FUNDAMENTALS OF LOGIC NO.4 PROOF

Tatsuya Hagino
hagino@sfc.keio.ac.jp

lecture URL

https://vu5.sfc.keio.ac.jp/slide/

So Far

- Proposition
  - Sentences of which truth does not change.
  - Propositional variables
  - Logical connectives（$\wedge$, $\vee$, $\rightarrow$, $\neg$）
  - Logical formula
  - Truth table
  - Tautology

- Normal form
  - Disjunctive normal form
  - Conjunctive normal form
  - Restricting logical connectives

Inference (Deduction)

- Using truth table to show the correctness of propositions
  - Calculate the truth value from the truth value of propositional variables.

- *Inference*
  - Infer new correct proposition from correct propositions already known
  - Apply inference rules to propositions
  - `Infer A from premises $B_1, \ldots, B_n$'

- *Inference rule*
  - Rule to infer correct proposition from correct premise propositions

- Example:
  - From A and $A \rightarrow B$, infer B.
  - *modus ponens* or *syllogism*
  - `All men are mortal' and `Socrates is a man', therefore `Socrates is mortal'.

Axiom and Theorem

- *Axiom*
  - Premises which we believe correct.
    - `There is only one straight line which goes through two different points.'
    - `Parallel straight lines never meet.'

- *Theorem*
  - Propositions which are inferred from axioms using inference rules
  - *Proof* is the inference steps of theorem
    - `The sum of internal angles of any triangle is 180 degrees.'
    - `Pythagorean theorem'

Formal Logical Framework

- Framework for handling logic formally
  - Framework for handling logical formulae
  - Consist of axioms and inference rules

- Frameworks for *Classical Propositional Logic*:
  - Hilbert framework (Hilbert style)
    - Axiomatic framework
    - Only one inference rule: modus ponens
  - Natural Deduction by Gentzen
    - NK framework (NK system)
    - Close to ordinary (human) inference
  - Sequent Calculus by Gentzen
    - LK framework (LK system)
    - Easy to formalize

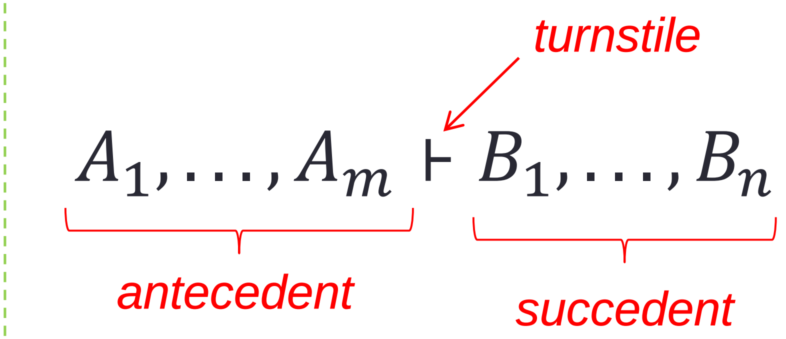
LK Sequent

- LK system uses ***sequent*** :

$$A_1, \dots, A_m \vdash B_1, \dots, B_n$$

- Intuitive meaning
  - If A_1 to A_m are true, at least one of B_1 to B_n is also true.

- \` $\vdash$ ' is called:
  - turnstyle
  - tee



- $A_1, \dots, A_m \vdash B_1, \dots, B_n$
  - $A_1, \dots, A_m$ antecedent (assumption)
  - $B_1, \dots, B_n$ succedent (consequence)
  - \`The succedent is inferred from the antecedent.'

Special Cases for Sequent

$$A_1, \ldots, A_m \vdash B_1, \ldots, B_n$$

- m or n can be 0
  - $\vdash B_1, \ldots, B_n$
    - At least one of B_1 to B_n is true.
  - $\vdash B$
    - B is true.
  - $A_1, \ldots, A_m \vdash$
    - If A_1 to A_m are true, contradicts.
    - At least one of A_1 to A_m is not true.
  - $A \vdash$
    - A is not true.
  - $\vdash$
    - Contradiction

An Example of LK Inference Rules

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,(\mathrm{CL})$$

premise sequent

name of the rule

conclusion sequent

- Given the premise sequent, infer the conclusion sequent.
  - A is a logical formula
  - Γ and Δ are sequence of logical formulae.
    - Γ or Δ or both may be empty.

- Examples:

$$\frac{A, A, B, C \vdash D}{A, B, C \vdash D}\,(\mathrm{CL}) \qquad \frac{A, A \vdash A \wedge B}{A \vdash A \wedge B}\,(\mathrm{CL}) \qquad \frac{A \vee B, A \vee B \vdash A \wedge B}{A \vee B \vdash A \wedge B}\,(\mathrm{CL})$$

$$\frac{p \to q \vee r, p \to q \vee r, p \wedge s \vdash s \to t}{p \to q \vee r, p \to q \vdash s \to t}\,(\mathrm{CL})$$

LK Axiom and Inference Rules

- Axiom: *Initial Sequent* and *Constants*

$$\frac{}{A \vdash A}\,(\mathrm{I}) \qquad \frac{}{\vdash \top}\,(\top) \qquad \frac{}{\bot \vdash}\,(\bot)$$

- Inference rules for structure: weakening, contraction, exchange, cut

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,(\mathrm{WL}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\,(\mathrm{WR})$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,(\mathrm{CL}) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,(\mathrm{CR})$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\,(\mathrm{EL}) \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\,(\mathrm{ER})$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,(\mathrm{Cut})$$

(where Γ, Δ are sequence of logical formulae)

Inference rules (cont.)

- Inference rules for logical connectives:

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge L_1)$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ (\vee R_1)$$

$$\frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge L_2)$$

$$\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ (\vee R_2)$$

$$\frac{A, \Gamma_1 \vdash \Delta_1 \quad B, \Gamma_2 \vdash \Delta_2}{A \vee B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\vee L)$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B}\ (\wedge R)$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{A \rightarrow B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\rightarrow L)$$

$$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}\ (\rightarrow R)$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ (\neg L)$$

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ (\neg R)$$

LK Axiom

$$\frac{}{A \vdash A}\ (\mathrm{I})$$

- Meaning:
  - A can be inferred from A .

- LK has only one axiom.
  - Since A can be any formula, there are infinitely many axioms.
  - Find an appropriate formula for A .

- Examples:

$$\frac{}{p \vdash p}\ (\mathrm{I}) \qquad \frac{}{p \wedge q \vdash p \wedge q}\ (\mathrm{I}) \qquad \frac{}{p \to q \vee r \vdash p \to q \vee r}\ (\mathrm{I})$$

$$\frac{}{A \vee B \vdash A \vee B}\ (\mathrm{I})$$

Axioms for Constants

$$\frac{}{\vdash \top}\ (\top) \qquad \frac{}{\bot \vdash}\ (\bot)$$

- Meaning:
  - True can be inferred always.
  - False cannot infer anything.

Weakening, Contraction and Exchange

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ (\mathrm{WL}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ (\mathrm{WR})$$

- weakening
  - Any formula may be added to antecedent or succedent.

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ (\mathrm{CL}) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ (\mathrm{CR})$$

- contraction
  - The same formulae can be merged (or contracted).

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ (\mathrm{EL}) \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ (\mathrm{ER})$$

- exchange
  - The order of formulae in antecedent or succedent may be changed.

Cut

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{(Cut)}$$

- cut
  - From a sequent having A in succedent and a sequent having A in antecedent, infer a sequent removing A .
  - When it is difficult to show $\Gamma_1 \vdash \Delta_2$,
    - Show $\Gamma_1 \vdash A$ first, and
    - Infer $A \vdash \Delta_2$.

$$\frac{\Gamma_1 \vdash A \qquad A \vdash \Delta_2}{\Gamma_1 \vdash \Delta_2} \text{(Cut)}$$

Left and Right

- Conclusion sequent has a logical connective.

$$\frac{A, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta}\ (\land L_1)$$

$$\frac{B, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta}\ (\land L_2)$$

$$\frac{A, \Gamma_1 \vdash \Delta_1 \quad B, \Gamma_2 \vdash \Delta_2}{A \lor B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\lor L)$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{A \to B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\to L)$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ (\neg L)$$

left : antecedent has a logical connective

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \lor B}\ (\lor R_1)$$

$$\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \lor B}\ (\lor R_2)$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \land B}\ (\land R)$$

$$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ (\to R)$$

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ (\neg R)$$

right : consequent has a logical connective

Inference Rules for ∧

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge L_1)$$

$$\frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge L_2)$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B}\ (\wedge R)$$

- Two left rules: $\wedge L_1$, $\wedge L_2$
  - Add ∧ to the antecedent
  - Make A to $A \wedge B$
  - Make B to $A \wedge B$
- One right rule: ∧R
  - Add ∧ to the succedent
  - Combine A and B, and make $A \wedge B$
  - From two premises, infer one conclusion.

Inference Rules for ∨

$$\frac{A, \Gamma_1 \vdash \Delta_1 \quad B, \Gamma_2 \vdash \Delta_2}{A \vee B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \ (\vee L)$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \ (\vee R_1)$$

$$\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \ (\vee R_2)$$

- One left rule: ∨L
  - Add ∨ to the antecedent.
  - Combine A and B , to make $A \vee B$
  - From two premises, infer one conclusion.

- Two right rules: $\vee R_1$, $\vee R_2$
  - Add ∨ to the succedent.
  - Make A to $A \vee B$
  - Make B to $A \vee B$

Inference Rules for →

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{A \to B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \ (\to L)$$

$$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B} \ (\to R)$$

- One left rule: →L
  - Add → to the antecedent.
  - Combine A in the succedent and B in the antecedent, make $A \to B$ in the antecedent.
  - From two premises, infer one conclusion.

- One right rule: →R
  - Add → to the succedent.
  - Combine A in the antecedent and B in the succedent, make $A \to B$ in the succedent.

Inference Rules for ¬

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ (\neg\mathrm{L}) \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ (\neg\mathrm{R})$$

- One left rule: ¬L
  - Add ¬ to the antecedent.
  - Move A in the succedent to $\neg A$ in the antecedent.
- One right rule: ¬R
  - Add ¬ to the succedent.
  - Move A in the antecedent to $\neg A$ in the succedent.

LK Proof Figure

- LK *Proof Figure*:
  - Start from initial sequent (or constants) and apply inference rules.
  - The bottom sequent is called *end sequent* of the proof figure.

- Example: proof figure

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I})}{\vdash A, \neg A}\,(\neg\mathrm{R})}{\vdash A, A \lor \neg A}\,(\lor\mathrm{R}_2)}{\vdash A \lor \neg A, A}\,(\mathrm{ER})}{\vdash A \lor \neg A, A \lor \neg A}\,(\lor\mathrm{R}_1)}{\vdash A \lor \neg A}\,(\mathrm{CR})
$$

initial sequent ($A \vdash A$)

end sequent ($\vdash A \lor \neg A$)

- When there is a proof figure of which end sequent is S, S is *provable* in LK.

Exercise

- Show proof figures of the following propositions:
  - $A \to \neg\neg A$
  - $\neg\neg A \to A$
  - $A \land B \to B \land A$
  - $A \lor B \to B \lor A$
  - $\neg(A \land \neg B) \to (A \to B)$
  - $(A \to B) \to \neg(A \land \neg B)$

Summary

- Inference
  - Axiom
  - Theorem

- LK System
  - Sequent calculus
  - Initial sequent
  - LK inference rules

- Proof
  - Proof figure