# FUNDAMENTALS OF LOGIC
# NO.8 SEMANTICS OF PREDICATE LOGIC

Tatsuya Hagino
hagino@sfc.keio.ac.jp

lecture URL

https://vu5.sfc.keio.ac.jp/slide/

# So Far

- Propositional Logic
  - Logical connectives ($\wedge$, $\vee$, $\rightarrow$, $\neg$)
  - Truth value table
  - Tautology
  - Normal form
  - Axiom and proof
  - LK framework and NK framework
  - Soundness and completeness

- Predicate Logic
  - Logical formula (language, term)
  - Quantifier ($\forall x\ P(x)$, $\exists x\ P(x)$)
  - Bound and free variables
  - Closed formula

# Predicate Logic

- From propositional logic to predicate logic
  - Extend to handle properties and relations of objects
  - Object variables, constants and predicates

- Predicate Logic
  - Use predicates instead of propositional variables
  - Logical connectives: $\land$, $\lor$, $\rightarrow$, $\neg$
  - Quantifiers: $\forall x$, $\exists x$

# Write in Predicate Logic

- Let $S, M, W, H, T$ and $L$ be as follows:
  - $S(x)$ $=$ "$x$ is an SFC student",
  - $B(x)$ $=$ "$x$ is a boy",
  - $G(x)$ $=$ "$x$ is a girl",
  - $H(x)$ $=$ "$x$ is handsome",
  - $T(x)$ $=$ "$x$ is tall", and
  - $L(x, y)$ $=$ "$x$ likes $y$".

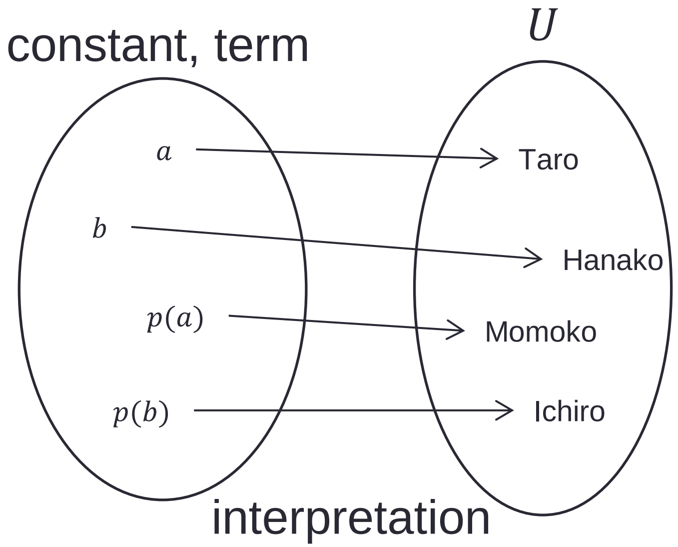
- Write the following statements in predicate logic:
  1. There are boy students in SFC.
  2. SFC students are handsome.
  3. SFC boy students are handsome.
  4. There are handsome boy SFC students.
  5. SFC girl students are tall.

# cont.

6. Some SFC boy students are not tall.
7. Handsome SFC students are tall.
8. Handsome SFC students are not necessarily tall.
9. SFC boy students are either tall or handsome.
10. Girls like tall boys.
11. Girls in SFC like boys who are tall and handsome.
12. A tall boy is liked by any girls.

# Semantics of Predicate Logic

- In order to determine truth value of predicate logic formulae, the set of objects need to be selected.




- *Domain*
  - A set $U$ of objects
- *Interpretation*
  - Each constant is mapped to an element in $U$
  - Each variable has any value in $U$
  - Each function symbol us mapped to a function on $U$
  - Each predicate symbol is mapped to a predicate on $U$
- *Structure*
  - A pair of domain $U$ and interpretation $\sigma$
  - $\langle U, \sigma \rangle$

# Definition: Structure

- For a language $L$, its *structure* $\mu = \langle U, \sigma \rangle$ is defined as follow:
  1. $U$ is a non empty set. (domain of $\mu$)
  2. $\sigma$ is a map which maps constants, function symbols and predicate symbols of $L$ to elements, functions and predicates on $U$, respectively.
     a. If $c$ is a constant, $c^\sigma \in U$.
     b. If $f$ is an $n$-ary function symbol, $f^\sigma$ is a function from $U^n$ to $U$:
     $$f^\sigma : U^n \to U$$
     c. If $P$ is a predicate with $n$ variables (except equality), $P^\sigma$ is a predicate on $U$:
     $$P^\sigma \subseteq U^n$$

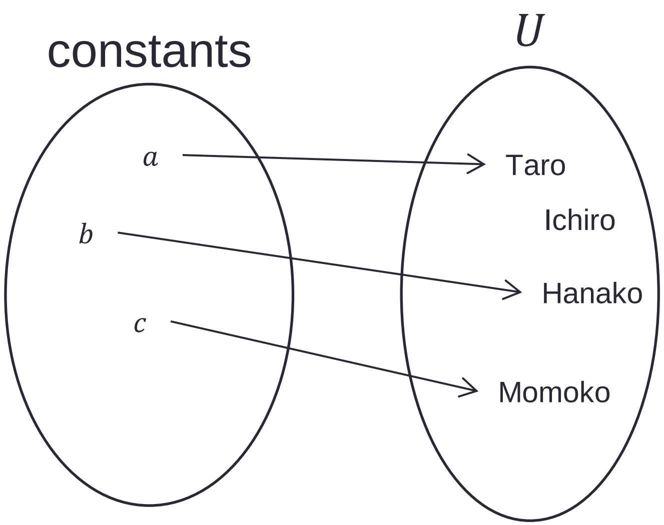
  $\sigma$ is called an *interpretation*.

- Language $L[\mu]$
  - Language $L$ with elements of domain $U$ of $\mu = \langle U, \sigma \rangle$ added as constants.
  - For $u \in U$, $u$ stands for its constant of $L[\mu]$.
  - $u^\sigma = u$

# Example of Structure

- Language: $L$
  - constant : $a, b, c$
  - function : $f(x)$
  - predicate : $S(x), P(x), L(x, y)$

- Structure: $\mu = \langle U, \sigma \rangle$
  - $U = \{\text{Taro, Ichiro, Hanako, Momoko}\}$
  - constant:
    - $a^\sigma = \text{Trao}$
    - $b^\sigma = \text{Hanako}$
    - $c^\sigma = \text{Momoko}$
  - function:
    - $f^\sigma(\text{Taro}) = \text{Ichiro}$
    - $f^\sigma(\text{Ichiro}) = \text{Taro}$
    - $f^\sigma(\text{Hanako}) = \text{Momoko}$
    - $f^\sigma(\text{Momoko}) = \text{Hanako}$
  - predicate:
    - $S^\sigma = \{\text{Taro, Hanako}\}$
    - $P^\sigma = \{\text{Hanako}\}$
    - $L^\sigma = \{(\text{Taro, Hanako}), (\text{Momoko, Ichiro}), (\text{Hanako, Taro})\}$




Interpretation $\sigma$

# Interpretation of Formulae

- For a structure $\mu = \langle U, \sigma \rangle$ , the meaning $\mu$ of a term $t$ of $L[\mu]$ without variables is defined as follows:
  1. If $t$ is a constant $c$, $t^{\mu} = c^{\sigma}$
  2. If $t$ is $f(t_1, \cdots, t_n)$, $t^{\mu} = f^{\sigma}(t_1{}^{\mu}, \cdots, t_n{}^{\mu})$

- For a closed formula $A$ of $L[\mu]$, $\mu \vDash A$ means $A$ holds in a structure $\mu = \langle U, I \rangle$, and $\mu \nvDash A$ means $A$ does not hold.
  1. $\mu \vDash P(t_1, \cdots, t_n) \iff (t_1{}^{\mu}, \cdots, t_n{}^{\mu}) \in P^{\sigma}$
     For the equality symbol, $\mu \vDash t_1 = t_2 \iff t_1{}^{\mu} = t_2{}^{\mu}$
  2. $\mu \vDash A \wedge B \iff \mu \vDash A$ and $\mu \vDash B$
  3. $\mu \vDash A \vee B \iff \mu \vDash A$ or $\mu \vDash B$
  4. $\mu \vDash A \rightarrow B \iff \mu \nvDash A$ or $\mu \vDash B$
  5. $\mu \vDash \neg A \iff \mu \nvDash A$
  6. $\mu \vDash \forall x\, A \iff$ for any element $u \in U$, $\mu \vDash A[u/x]$
  7. $\mu \vDash \exists x\, A \iff$ there is an element $u \in U$ which makes $\mu \vDash A[u/x]$

- If $A$ is not closed, use its closure $A^*$ and
  - $\mu \vDash A \iff \mu \vDash A^*$

# Example of Interpretation

- Terms:
  - $f(a)^{\mu} = f^{\sigma}(a^{\mu}) = f^{\sigma}(a^{\sigma}) = f^{\sigma}(\mathrm{Taro}) = \mathrm{Ichiro}$
  - $f(f(b))^{\mu} =$
- Formulae:
  - $\mu \vDash S(a) \Longleftrightarrow a^{\mu} \in S^{\sigma} \Longleftrightarrow \mathrm{Taro} \in \{\mathrm{Taro}, \mathrm{Hanako}\}$
  - $\mu \vDash L(a, f(c)) \Longleftrightarrow (a^{\mu}, f(c)^{\mu}) \in L^{\sigma} \Longleftrightarrow$
  - $\mu \vDash \forall x (P(x) \rightarrow S(x)) \Longleftrightarrow$
  - $\mu \vDash \forall x (S(x) \rightarrow \exists y\, L(x, y)) \Longleftrightarrow$

# Valid Formulae

- $A$ is *valid* $\Longleftrightarrow$
  - For any structure $\mu = \langle U, I \rangle$, $\mu \vDash A$

- Valid formulae

  (where $A$ does not contain $x$ as a free variable, and $y$ is a variable which does not appear in $B$.)

  1. $\forall x\, A \equiv A$ , $\exists x\, A \equiv A$
  2. $\forall x\, B \equiv \forall y\, B[y/x]$ , $\exists x\, B \equiv \exists y\, B[y/x]$
  3. $A \wedge \forall x\, B \equiv \forall x(A \wedge B)$ , $A \wedge \exists x\, B \equiv \exists x(A \wedge B)$
  4. $A \vee \forall x\, B \equiv \forall x(A \vee B)$ , $A \vee \exists x\, B \equiv \exists x(A \vee B)$
  5. $\forall x\, B \wedge \forall x\, C \equiv \forall x(B \wedge C)$ , $\exists x\, B \vee \exists x\, C \equiv \exists x(B \vee C)$
  6. $\forall x\, B \vee \forall x\, C \rightarrow \forall x(B \vee C)$ , $\exists x\, (B \wedge C) \rightarrow \exists x\, B \wedge \exists x\, C$
  7. $\forall x \forall y\, D \equiv \forall y \forall x\, D$ , $\exists x \exists y\, D \equiv \exists y \exists x\, D$
  8. $\exists x \forall y\, D \rightarrow \forall y \exists x\, D$
  9. $\forall x\, B \rightarrow \exists x\, B$
  10. $\neg \forall x\, B \equiv \exists x \neg B$ , $\neg \exists x\, B \equiv \forall x \neg B$
  11. $A \rightarrow \forall x\, B \equiv \forall x(A \rightarrow B)$ , $A \rightarrow \exists x\, B \equiv \exists x(A \rightarrow B)$
  12. $\forall x\, B \rightarrow A \equiv \exists x(B \rightarrow A)$ , $\exists x\, B \rightarrow A \equiv \forall x(B \rightarrow A)$
  13. $\exists x\, (B \rightarrow C) \equiv \forall x\, B \rightarrow \exists x\, C$
  14. $\forall x\, (B \rightarrow C) \rightarrow (\forall x\, B \rightarrow \forall x\, C)$
  15. $\forall x\, (B \rightarrow C) \rightarrow (\exists x\, B \rightarrow \exists x\, C)$

$$\boxed{A \equiv B}$$
$$\Updownarrow$$
$$\boxed{(A \rightarrow B) \wedge (B \rightarrow A)}$$
$$\Updownarrow$$
$$\boxed{\mu \vDash A \Longleftrightarrow \mu \vDash B}$$

- Note: Followings are not valid:
  - $\forall x(B \vee C) \rightarrow \forall x\, B \vee \forall x\, C$
  - $\exists x\, B \wedge \exists x\, C \rightarrow \exists x\, (B \wedge C)$
  - $\exists x\, B \rightarrow \forall x\, B$
  - $\forall x \exists y\, D \rightarrow \exists y \forall x\, D$

# Example of Valid and not Valid Formulae

- $S(x) =$ "$x$ is a student", $T(x) =$ "$x$ is a teacher".
  - ○ $\exists x\,(S(x) \wedge T(x)) \rightarrow \exists x\, S(x) \wedge \exists x\, T(x)$
    English " "
  - × $\exists x\, S(x) \wedge \exists x\, T(x) \rightarrow \exists x\,(S(x) \wedge T(x))$
    English " "
- $M(x) =$ "$x$ is a boy", $F(x) =$ "$x$ is a girl".
  - ○ $\forall x\, M(x) \vee \forall x\, F(x) \rightarrow \forall x(M(x) \vee F(x))$
    English " "
  - × $\forall x(M(x) \vee F(x)) \rightarrow \forall x\, M(x) \vee \forall x\, F(x)$
    English " "
- $L(x, y) =$ "$x$ likes $y$"
  - ○ $\exists x \forall y\, L(x, y) \rightarrow \forall y \exists x\, L(x, y)$
    English " "
  - × $\forall x \exists y\, L(x, y) \rightarrow \exists y \forall x\, L(x, y)$
    English " "

# Valid and Satisfiable

- Satisfiable
  - Let $x_1, \cdots, x_n$ be free variables of $A$, $A$ is *satisfiable* $\Longleftrightarrow$
    - For a structure $\mu = \langle U, I \rangle$ and elements $u_1, \cdots u_n$, $\mu \vDash A[u_1/x_1, \cdots u_n/x_n]$

- The necessary and sufficient condition of $A$ not being satisfiable is $\neg A$ being valid.

# Prenex Formula

- Prenex formula
  - Let $Q_1, \cdots, Q_n$ be ∀ or ∃ and $A$ be a formula without quantifiers:
$$Q_1 x_1 \cdots Q_n x_n\, A$$
is called a *prenex formula*.

- $A \sim B$
  - If $A \equiv B$ is valid, $A$ and $B$ are *logically equivalent*, and write it as $A \sim B$.
  - $\sim$ is an equivalent relation.

- **Theorem**: For any formula $A$, there is a prenex formula $A^+$ and $A \sim A^+$.

- For a formula $A$, a prenex formula $A^+$ where $A \sim A^+$ is called its *prenex normal form*.
  - A prenex normal form may not be unique.

# Example (1)

- Find an equivalent prenex normal form:

1. $(\exists y\, P(y) \vee Q(x)) \to \exists x\, R(x)$

# Example (2)

- Find an equivalent prenex normal form:

2. $\exists x\, R(x, y) \rightarrow \forall y (P(y) \land \neg \forall z\, Q(z))$

# Example (3)

- Find an equivalent prenex normal form:

3. $\exists x(\forall y\big(P(y) \rightarrow Q(x,z)\big) \vee \exists z\big(\neg\exists u\, R(z,u) \wedge Q(x,z)\big))$

# Summary

- Semantics of predicate logic
  - domain
  - interpretation
  - structure ＝domain＋interpretation
- $\mu \vDash A$
- Valid formulae
  - Satisfiable
- Prenex normal form