FUNDAMENTALS OF LOGIC
NO.3 NORMAL FORMS

Tatsuya Hagino

hagino@sfc.keio.ac.jp

lecture URL

https://vu5.sfc.keio.ac.jp/slide/

So Far

- What is Logic?
  - mathematical logic
  - symbolic logic

- Proposition
  - A statement of which truth does not change.
  - propositional variables
  - logical connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$
  - logical formula

- Truth table
  - truth value of logical connectives
  - tautology = always true

Sub Formula

- The truth value of A can be calculated from truth values of *sub formulae* of A.

- Definition: Sub Formulae
  1. A itself is a sub formulae of A.
  2. If A is $(B \wedge C)$, $(B \vee C)$ or $(B \rightarrow C)$, sub formulae of B and C are sub formulae of A.
  3. If A is $(\neg B)$, sub formulae of B are sub formulae of A.

- Example
  - List of the sub formulae of $(p \rightarrow \neg q) \vee (q \wedge r)$.

Assignment

- An *assignment* is a map from the set of propositional variables V to the set of truth value $\{T, F\}$.
  - It assigns true or false to all the propositional variables.
  - Example: When $V = \{p, q\}$, if $v(p) = T \text{ and } v(q) = F$, v is an assignment.

- An assignment v can be uniquely extended to a map from the set of logical formulae Φ to $\{T, F\}$.
  1. $v(A \wedge B) = T \Leftrightarrow v(A) = v(B) = T$
  2. $v(A \vee B) = T \Leftrightarrow v(A) = T$ or $v(B) = T$
  3. $v(A \rightarrow B) = T \Leftrightarrow v(A) = F$ or $v(B) = T$
  4. $v(\neg A) = T \Leftrightarrow v(A) = F$

- Here, `⇔' is a meta symbol expressing necessary and sufficient condition.

- A logical formula A is a tautology.

  ⇔ For any assignment v, $v(A) = T$.

Necessary and Sufficient Condition

- When $A \to B$ holds;
  - A is a *sufficient condition* for B.
  - B is a *necessary condition* for A.

- Example:
  - $x = 2 \to x^2 = 4$
    - $x = 2$ is sufficient for $x^2 = 4$ to hold.
    - $x^2 = 4$ is not sufficient for $x = 2$ to hold. It is just a necessary condition.

  - When `If Taro likes Hanako, Hanako likes Taro' holds:
    - `Taro likes Hanako' is sufficient for 'Hanako liked Taro', but
    - 'Hanako likes Taro' is just necessary for `Taro likes Hanako'.

Satisfiability

- Dual concept of tautology
  - A formula A is *satisfiable* if there is an assignment v and $v(A) = T$.
  - If a formula is not satisfiable, it is *unsatisfiable*.

- Theorem:
  - A necessary and sufficient condition of a formula A being unsatisfiable is $\neg A$ is a tautology.

- Exercise:
  - Find all the combinaron of p, q, r assignment to make the value

$$((p \vee q) \rightarrow r) \vee (p \wedge q)$$

false.

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	$p \wedge q$	$((p \vee q) \rightarrow r) \vee (p \wedge q)$
						F

Equivalent Formula

- $(A \to B) \land (B \to A)$ will be abbreviated to $A \equiv B$.
  - A and B are *equivalent*.
  - $v(A \equiv B) = T \Leftrightarrow v(A) = v(B)$

- Theorem : The following formulae are tautologies:
  - $A \land A \equiv A$
  - $A \lor A \equiv A$

    Idempotent Law
  - $A \land (B \land C) \equiv (A \land B) \land C$
  - $A \lor (B \lor C) \equiv (A \lor B) \lor C$

    Associative Law
  - $A \land B \equiv B \land A$
  - $A \lor B \equiv B \lor A$

    Commutative Law
  - $A \land (A \lor B) \equiv A$
  - $A \lor (A \land B) \equiv A$

    Absorption Law
  - $A \land (B \lor C) \equiv (A \land B) \lor (A \land C)$
  - $A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$

    Distribution Law
  - $\neg\neg A \equiv A$
  - $\neg(A \lor B) \equiv \neg A \land \neg B$
  - $\neg(A \land B) \equiv \neg A \lor \neg B$

    Law of De Morgan
  - $A \to B \equiv \neg A \lor B$

Examples

- *Idempotent*
  - $A \wedge A \ \equiv \ A$
  - p = `Taro likes Hanako'
  - $p \wedge p \ \equiv \ p$
- *Contraposition*
  - $A \rightarrow B \ \equiv \ \neg B \rightarrow \neg A$
  - $\neg B \rightarrow \neg A$ is the contraposition of $A \rightarrow B$.
  - What is the contraposition of `If you are scolded by your teacher, yu study hard'?
- *Double Negation*
  - $\neg\neg A \ \equiv \ A$
  - `I don't know nothing.' ≡ `I know something.'?
  - `I don't dislike you.' ≡ `I like you.'?

Propositional Constant

- For convenience, we have two *propositional constants* representing true and false.
  - $\top$ and $\bot$ are formulae.
  - For any assignment v, $v(\top) = T$ and $v(\bot) = F$.

- Tautologies
  - $A \wedge \neg A \equiv \bot$
  - $A \vee \neg A \equiv \top$
  - $A \vee \bot \equiv A$
  - $A \vee \top \equiv \top$
  - $A \wedge \top \equiv A$
  - $A \wedge \bot \equiv \bot$
  - $\neg \top \equiv \bot$
  - $\neg \bot \equiv \top$
  - $\neg A \equiv A \rightarrow \bot$
  - $A \equiv \top \rightarrow A$

Logical Equivalence

- If $A \equiv B$ is a tautology, A and B are *logically equivalent*.

- $A \sim B$ is used when A and B are logically equivalent.
  - \`~' is not a logical symbol in the logic, but is a meta symbol which represents \`logically equivalent'.

- Theorem: The followings hold for logical equivalence:
  1. $A \sim A$
  2. If $A \sim B$, then $B \sim A$.
  3. If $A \sim B$ and $B \sim C$, then $A \sim C$.
  4. If $A \sim B$, then $C[A/p] \sim C[B/p]$

     where $C[A/p]$ stand for replacing all the occurrence of logical variables p inside C with a formula A.

- This means that logically equivalent formulae can be replaced each other without changing the meaning.

Extending Disjunction and Conjunction

- For n formulae $A_1, . . . , A_n$,
  - $\bigvee_{i=1}^{n} A_i$ represents $(\cdots((A_1 \vee A_2) \vee A_3) \vee \cdots \vee A_n)$, and
  - $\bigwedge_{i=1}^{n} A_i$ represents $(\cdots((A_1 \wedge A_2) \wedge A_3) \wedge \cdots \wedge A_n)$.

- Under the logical equivalence, parentheses may be omitted.
  - $\bigvee_{i=1}^{n} A_i \sim A_1 \vee A_2 \vee A_3 \vee \cdots \vee A_n$
  - $\bigwedge_{i=1}^{n} A_i \sim A_1 \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_n$

Normal Form

- *Literal*
  - A propositional variable or a propositional variable with $\neg$ is called *literal*.
  - Example: p and $\neg q$ are literals, but $\neg\neg r$ is not.

- *Disjunctive Normal Form*
  - For any formula, there is an equivalent logical formula of the form $\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n_i} A_{i,j}$ where $A_{i,j}$ are literals.
  - $(A_{1,1} \wedge A_{1,2} \wedge \cdots \wedge A_{1,n_1}) \vee (A_{2,1} \wedge A_{2,2} \wedge \cdots \wedge A_{2,n_2}) \vee \cdots \vee (A_{m,1} \wedge A_{m,2} \wedge \cdots \wedge A_{m,n_m})$

- *Conjunctive Normal Form*
  - For any formula, there is an equivalent logical formula of the form $\bigwedge_{i=1}^{m} \bigvee_{j=1}^{n_i} A_{i,j}$ where $A_{i,j}$ are literals.
  - $(A_{1,1} \vee A_{1,2} \vee \cdots \vee A_{1,n_1}) \wedge (A_{2,1} \vee A_{2,2} \vee \cdots \vee A_{2,n_2}) \wedge \cdots \wedge (A_{m,1} \vee A_{m,2} \vee \cdots \vee A_{m,n_m})$

Converting to Disjunctive Normal Form

- How to convert a give logical formula to a disjunctive normal form:
  1. Using $A \to B \sim \neg A \vee B$, remove `→'.
  2. Using $\neg(A \vee B) \sim \neg A \wedge \neg B$ and $\neg(A \wedge B) \sim \neg A \vee \neg B$, move `¬' inward until placed in front of propositional variables.
  3. Using $\neg\neg A \sim A$, replace more than two `¬' with only one or none.
  4. Using $A \wedge (B \vee C) \sim (A \wedge B) \vee (A \wedge C)$, move `∧' inside `∨'.

- Examples:
  - $(p \to q) \to r$
    1.
    2.
    3.
    4.
  - $\neg(\mathrm{p} \to q \wedge r)$
    1.
    2.
    3.
    4.

Exercises:

- Find a disjunctive normal form of $\big((p \to q) \to p\big) \to p$.

- Find a disjunctive normal form of $(p \to p \wedge \neg q) \wedge (q \to q \wedge \neg p)$.

Exercise

- Find a conjunctive normal form of $\big((p \to q) \to p\big) \to p$.

- Find a conjunctive normal form of $\neg(p \to q) \land ((q \to s) \to r)$.

Conversion Using Truth Table

- A disjunctive normal form $\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n_i} A_{i,j}$ expressing the condition when the formula becomes true.

- Using the truth table of $\neg(p \to q \wedge r)$, find an equivalent disjunctive normal form.

p	q	r	$q \wedge r$	$p \to q \wedge r$	$\neg(p \to q \wedge r)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

- Picking up the lines with T, a disjunctive normal form of $\neg(p \to q \wedge r)$ is:

Restricting Logical Connectives

- A formula may use four kinds of logical connectives:
  - $\land$, $\lor$, $\rightarrow$, $\neg$
- Using $A \rightarrow B \sim \neg A \lor B$, `$\rightarrow$' is not necessary.
  - $\land$, $\lor$, $\neg$
- Using $A \land B \sim \neg(\neg A \lor \neg B)$, `$\land$' can be expressed by `$\neg$' and `$\lor$'.
  - $\lor$, $\neg$
- Using $A \lor B \sim \neg(\neg A \land \neg B)$, `$\lor$' can be expressed by `$\neg$' and `$\land$'.
  - $\land$, $\neg$

Summary

- Logical Formula
  - sub formula
  - assignment
  - equivalent logical formula
- Normal Form
  - Disjunctive Normal Form
  - Conjunctive Normal Form
- Restricting Logical Connectives