FUNDAMENTALS OF LOGIC
NO.9 PROOF IN PREDICATE LOGIC

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lecture URL

https://vu5.sfc.keio.ac.jp/slide/

So Far

- Propositional Logic
  - Logical connectives ($\wedge$, $\vee$, $\rightarrow$, $\neg$)
  - Truth table
  - Tautology
  - Normal form
  - Axiom and theorem
  - LK frame work
  - Soundness and completeness

- Predicate Logic
  - Logical Formulas (language, term)
  - Quantifiers ($\forall x\ P(x)$, $\exists x\ P(x)$)
  - Closed formulae (bound and free variables)
  - Semantics of predicate logic (domain, interpretation, structure)
  - Valid formulae
  - Prenex formulae

Exercise: express in predicate logic

- Let S, P, J, M, L, T, H be as follows:
  - $S(x)$ $=$ "x is an SFC student."
  - $P(x)$ $=$ "x is an SFC professor."
  - $J(x)$ $=$ "x is a lecture at SFC."
  - $M(x)$ $=$ "x is a math lecture."
  - $L(x, y)$ $=$ "x likes y."
  - $T(x, y)$ $=$ "x takes lecture y."
  - $H(x)$ $=$ "x is happy."

- Write the following sentences in predicate logic:
  1. SFC students are happy.
  2. All the SFC lectures are math.
  3. SFC students always take lectures which they like.
  4. SFC students must take a math lecture.

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5. SFC students like any math lectures.

6. There are some SFC students who do not like math lectures.

7. SFC students who take math lectures will like them.

8. If there are some SFC students who like math lectures, SFC professors are happy.

9. If all the SFC students like math lectures, SFC professors are happy.

LK Framework

- Sequent

$$A_1, \ldots, A_m \vdash B_1, \ldots, B_n$$

- meaning: If all of $A_1, \ldots, A_m$ hold, one of $B_1, \ldots, B_n$ holds.

- Inference rules for axioms and structure:

$$\frac{}{A \vdash A}\,(\mathrm{I}) \qquad \frac{}{\vdash \top}\,(\top) \qquad \frac{}{\bot \vdash}\,(\bot)$$

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,(\mathrm{WL}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\,(\mathrm{WR}) \qquad \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,(\mathrm{CL}) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,(\mathrm{CR})$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\,(\mathrm{EL}) \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\,(\mathrm{ER}) \qquad \frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,(\mathrm{Cut})$$

LK Inference Rules

- Inference rules for logical connectives:

$$\frac{A, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta}\ (\land L_1) \qquad \frac{B, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta}\ (\land L_2) \qquad \frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \land B}\ (\land R)$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \lor B}\ (\lor R_1) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \lor B}\ (\lor R_2) \qquad \frac{A, \Gamma_1 \vdash \Delta_1 \quad B, \Gamma_2 \vdash \Delta_2}{A \lor B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\lor L)$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{A \to B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\to L) \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ (\to R)$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ (\neg L) \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ (\neg R)$$

LK Inference Rules for Predicate Logic

- Add the following inference rules to the LK framework for propositional logic:

$$\frac{A[t/x], \Gamma \vdash \Delta}{\forall x\, A, \Gamma \vdash \Delta}\ (\forall L) \qquad \frac{\Gamma \vdash \Delta, A[z/x]}{\Gamma \vdash \Delta, \forall x\, A}\ (\forall R)$$

$$\frac{A[z/x], \Gamma \vdash \Delta}{\exists x\, A, \Gamma \vdash \Delta}\ (\exists L) \qquad \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x\, A}\ (\exists R)$$

- where t is a term and z is a variable.
- z is called *eigenvariable* and the rules are only applicable when z does not appear as free in the blow sequent: $\Gamma, \Delta, \forall x\, A, \exists x\, A$

Example

- Let $P(x), Q(x), R(x, y)$ be predicates.

$$\frac{P(\mathrm{Taro}), \Gamma \vdash \Delta}{\forall x\, P(x), \Gamma \vdash \Delta}\ (\forall \mathrm{L}) \qquad \frac{\Gamma \vdash \Delta, P(z)}{\Gamma \vdash \Delta, \forall x\, P(x)}\ (\forall \mathrm{R})$$

$$\frac{P(z) \wedge Q(z), \Gamma \vdash \Delta}{\exists x\big(P(x) \wedge Q(x)\big), \Gamma \vdash \Delta}\ (\exists \mathrm{L}) \qquad \frac{\Gamma \vdash \Delta, P(\mathrm{Taro}) \to R(\mathrm{Taro}, Hanako)}{\Gamma \vdash \Delta, \exists x\,\big(P(x) \to R(x, Hanako)\big)}\ (\exists \mathrm{R})$$

- The following applications are wrong because they do not satisfy the eigenvariable condition.

$$\frac{P(z) \vdash P(z)}{P(z) \vdash \forall x\, P(x)}\ (\forall \mathrm{R}) \qquad \frac{P(z) \wedge Q(z) \vdash R(x, z)}{\exists x(P(x) \wedge Q(z))\ \vdash R(x, z)}\ (\exists \mathrm{L})$$

Proof Figure

- $\forall x(A \to B) \vdash A \to \forall x\, B$ (where A does not contain x as a free variable)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I}) \qquad \dfrac{}{B \vdash B}\,(\mathrm{I})}{A \to B, A \vdash B}\,(\to\mathrm{L})}{\forall x(A \to B), A \vdash B}\,(\forall\mathrm{L})}{\forall x(A \to B), A \vdash \forall x\, B}\,(\forall\mathrm{R})}{\forall x(A \to B) \vdash A \to \forall x\, B}\,(\to\mathrm{R},\mathrm{EL})$$

$$\dfrac{A[t/x], \Gamma \vdash \Delta}{\forall x\, A, \Gamma \vdash \Delta}\,(\forall\mathrm{L})$$

Choose x as term t

Choose x as the eigenvariable
Since x is not free in A

$$\dfrac{\Gamma \vdash \Delta, A[z/x]}{\Gamma \vdash \Delta, \forall x\, A}\,(\forall\mathrm{R})$$

$B[x/x]$ は B

Wrong Proof Figure

- $\forall x(A \to B) \vdash A \to \forall x\, B$ (where A does not contain x as a free variable)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I}) \qquad \dfrac{}{B \vdash B}\,(\mathrm{I})}{A \to B, A \vdash B}\,(\to\mathrm{L})}{A \to B, A \vdash \forall x\, B}\,(\forall\mathrm{R})}{A \to B \vdash A \to \forall x\, B}\,(\to\mathrm{R,EL})}{\forall x(A \to B) \vdash A \to \forall x\, B}\,(\forall\mathrm{L})$$

$$\dfrac{\Gamma \vdash \Delta, A[z/x]}{\Gamma \vdash \Delta, \forall x\, A}\,(\forall\mathrm{R})$$

Choose x as the eigenvariable?

$$\dfrac{A[t/x], \Gamma \vdash \Delta}{\forall x\, A, \Gamma \vdash \Delta}\,(\forall\mathrm{L})$$

Choose x as term t

Socrates Example

- Prove Socrates problem.
  - $P(x)$ = "x is a man."
  - $Q(x)$ = "x is mortal."
  - Let s be an object constant for Socrates.

  - Major premise: All men are mortal.
    - $\forall x\ (P(x) \rightarrow Q(x))$
  - Minor premise: Socrates is a man.
    - $P(s)$
  - Conclusion: Therefore, Socrates is mortal.
    - $Q(s)$

$$\forall x\big(P(x) \rightarrow Q(x)\big), P(s) \vdash Q(s)$$

Prove the following formulae (1)

- $\forall x\, B \land \forall x\, C \vdash \forall x(B \land C)$

Prove the following formulae (2)

- $\forall x(B \land C) \vdash \forall x\, B \land \forall x\, C$

Prove the following formulae (3)

- $\forall x\, B \vee \forall x\, C \vdash \forall x (B \vee C)$

Prove the following formulae (4)

- $\neg\forall x\, B \vdash \exists x \neg B$

Prove the following formulae (5)

- $\exists x \neg B \vdash \neg \forall x\, B$

Prove the following formulae (6)

- $\exists x \forall y\ D \vdash \forall y \exists x\ D$

Theorem about Proofs in LK

- *cut Elimination Theorem:*
  - If a sequent $\Gamma \vdash \Delta$ is provable in LK, there is a proof figure of $\Gamma \vdash \Delta$ which does not contain cut inference rules.

- Theorem about sub formulae appearing in the proof figure:
  - Any logical formula appearing in a proof figure without cut of $\Gamma \vdash \Delta$ is a sub formulae of $\Gamma \vdash \Delta$.

- *Undecidability* of predicate logic:
  - Whether a given formula is provable in LK or not is undecidable.
  - This means that there is no finite procedure (i.e. algorithm) of proving logical formulae.
  - Proved by Alonzo Church.

Sound and Completeness of LK

- Soundness of LK
  - For any sequent $\Gamma \vdash \Delta$, if $\Gamma \vdash \Delta$ is provable in LK, $\Gamma \vdash \Delta$ is valid.
  - For each inference rule, if the above sequents are valid, the blow sequent is also valid.

- Completeness of LK
  - For any sequent $\Gamma \vdash \Delta$, if $\Gamma \vdash \Delta$ is valid, $\Gamma \vdash \Delta$ is provable in LK.
  - Proved by Kurt Gödel (Completeness Theorem of K. Gödel)

Formal Theory

- A set T of closed formulae of language L is called *formal theory*:
  - A sequent $\Gamma \vdash \Delta$ is provable in T
    $\Longleftrightarrow$ with a finite number of logical formulae $B_1, \ldots, B_n$ of T, $B_1, \ldots, B_n, \Gamma \vdash \Delta$ is provable in LK.
  - $T, \Gamma \vdash \Delta$

- T consists of formulae which we believe true. Axioms of axiomatic theory.

- T is *inconsistent* $\Longleftrightarrow$
  - $T \vdash$

- T is *consistent* $\Longleftrightarrow$
  - T is not inconsistent.

- *Model* of T
  - For a formal theory T of L and a structure μ, if any formula A in T satisfies $\mu \vDash A$, μ is a *model* of T.

Axioms of Equality

- Let $=$ be a predicate symbol with two variables in L, the following set of formulae is called the equality axiom.
  1. $\forall x(x = x)$
  2. $\forall x \forall y(x = y \rightarrow y = x)$
  3. $\forall x \forall y \forall z(x = y \wedge y = z \rightarrow x = z)$
  4. For each function symbol f with m varaibles,
     $\forall x_1 \dots \forall x_m \forall y_1 \dots \forall y_m (x_1 = y_1 \wedge \cdots \wedge x_m = y_m \rightarrow f(x_1, \dots, x_m) = f(y_1, \dots, y_m))$
  5. For each predicate symbol P of n variables,
     $\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n \left(x_1 = y_1 \wedge \cdots \wedge x_n = y_n \rightarrow \left(P(x_1, \dots, x_n) \rightarrow P(y_1, \dots, y_n)\right)\right)$

- Form now on, if the language L contains the equality symbol, we assume any theorem T contains equality axioms.

Example: Theory of Group

- A language L consists of a constant e, a unary symbol $^{-1}$ and two binary function symbols $\cdot$ and $=$. The theory T of group has equality axioms and the following formulae:
  1. $\forall x \forall y \forall z \big((x \cdot y) \cdot z = x \cdot (y \cdot z)\big)$
  2. $\forall x(e \cdot x = x \wedge x \cdot e = x)$
  3. $\forall x(x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e)$

- For any group G, if I maps e to the identity element, $\cdot$ to the operation and $^{-1}$ to the inverse, the structure $\langle G, I \rangle$ is a model of T.

- Since a model of T naturally defines a group, we consider the group G and the structure $\langle G, I \rangle$ as the same thing.

Strong Completeness and Compactness Theorem

- *Strong Completeness*
  - For any theory T, if T is consistent, T has a model.

- Proving Completeness form Strong Completeness.
  - Assume $\Gamma \vdash \Delta$ is not probable in LK.
  - $\neg C \vdash$ is not probable in LK where C is the closure $\Gamma^* \rightarrow \Delta_*$.
  - Let T be a theory consisting of $\neg C$ only. Since $\neg C \vdash$ is not provable in LK, T is consistent.
  - From strong completeness, T has a model μ.
  - $\mu \vDash \neg C$
  - $\mu \nvDash C$
  - C is not valid and $\Gamma \vdash \Delta$ is not valid either.

- *Compactness Theorem*
  - For any theory T, the necessary and sufficient condition of T having a model is for any finite subject of T to have a model.

Summary

- LK frame work for predicate logic
  - Inference rules
    - eigenvariable
  - Sound and completeness

- Formal theory
  - Strong completeness
  - Compact theorem