FUNDAMENTALS OF LOGIC
NO.5 PROOF (EXERCISE)

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lecture URL

https://vu5.sfc.keio.ac.jp/slide/

So Far

- Proposition
  - Logical connectives（$\wedge$, $\vee$, $\rightarrow$, $\neg$）
  - Truth table
  - Tautology

- Normal form
  - Disjunctive normal form
  - Conjunctive normal form

- Proof
  - Axiom and theorem
  - LK logical framework

Syntactic Meaning of Sequent

Theorem: The followings are equivalent:

1. A sequent $A_1, \dots, A_m \vdash B_1, \dots, B_n$ is provable in LK.
2. A sequent $A_1 \wedge \cdots \wedge A_m \vdash B_1 \vee \cdots \vee B_n$ is provable in LK.
3. A formula $A_1 \wedge \cdots \wedge A_m \to B_1 \vee \cdots \vee B_n$ is provable in LK.

Proof: For simplicity, let us show the case of $m = n = 2$.

- First, if (1) holds, (2) also holds.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_1, A_2 \vdash B_1, B_2}{A_1, A_2 \vdash B_1, B_1 \vee B_2}(\vee R_2)}{A_1, A_2 \vdash B_1 \vee B_2, B_1}(ER)}{A_1, A_2 \vdash B_1 \vee B_2, B_1 \vee B_2}(\vee R_1)}{A_1, A_2 \vdash B_1 \vee B_2}(CR)}{A_1 \wedge A_2, A_2 \vdash B_1 \vee B_2}(\wedge L_1)}{A_2, A_1 \wedge A_2 \vdash B_1 \vee B_2}(EL)}{A_1 \wedge A_2, A_1 \wedge A_2 \vdash B_1 \vee B_2}(\wedge L_2)}{A_1 \wedge A_2 \vdash B_1 \vee B_2}(CL)
$$

Proof (cont.)

Theorem: The followings are equivalent:

1. A sequent $A_1, \dots, A_m \vdash B_1, \dots, B_n$ is provable in LK.
2. A sequent $A_1 \land \cdots \land A_m \vdash B_1 \lor \cdots \lor B_n$ is provable in LK.
3. A formula $A_1 \land \cdots \land A_m \to B_1 \lor \cdots \lor B_n$ is provable in LK.

Proof (cont.):

- Secondly, if (2) holds, (3) also holds.

$$\dfrac{A_1 \land A_2 \vdash B_1 \lor B_2}{\vdash A_1 \land A_2 \to B_1 \lor B_2}\ (\to\text{R})$$

Proof (cont.)

Theorem: The followings are equivalent:

1. A sequent $A_1, \dots, A_m \vdash B_1, \dots, B_n$ is provable in LK.
2. A sequent $A_1 \land \cdots \land A_m \vdash B_1 \lor \cdots \lor B_n$ is provable in LK.
3. A formula $A_1 \land \cdots \land A_m \rightarrow B_1 \lor \cdots \lor B_n$ is provable in LK.

Proof (cont.):

- Finally, if (3) holds, (1) also holds.
- Therefore, (2) follows from (1), (3) follows from (2) and (1) follows from (3).
- Hence, (1), (2) and (3) are equivalent.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A_1 \vdash A_1}\,(\mathrm{I})}{A_2, A_1 \vdash A_1}\,(\mathrm{WL})}{A_1, A_2 \vdash A_1}\,(\mathrm{EL}) \qquad \dfrac{\dfrac{}{A_2 \vdash A_2}\,(\mathrm{I})}{A_1, A_2 \vdash A_1}\,(\mathrm{WL})}{A_1, A_2 \vdash A_1 \land A_2}\,(\land\mathrm{R})$$

$$\dfrac{\dfrac{\dfrac{}{B_1 \vdash B_1}\,(\mathrm{I})}{B_1 \vdash B_1, B_2}\,(\mathrm{WR}) \qquad \dfrac{\dfrac{\dfrac{}{B_2 \vdash B_2}\,(\mathrm{I})}{B_2 \vdash B_2, B_1}\,(\mathrm{WR})}{B_2 \vdash B_1, B_2}\,(\mathrm{ER})}{B_1 \lor B_2 \vdash B_1, B_2}\,(\lor\mathrm{L})$$

$$\dfrac{A_1, A_2 \vdash A_1 \land A_2 \qquad B_1 \lor B_2 \vdash B_1, B_2}{A_1 \land A_2 \rightarrow B_1 \lor B_2, A_1, A_2 \vdash B_1, B_2}\,(\rightarrow\mathrm{L})$$

$$\dfrac{\vdash A_1 \land A_2 \rightarrow B_1 \lor B_2 \qquad A_1 \land A_2 \rightarrow B_1 \lor B_2, A_1, A_2 \vdash B_1, B_2}{A_1, A_2 \vdash B_1, B_2}\,(\mathrm{Cut})$$

Meaning of LK Sequent

- LK sequent

$$A_1, \ldots, A_m \vdash B_1, \ldots, B_n$$

- Intuitive meaning:
  - If we assume A_1 to A_m , we can infer one of B_1 to B_n .

- Syntactical meaning:
  - $A_1 \wedge \cdots \wedge A_m \vdash B_1 \vee \cdots \vee B_n$
  - $A_1 \wedge \cdots \wedge A_m \rightarrow B_1 \vee \cdots \vee B_n$

- Interpretation of syntax:
  - Antecedent $A_1, \ldots, A_m$ are connected by `and'.
  - Succedent $B_1, \ldots, B_n$ are connected by `or'.
  - $\vdash$ is `imply'.

Tautology for Sequent

- Extend the notion of tautology to sequent.

- Let Γ be a sequence of formulae $A_1, \ldots, A_m$:

$$\Gamma^* = \begin{cases} A_1 \vee \cdots \vee A_m & \text{when } m > 0 \\ \bot & \text{when } m = 0 \end{cases}$$

$$\Gamma_* = \begin{cases} A_1 \wedge \cdots \wedge A_m & \text{when } m > 0 \\ \top & \text{when } m = 0 \end{cases}$$

- Sequent $\Gamma \vdash \Delta$ is a tautology $\Leftrightarrow$ $\Gamma_* \to \Delta^*$ is a tautology.

Sequent

$$\Gamma \vdash \Delta$$

=

Logical Formula

$$\Gamma_* \to \Delta^*$$

Extending Inference Rules

- Applying inference rules to formula in the sequent other than left-most or right-most one.
  - Using exchange rules, formula at any position can be moved to left-most or right-most position.
  - Extend inference rules of contraction, weakening and logical connectives to formula at any position.

- *Lemma*
  - When S can be inferred from $S_1, S_2, \ldots, S_n$ using LK inference rules, we write:

$$\frac{S_1 \quad S_2 \quad \ldots \quad S_n}{S}$$

The above inference can be used in other proofs as a lemma.

- Example: lemma

$$\frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, A \vee B, \Delta_2} \qquad \frac{\Gamma_1, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A \vee B, \Gamma_2 \vdash \Delta}$$

Exercises

- Prove the following tautologies:
  - $A \to (B \to A)$
  - $\big(A \to (B \to C)\big) \to \big((A \to B) \to (A \to C)\big)$
  - $(A \to B) \to \big((B \to C) \to (A \to C)\big)$
  - $(A \to C) \to \big((B \to C) \to (A \vee B \to C)\big)$
  - $(A \to B) \to \big((A \to \neg B) \to \neg A\big)$
  - $(A \wedge B) \wedge C \to A \wedge (B \wedge C)$
  - $(A \vee B) \vee C \to A \vee (B \vee C)$
  - $\neg(A \wedge B) \to (\neg A \vee \neg B)$
  - $(\neg A \vee \neg B) \to \neg(A \wedge B)$
  - $(A \to B) \to (\neg B \to \neg A)$
  - $(\neg B \to \neg A) \to (A \to B)$

Bottom-up and Top-down Proof

- Bottom-up proof
  - Starting with initial sequent, apply inference rules until it becomes the end sequent.
  - Intermediate state is also a valid proof figure.
- Top-down proof
  - Starting with the end sequent, apply inference rules backward until it becomes the initial sequent.
  - Intermediate state is not a proof figure, but it is just a lemma.

$$\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I}) \qquad \dfrac{}{B \to C \vdash B \to C}\,(\mathrm{I})}{A \to (B \to C), A \vdash B \to C}\,(\to\mathrm{L})$$

Bottom-up

$$\dfrac{\dfrac{}{B \vdash B}\,(\mathrm{I}) \qquad \dfrac{}{C \vdash C}\,(\mathrm{I})}{B \to C, B \vdash C}\,(\to\mathrm{L})$$

$$\dfrac{\dfrac{\dfrac{A, A \to B, A \to (B \to C) \vdash C}{A \to B, A \to (B \to C) \vdash A \to C}\,(\to\mathrm{R})}{A \to (B \to C) \vdash (A \to B) \to (A \to C)}\,(\to\mathrm{R})}{\vdash \big(A \to (B \to C)\big) \to \big((A \to B) \to (A \to C)\big)}\,(\to\mathrm{R})$$

Top-down

Summary

- LK Sequent
  - Syntactical meaning
  - Tautology
- Inference
  - Lemma