FUNDAMENTALS OF LOGIC
NO.4 PROOF

Tatsuya Hagino
hagino@sfc.keio.ac.jp

lecture URL

https://vu5.sfc.keio.ac.jp/slide/

So Far

- Proposition
  - Sentences of which truth does not change.
  - Propositional variables
  - Logical connectives（∧, ∨, →, ¬）
  - Logical formula
  - Truth table
  - Tautology

- Normal form
  - Disjunctive normal form
  - Conjunctive normal form
  - Restricting logical connectives

Inference (Deduction)

- Using truth table to show the correctness of propositions
  - Calculate the truth value from the truth value of propositional variables.

- *Inference*
  - Infer new correct proposition from correct propositions already known
  - Apply inference rules to propositions
  - `Infer A from premises $B_1, \dots, B_n$'

- *Inference rule*
  - Rule to infer correct proposition from correct premise propositions

- Example:
  - From A and $A \to B$, infer B.
  - *modus ponens* or *syllogism*
  - `All men are mortal' and `Socrates is a man', therefore `Socrates is mortal'.

Axiom and Theorem

- *Axiom*
  - Premises which we believe correct.
    - `There is only one straight line which goes through two different points.'
    - `Parallel straight lines never meet.'

- *Theorem*
  - Propositions which are inferred from axioms using inference rules
  - *Proof* is the inference steps of theorem
    - `The sum of internal angles of any triangle is 180 degrees.'
    - `Pythagorean theorem'

Formal Logical Framework

- Framework for handling logic formally
  - Framework for handling logical formulae
  - Consist of axioms and inference rules

- Frameworks for *Classical Propositional Logic*:
  - Hilbert framework (Hilbert style)
    - Axiomatic framework
    - Only one inference rule: modus ponens
  - Natural Deduction by Gentzen
    - NK framework (NK system)
    - Close to ordinary (human) inference
  - Sequent Calculus by Gentzen
    - LK framework (LK system)
    - Easy to formalize

LK Sequent

- LK system use *sequent*: $\underbrace{A_1, \ldots, A_m}_{\textit{antecedent}} \overset{\textit{turnstile}}{\vdash} \underbrace{B_1, \ldots, B_n}_{\substack{\textit{succedent} \\ \textit{consequent}}}$

- Intuitive meaning
  - If A_1 to A_m are true, at least one of B_1 to B_n is also true.

- m and/or n may be zero.
  - $\vdash B_1, \ldots, B_n$
    - At least one of B_1 to B_n is true.
  - $\vdash B$
    - B is true.
  - $A_1, \ldots, A_m \vdash$
    - If A_1 to A_m are true, contradicts.
    - At least one of A_1 to A_m is not true.
  - $A \vdash$
    - A is not true.
  - $\vdash$
    - Contradiction.

LK Axiom and Inference Rules

- Axiom: *Initial Sequent*

$$\frac{}{A \vdash A}\ \text{(I)}$$

- Inference rules for structure: weakening, contraction, exchange, cut

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \text{(WL)} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ \text{(WR)}$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \text{(CL)} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \text{(CR)}$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ \text{(EL)} \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ \text{(ER)}$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \text{(Cut)}$$

(where Γ, Δ are sequence of logical formulae)

Inference rules (cont.)

- Inference rules for logical connectives:

$$\frac{A, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta}\ (\land L_1) \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \lor B}\ (\lor R_1)$$

$$\frac{B, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta}\ (\land L_2) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \lor B}\ (\lor R_2)$$

$$\frac{A, \Gamma_1 \vdash \Delta_1 \quad B, \Gamma_2 \vdash \Delta_2}{A \lor B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\lor L) \qquad \frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \land B}\ (\land R)$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{A \to B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\to L) \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ (\to R)$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ (\neg L) \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ (\neg R)$$

LK Proof Figure

- LK *Proof Figure*:
  - Start from initial sequent and apply inference rules.
  - The bottom sequent is called *end sequent* of the proof figure.

- Example: proof figure

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I})}{\vdash A, \neg A}\,(\neg\mathrm{R})}{\vdash A, A \vee \neg A}\,(\vee\mathrm{R}_2)}{\vdash A \vee \neg A, A}\,(\mathrm{ER})}{\vdash A \vee \neg A, A \vee \neg A}\,(\vee\mathrm{R}_1)}{\vdash A \vee \neg A}\,(\mathrm{CR})
$$

end sequent

- When there is a proof figure of which end sequent is S, S is *provable* in LK.

Extending Inference Rules

- Applying inference rules to formula in the sequent other than left-most or right-most one.
  - Using exchange rules, formula at any position can be moved to left-most or right-most position.
  - Extend inference rules of contraction, weakening and logical connectives to formula at any position.

- *Lemma*
  - When S can be inferred from $S_1, S_2, \ldots, S_n$ using LK inference rules, we write:

$$\frac{S_1 \quad S_2 \quad \ldots \quad S_n}{S}$$

The above inference can be used in other proofs as a lemma.

- Example: lemma

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \lor B} \qquad \frac{\Gamma \vdash \Delta, A \lor B}{\Gamma \vdash \Delta, A, B} \qquad \frac{\Gamma \vdash \Delta, A \lor B}{\Gamma \vdash \Delta, B \lor A}$$

Syntactic Meaning of Sequent

Theorem: The followings are equivalent:

1. A sequent $A_1, \dots, A_m \vdash B_1, \dots, B_n$ is provable in LK.
2. A sequent $A_1 \land \cdots \land A_m \vdash B_1 \lor \cdots \lor B_n$ is provable in LK.
3. A formula $A_1 \land \cdots \land A_m \to B_1 \lor \cdots \lor B_n$ is provable in LK.

Proof:

1⇒2

$$\dfrac{\dfrac{A_1, \dots, A_m \vdash B_1, \dots, B_n}{A_1, \dots, A_m \vdash B_1 \lor \cdots \lor B_n}\ (\lor\mathrm{R},\mathrm{CR})}{A_1 \land \cdots \land A_m \vdash B_1 \lor \cdots \lor B_n}\ (\land\mathrm{L},\mathrm{CL})$$

2⇒3

$$\dfrac{A_1 \land \cdots \land A_m \vdash B_1 \lor \cdots \lor B_n}{\vdash A_1 \land \cdots \land A_m \to B_{11} \lor \cdots \lor B_n}\ (\to\mathrm{R})$$

3⇒1

$$\dfrac{\vdash A_1 \land \cdots \land A_m \to B_1 \lor \cdots \lor B_n \qquad \dfrac{(\land\mathrm{R})\ \dfrac{(\mathrm{WL})\ \dfrac{(\mathrm{I})\ \dfrac{}{A \vdash A_1}}{A_1, \dots, A_m \vdash A_1} \quad \cdots}{A_1, \dots, A_m \vdash A_1 \land \cdots \land A_m} \qquad \dfrac{\dfrac{\dfrac{}{B_1 \vdash B_1}\ (\mathrm{I})}{B_1 \vdash B_1, \dots, B_n}\ (\mathrm{WR}) \quad \cdots}{B_1 \lor \cdots \lor B_n \vdash B_1, \dots, B_n}\ (\lor\mathrm{L})}{A_1 \land \cdots \land A_m \to B_1 \lor \cdots \lor B_n, A_1, \dots, A_m \vdash B_1, \dots, B_n}\ (\to\mathrm{L})}{A_1, \dots, A_m \vdash B_1, \dots, B_n}\ (\mathrm{Cut})$$

Exercise (1)

- Show a proof figure of $\neg(A \land \neg B) \vdash A \to B$

$$\frac{}{\neg(A \land \neg B) \vdash A \to B}$$

Exercise (2)

- Show a proof figure of $A \to B, A \to \neg B \vdash \neg A$

$$\frac{}{A \to B, A \to \neg B \vdash \neg A}$$

Exercise (3)

- Show a proof figure of $A \to B \vdash \neg(A \land \neg B)$

$$\frac{}{A \to B \vdash \neg(A \land \neg B)}$$

Exercise (4)

- Show a proof figure of $A \vdash \neg\neg A$

$$\frac{}{A \vdash \neg\neg A}$$

Exercise (5)

- Show a proof figure of $\neg\neg A \vdash A$

$$\frac{}{\neg\neg A \vdash A}$$

Summary

- Inference
  - Axiom
  - Theorem
- LK System
  - Sequent calculus
  - Initial sequent
  - LK inference rules
- Proof
  - Proof figure