FUNDAMENTALS OF LOGIC
NO.2 PROPOSITION AND TRUTH VALUE

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lecture URL

https://vu5.sfc.keio.ac.jp/slide/

Proposition

- A *Proposition* is a statement of which truth does not change (i.e. always true or always false).
  - \`$1 < 2$'
  - \`There are infinitely many prime numbers.'
  - \`A triangle has equal edges.'
  - \`Any even number can be expressed as a sum of two prime numbers.' (Gold Bach's Conjecture)
  - \`Taro likes Hanako.'
  - \`The headquarter of Keio University is at SFC.'
  - \`Keio University is a national university.'

- If a statement contains a variable, it is not a proposition. The truth may change depending on the variable.
  - \`$x < 5$'
  - \`Taro likes A.'

Propositional Variable

- A *propositional variable* represents a proposition which cannot be decomposed.
  - It represents an atomic statement.
  - $p, q, r, \cdots$

- Non atomic statements:
  - A compound statement is not basic because it can be decomposed into smaller statements.
  - `Taro likes Hanako and Hanako likes Taro.'
  - `If the wind blows, the bucket makers make money.'
  - `Taro comes to SFC using bus or bicycle'

Compound Proposition

- Propositions can be combined.
  - A compound statement is composed of some atomic statements.

- There are four ways of composition:

Connective	Symbol	Name	Meaning	Other Symbols
and	$\wedge$	conjunction	both hold	$\cap$ &
or	$\vee$	disjunction	one of them holds	$\cup$ |
imply	$\rightarrow$	conditional	under some condition, it holds	$\supset$ $\Rightarrow$
not	$\neg$	negation	the reverse holds	$\sim$

Logical Formula

- A *logical formula* represents a compound proposition.

- **Definition**
  - A propositional variable is a logical formula.
  - If A and B are logical formulae, the followings are also logical formulae:
    - $(A \land B)$
    - $(A \lor B)$
    - $(A \rightarrow B)$
    - $(\neg A)$

- Example
  - $(p \rightarrow q)$
  - $(p \rightarrow (q \lor (\neg r)))$
  - $(\neg((p \land q) \rightarrow (r \lor p)))$

Omission of Parenthesis

- Too many parentheses.
  - Omit some of them if there is no confusion.
  - Omit out most parentheses.
  - Give priority to connectives.
    - $\neg \ > \ \wedge \ > \ \vee \ > \ \rightarrow$
  - $\wedge$ and $\vee$ are left associative and $\rightarrow$ is right associative.
    - $p \wedge q \wedge r \equiv (p \wedge q) \wedge r$
    - $p \vee q \vee r \equiv (p \vee q) \vee r$
    - $p \rightarrow q \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

- Example:
  - \`$\neg$' has more priority than \`$\wedge$', \`$\neg$' connects stronger than \`$\wedge$'. Therefore, \`$\neg p \wedge q$' means \`$((\neg p) \wedge q)$'.
  - $p \ \rightarrow \ q \ \vee \ \neg \, r$ means
  - $\neg \neg \, p \ \rightarrow \ p$ means
  - $p \ \vee \ \neg \, (q \ \rightarrow \ p)$ means

Creating Logical Formulae

- Let p represent `Taro likes Hanako', $q$ represent `Taro likes Momoko' and r represent `Hanako likes Taro'.
- Write the following statements as logical formulae.
  - `Taro likes both Hanako and Momoko.'
  - `Taro likes Hanako or Momoko.'
  - `Taro likes Hanako, but Hanako does not like him.'
  - `If Taro likes Hanako, Hanako also likes him.'
  - `If Taro likes Hanako and not Momoko, Hanako likes him.'
  - `Hanako likes Taro who likes Momoko.'

Truth Table

- A proposition has a value of *true* (T) or *false* (F).
  - A proposition is either true or false, not both.
  - The negation of true is false, and the negation of false is true.

- The truth value of a logical formula depends on the true value of propositional variables in the formula.

- If a formula consists of connecting two formulae with a logical connective, its truth value can be determined from the true value of two sub formulae.

- The following *truth tables* show truth value of each logical connective.

$A \wedge B$

A \ B	T	F
T	T	F
F	F	F

$A \vee B$

A \ B	T	F
T	T	T
F	T	F

$A \rightarrow B$

A \ B	T	F
T	T	F
F	T	T

$\neg A$

A	$\neg A$
T	F
F	T

Exclusive Or

- $A \lor B$ is true if A or B is true.
  - When A and B are true, $A \lor B$ is true.

- *exclusive or*, xor
  - Exclude the case when both A and B are true.
  - It is true when only one of A or B is true.

$A \lor B$

$A \backslash B$	T	F
T	T	T
F	T	F

$A \veebar B$

$A \backslash B$	T	F
T	F	T
F	T	F

$A \oplus B$ is also used for exclusive or.

Calculating Truth Value of a Formula

- When a formula A contains propositional variables $p_1, p_2, \ldots, p_n$, the truth value of A can be calculated from the truth value of $p_1, p_2, \ldots, p_n$.
  - Starting form the truth value of $p_1, p_2, \ldots, p_n$, using the truth value table of each connective, the truth value of the formula can be calculated.

- Example
  - Truth value of $p \to q \vee \neg p$

p	q	$\neg p$	$q \vee \neg p$	$p \to q \vee \neg p$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Exercise

- Calculate the truth value of $p \vee \neg(q \rightarrow p)$.

p	q	$q \rightarrow p$	$\neg(q \rightarrow p)$	$p \vee \neg(q \rightarrow p)$
T	T			
T	F			
F	T			
F	F			

Tautology

- A is a *tautology* if A is always true no matter what the truth value of propositional variables $p_1, p_2, \ldots, p_n$ inside A.
  - A is also called *valid*.
  - There are 2^n comination of truth value of $p_1, p_2, \ldots, p_n$.
  - By checking all the cases, we can determine whether A is a tautology or not.

- **Theorem:**
  - It is *decidable* whether a given logical formula is a tautology or not.

- Example: $p \land (p \rightarrow q) \rightarrow q$ is a tautology.

p	q	$p \rightarrow q$	$p \land (p \rightarrow q)$	$p \land (p \rightarrow q) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Exercise

- Show that $p \rightarrow p$ is a tautology.

p	$p \rightarrow p$
T	
F	

- Show that $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$ is also a tautology.

p	q	$p \rightarrow q$	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
T	T					
T	F					
F	T					
F	F					

Summary

- Mathematical Logic
- Proposition
  - Propositional variables
  - Logical connectives
  - Logical formulae
- Truth value
  - Truth table of logical connectives
  - Tautologies