagoras, 123, SFC, Keio, ...

Predicate

- *Predicate*
  - Object x has property P: $P(x)$
  - Relation R holds between object x and object y: $R(x, y)$
- $Q(x_1, x_2, \ldots, x_n)$
  - Q holds for objects $x_1, x_2, \ldots, x_n$
  - Q is a predicate with n variables.
- $P(x) =$ "x is a man"
  - $P(\text{Socrates}) =$ "Socrates is a man"
  - $P(\text{Pythagoras}) =$ "Pythagoras is a man"
  - $P(\text{Taro}) =$ "Taro is a man"
- $R(x, y) =$ "x likes y"
  - $R(\text{Taro}, \text{Hanako}) =$ "Taro likes Hanako"
  - $R(\text{Taro}, \text{Momoko}) =$ "Taro likes Momoko"
  - $R(\text{Hanako}, \text{Taro}) =$ "Hanako likes Taro"

Quantifier

- $P(x)$
  - Which x makes P hold?
  - Does it hold for any x?
  - Does it only hold for some x?
- *Quantifier*
  - $\forall x P(x)$
    - *Universal quantifier*
    - For all x, $P(x)$ holds.
  - $\exists x P(x)$
    - *Existential quantifier*
    - For some x, $P(x)$ holds.
    - There exists x which makes $P(x)$ hold.
- $Q(x) =$ "x is mortal"
  - $\forall x Q(x) =$ "Everybody is mortal"
  - $\exists x Q(x) =$ "Someone is mortal", "There is someone who is mortal"

Predicate Logic

- *Predicate Logic*
  - Use predicates instead of propositional variables.
  - Four logical connectives: $\wedge, \vee, \Rightarrow, \neg$
  - Two quantifiers: $\forall, \exists$

- Socrates example: $P(x) =$ "x is a man", $Q(x) =$ "x is mortal"
  - $P(\text{Socrates}) =$ "Socrates is a man"
  - $\forall x(P(x) \Rightarrow Q(x)) =$ "All men are mortal"
  - $Q(\text{Socrates}) =$ "Socrates is mortal"

- Math example: $P(x) =$ "x is a prime number bigger than 2", $Q(x) =$ "x is an odd number"
  - $P(7) =$ "7 is a prime number bigger than 2"
  - $\forall x(P(x) \Rightarrow Q(x)) =$
    "Any prime number bigger than 2 is an odd number"
  - $Q(7) =$ "7 is an odd number"

Example (1)

- Let $S(x)$ and $M(x)$ be as follows:
  - $S(x) =$ "x is an SFC student"
  - $M(x) =$ "x likes mathematics"
- Write the meaning of the following formulae:
  - $\forall x(S(x) \Rightarrow M(x)) =$
    "All the SFC students "
  - $\exists x(S(x) \wedge M(x)) =$
    "There is an SFC student "
  - $\forall x(S(x) \Rightarrow \neg M(x)) =$
    " "
  - $\neg \forall x(S(x) \Rightarrow M(x)) =$
    " "
  - $\forall x \neg (S(x) \Rightarrow M(x)) =$
    " "

Example (2)

- Let $L(x, y)$ mean "x likes y". Write the meaning of following formulae?
  - $\forall x L(\text{Taro}, x) =$ "Taro likes "
  - $\exists x L(\text{Taro}, x) =$ "Taro likes "
  - $\forall x L(x, \text{Taro}) =$ " "
  - $\exists x L(x, \text{Taro}) =$ " "
  - $\forall x \forall y L(x, y) =$ " "
  - $\forall x \exists y L(x, y) =$ " "
  - $\exists x \forall y L(x, y) =$ " "
  - $\exists y \forall x L(x, y) =$ " "
  - $\exists x \exists y L(x, y) =$ " "
  - $\forall x \forall y (S(x) \Rightarrow L(x, y)) =$ " "
  - $\forall x \forall y (S(y) \Rightarrow L(x, y)) =$ " "
  - $\forall x (\forall y L(x, y) \Rightarrow S(x)) =$ " "

Language for Predicate Logic

- A set of symbols for predicate logic is called *language*.
  - It is different from linguistic language.
  - It is closer to *vocabulary*.

- A language $\mathcal{L}$ of predicate logic consists of the followings:
  (1) Logical connectives: $\wedge, \vee, \Rightarrow, \neg$
  (2) Quantifiers: $\forall, \exists$
  (3) Object variables: $x, y, z, \ldots$
  (4) Object constants: $c, d, \ldots$
  (5) Function symbols: $f, g, \ldots$
  (6) Predicate symbols: $P, Q, \ldots$

Terms

- *Terms* of a language $\mathcal{L}$ is defined as follows:
  (1) Object variables and constants of $\mathcal{L}$ are terms.
  (2) For a function symbol f of m variables (arity m) in $\mathcal{L}$, if $t_1, \ldots, t_m$ are terms, $f(t_1, \ldots, t_m)$ is also a term.

- Example: Natural Number Theory
  - Object constants: 0, 1, etc.
  - Function symbols: $S(x)$, $+$, $\times$, etc.
  - Predicate symbols: $=$, $<$, etc.
  - Terms
    - x
    - 0
    - $S(x) + (1 \times S(S(y)))$

Logical Formulae

- *Logical Formulae* of $\mathcal{L}$ is defined as follows:
  (1) For a predicate symbol P of n variables in $\mathcal{L}$, if $t_1, \ldots, t_n$ are terms, $P(t_1, \ldots, t_n)$ is a formula (*atomic formula*).
  (2) For formulae A and B, $(A \wedge B)$, $(A \vee B)$, $(A \Rightarrow B)$ and $(\neg A)$ are formulae.
  (3) For a formula A and an object variable x, $(\forall x A)$ and $(\exists x A)$ are formulae.

- Example: Natural Number Theorem
  - $\exists z(x \times z = y)$
  - $\forall x \forall y((x + S(y)) = S(x + y))$

Bound and Free Variables

- *Bound variables*
  - In $\exists z(x \times z = y)$, z of $x \times z = y$ is bound by $\exists z$.
  - Bound variables can be renamed without changing the meaning.
  - $\exists w(x \times w = y)$
- Variables which are not bound are *free variables*
  - In $\exists z(x \times z = y)$, x and y are free variables.
- Variables may be bound or free depending on their *occurrence*.
  - $\exists z(x \times z = y) \wedge \exists y(x + x = y)$

Closed Formulae

- When a logical formulae A do not contain free variables, A is called a *closed* logical formulae.
  - $\forall x(S(x) \Rightarrow \forall y L(x,y))$
- If $x_1, \ldots, x_n$ are the free variables of a logical formulae A,
  - $\forall x_1 \cdots \forall x_n A$
  - is called *universal closure* of A.
- In mathematics, universal quantifiers are often omitted.
  - Commutative law of addition: $x + y = y + x$
  - Its universal closure: $\forall x \forall y(x + y = y + x)$

Assignment of Terms

- For a logical formula A, when all the free occurrence of x are replaced with a term t, it is called an *assignment* of t to x.
  - $A[t/x]$

- Example:
  - Let A be $\exists z(x \times z = y)$.
  - $A[w/y]$ is $\exists z(x \times z = w)$.
  - $A[x/y]$ is $\exists z(x \times z = x)$.
  - $A[(x + w)/x]$ is $\exists z((x + w) \times z = y)$.

- If bound relationship is affected by an assignment, the bound variable must be changed before the assignment.
  - $A[z/y]$ is not $\exists z(x \times z = z)$, but $\exists w(x \times w = z)$.
  - In general, $(\forall xA)[t/x]$ is $\forall u(A[u/x][t/x])$ where u is a variable which does not occur in A or t.

Sub-formulae

- Define sub-formulae similar to propositional logic.
  (1) A is a sub-formula of A.
  (2) A and B are sub-formulae of $(A \wedge B)$.
  (3) A and B are sub-formulae of $(A \vee B)$.
  (4) A and B are sub-formulae of $(A \Rightarrow B)$.
  (5) A is a sub-formula of $(\neg A)$.
  (6) For any term t, $A[t/x]$ is a sub-formula of $\forall x A$.
  (7) For any term t, $A[t/x]$ is a sub-formula of $\exists x A$.

- When a formula contains quantifiers, there are infinitely many sub-formulae.
  - Sub-formulae of $\forall x Q(x)$ are $\forall x Q(x)$, $Q(\text{Socrates})$, $Q(\text{Taro})$, $Q(\text{mother}(\text{Taro}))$, ...

Summary

- Predicate Logic
  - Limitation of propositional logic
  - Description about objects
- Logical Formulae for Predicate Logic
  - Language
  - Terms
  - Logical Formulae
- Quantifiers
  - Bound and free variables
  - Closed formulae
  - Universal closure