FUNDAMENTALS OF LOGIC
NO.6 SOUND AND COMPLETENESS

Tatsuya Hagino
hagino@sfc.keio.ac.jp

lecture URL

https://vu5.sfc.keio.ac.jp/slide/

So Far

- Propositional Logic
  - Logical Connectives（$\wedge$，$\vee$，$\rightarrow$，$\neg$）
  - Truth Table
  - Tautology

- Normal From
  - Disjunctive Normal Form
  - Conjunctive Normal Form

- Proof
  - Axiom and Theorem
  - LK Framework

Tautology for Sequent

- Extend the notion of tautology to sequent.

- Let Γ be a sequence of formulae $A_1, \ldots, A_m$:

$$\Gamma^* = \begin{cases} A_1 \vee \cdots \vee A_m & \text{when } m > 0 \\ \bot & \text{when } m = 0 \end{cases}$$

$$\Gamma_* = \begin{cases} A_1 \wedge \cdots \wedge A_m & \text{when } m > 0 \\ \top & \text{when } m = 0 \end{cases}$$

- Sequent $\Gamma \vdash \Delta$ is a tautology $\Leftrightarrow$ $\Gamma_* \rightarrow \Delta^*$ is a tautology.

Soundness and Completeness

- $\Gamma \vdash \Delta$ is a tautology:
  - For any assignment v, $v(\Gamma_* \to \Delta^*) = T$
  - For any assignment of truth values to propositional variables, we can
  - calculate the truth value of $\Gamma_* \to \Delta^*$ using the truth tables and check whether it is always true.

- $\Gamma \vdash \Delta$ is provable in LK:
  - There is a proof figure of which end sequent is $\Gamma \vdash \Delta$.
  - Starting from initial sequents, we can construct a proof figure of $\Gamma \vdash \Delta$.

- Soundness
  - Any sequent which is provable in LK is a tautology.

- Completeness
  - Any tautology is provable in LK.

Soundness Completeness

Relation between tautology and proof

Proved formulae are correct.

Correct formulae can be proved.

Axioms and inference rules are consistent.

Axioms and inference rules are sufficient.

- Both are important for any logical framework.
- We should not use unsound frameworks.
- A framework may not be complete.

LK Soundness Theorem

Theorem: For any sequent $\Gamma \vdash \Delta$, if $\Gamma \vdash \Delta$ is provable in LK, $\Gamma \vdash \Delta$ is a tautology.

Proof: We need to show the following two things:

1. An initial sequent is a tautology.
2. For each inference rule, if all the sequents above are tautologies, the bottom sequent is also a tautology.

For (1), an initial sequent $\mathrm{A} \vdash \mathrm{A}$ is $A \to A$, and it is a tautology.

For (2), we need to check for each inference rule. For example,

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad B, \Gamma_2 \vdash \Delta_2}{A \to B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \ (\to\mathrm{L})$$

we need to show when ${\Gamma_1}_* \to {\Delta_1}^* \vee A$ and $\mathrm{B} \wedge {\Gamma_2}_* \to {\Delta_2}^*$ are tautologies, $((A \to B) \wedge {\Gamma_1}_* \wedge {\Gamma_2}_*) \to ({\Delta_1}^* \vee {\Delta_2}^*)$ is also a tautology.

cont.

For an assignment v if $v((A \rightarrow B) \wedge \Gamma_{1*} \wedge \Gamma_{2*}) = T$, then $v(A \rightarrow B) = T,\ v(\Gamma_{1*}) = T,$ and $v(\Gamma_{2*}) = T$.

1. If $v(A) = F$, from $v(\Gamma_{1*} \rightarrow \Delta_1{}^* \vee A) = T$ and $v(\Gamma_{1*}) = T$, $v(\Delta_1{}^*) = T$.
2. If $v(A) = T$, from $v(A \rightarrow B) = T$, $v(B) = T$. Since $v(\mathrm{B} \wedge \Gamma_{2*} \rightarrow \Delta_2{}^*) = T$ and $v(\Gamma_{2*}) = T$, $v(\Delta_2{}^*) = T$.

In both cases, $v(\Delta_1{}^* \vee \Delta_2{}^*) = T$. Therefore, $((A \rightarrow B) \wedge \Gamma_{1*} \wedge \Gamma_{2*}) \rightarrow (\Delta_1{}^* \vee \Delta_2{}^*)$ is a tautology

For other rules, we can show the bottom sequent is a tautology when the above sequents are tautologies.

Therefore LK framework is sound. QED

Soundness Check (1)

- weakening left

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ (\mathrm{WL})$$

$v(\Gamma_* \to \Delta^*) = T$ then $v(A \wedge \Gamma_* \to \Delta^*) = T$

or

$v(\Gamma_*) = F$

$v(\Delta^*) = T$

$v(A \wedge \Gamma_*) = F$

- contraction right

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ (\mathrm{CR})$$

$v(\Gamma_* \to \Delta^* \vee A \vee A) = T$ then $v(\Gamma_* \to \Delta^* \vee A) = T$

or

$v(\Gamma_*) = F$

$v(\Delta^* \vee A \vee A) = T$

$v(\Delta^* \vee A) = T$

or

$v(\Delta^*) = T$

$v(A \vee A) = T$

$v(A) = T$

Soundness Check (2)

- cut

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,(\text{Cut})$$

$$v\left(\Gamma_{1*} \to {\Delta_1}^* \vee A\right) = T$$

and

$$v\left(A \wedge \Gamma_{2*} \to {\Delta_2}^*\right) = T$$

then

$$v\left(\Gamma_{1*} \wedge \Gamma_{2*} \to {\Delta_1}^* \vee {\Delta_2}^*\right) = T$$

or

$$v\left(\Gamma_{1*}\right) = F \qquad v\left(\Gamma_{1*} \wedge \Gamma_{2*}\right) = F$$

$$v({\Delta_1}^* \vee A) = T$$

or

$$v({\Delta_1}^*) = T$$

$$v(A) = T$$

or

$$v\left(A \wedge \Gamma_{2*}\right) = F \qquad v\left(\Gamma_{2*}\right) = F$$

$$v({\Delta_2}^*) = T \qquad v({\Delta_1}^* \vee {\Delta_2}^*) = T$$

Soundness Check (3)

- $\wedge$L$_1$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge \mathrm{L}_1)$$

$v(A \wedge \Gamma_* \to \Delta^*) = T$ then $v\big((A \wedge B) \wedge \Gamma_* \to \Delta^*\big) = T$

or

$v(A \wedge \Gamma_*) = F$ $\qquad$ $v(\Delta^*) = T$

or

$v(A) = F$ $\qquad$ $v(\Gamma_*) = F$

$v(A \wedge B) = F$

$v\big((A \wedge B) \wedge \Gamma_*\big) = F$

Soundness Check (4)

- →R

$$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ (\to\mathrm{R})$$

$v(A \wedge \Gamma_* \to \Delta^* \vee B) = T$

then

$v\big(\Gamma_* \to \Delta^* \vee (A \to B)\big) = T$

or

$v(A \wedge \Gamma_*) = F$

$v(\Delta^* \vee B) = T$

or

or

$v(A) = F$

$v(\Gamma_*) = F$

$v(\Delta^*) = T$

$v(B) = T$

$v(A \to B) = T$

$v\big(\Delta^* \vee (A \to B)\big) = T$

LK Completeness Theorem

Theorem: If $\Gamma \vdash \Delta$ is a tautology, $\Gamma \vdash \Delta$ is provable in LK without using cut inference rule.

Proof: First, define the decomposition of a sequent as follows:

$$\begin{tikzcd}[column sep=tiny] & \Gamma \vdash \Delta_1, A \wedge B, \Delta_2 \arrow[dl] \arrow[dr] & \\ \Gamma \vdash \Delta_1, A, \Delta_2 & & \Gamma \vdash \Delta_1, B, \Delta_2 \end{tikzcd} \qquad \begin{tikzcd} \Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta \arrow[d] \\ \Gamma_1, A, B, \Gamma_2 \vdash \Delta \end{tikzcd}$$

$$\begin{tikzcd} \Gamma \vdash \Delta_1, A \vee B, \Delta_2 \arrow[d] \\ \Gamma \vdash \Delta_1, A, B, \Delta_2 \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny] & \Gamma_1, A \vee B, \Gamma_2 \vdash \Delta \arrow[dl] \arrow[dr] & \\ \Gamma_1, A, \Gamma_2 \vdash \Delta & & \Gamma_1, B, \Gamma_2 \vdash \Delta \end{tikzcd}$$

$$\begin{tikzcd} \Gamma \vdash \Delta_1, A \rightarrow B, \Delta_2 \arrow[d] \\ A, \Gamma \vdash \Delta_1, B, \Delta_2 \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny] & \Gamma_1, A \rightarrow B, \Gamma_2 \vdash \Delta \arrow[dl] \arrow[dr] & \\ \Gamma_1, \Gamma_2 \vdash \Delta, A & & \Gamma_1, B, \Gamma_2 \vdash \Delta \end{tikzcd}$$

$$\begin{tikzcd} \Gamma \vdash \Delta_1, \neg A, \Delta_2 \arrow[d] \\ A, \Gamma \vdash \Delta_1, \Delta_2 \end{tikzcd} \qquad \begin{tikzcd} \Gamma_1, \neg A, \Gamma_2 \vdash \Delta \arrow[d] \\ \Gamma_1, \Gamma_2 \vdash \Delta, A \end{tikzcd}$$

A sequent is decomposed into one or two sequents.

Two things hold for any decomposition:

- The number of logical connectives decreases.
- If the original sequent is a tautology, the decompositions are tautologies.

cont.

Apply decomposition repeatedly:

$$\begin{tikzcd}
 & & \Gamma \vdash \Delta \arrow[dl] \arrow[dr] & & \\
 & \Gamma_1 \vdash \Delta_1 \arrow[dl] & & \Gamma_2 \vdash \Delta_2 \arrow[dl] \arrow[dr] & \\
\Gamma_3 \vdash \Delta_3 & & \Gamma_4 \vdash \Delta_4 & & \Gamma_5 \vdash \Delta_5
\end{tikzcd}$$

Sequents at the bottom row do not contain logical connectives.

- Complete decomposition tree
- If $\Gamma \vdash \Delta$ is a tautology, the bottom row of the complete decomposition contains tautology sequents containing propositional variables (no logical connectives).

A sequent $p_1, \cdots, p_m \vdash q_1, \cdots q_n$ containing only propositional variables is a tautology if and only if one of p_i matches with one of q_j.

$\Gamma_1, A, \Gamma_2 \vdash \Delta_1, A, \Delta_2$ is provable without cut rule.

- Apply weakening and exchange rules to initial sequent $A \vdash A$.

cont.

When $\Gamma \vdash \Delta$ is decomposed into $\Gamma_1 \vdash \Delta_1$ and $\Gamma_2 \vdash \Delta_2$, if $\Gamma_1 \vdash \Delta_1$ and $\Gamma_2 \vdash \Delta_2$ are provable without cut, $\Gamma \vdash \Delta$ is also provable without cut.

- For each decomposition, use rules for logical connectives, exchange and contraction.

Decomposition

$$\begin{array}{ccc} & \Gamma_1, A \to B, \Gamma_2 \vdash \Delta & \\ \swarrow & & \searrow \\ \Gamma_1, \Gamma_2 \vdash \Delta, A & & \Gamma_1, B, \Gamma_2 \vdash \Delta \end{array}$$

$\Leftrightarrow$

Inference rule

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad B, \Gamma_2 \vdash \Delta_2}{A \to B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \ (\to\mathrm{L})$$

Therefore, if a sequent is a tautology, the bottom row sequents of its complete decomposition are tautologies and they are provable without cut. By traversing back the decomposition, we can see that the original sequent is also provable without cut.

- On the other hand, if a given sequent is not a tautology, no matter how we may decompose it, there exists a non tautology sequent at the bottom row.

LK framework is complete. QED

This proof provides an algorithm to check whether a sequent is provable or not.

Example

- Find a complete decomposition of $\vdash \big((p \to q) \to p\big) \to p$ and create a proof figure without cut.

Decomposition

$\vdash \big((p \to q) \to p\big) \to p$

$\downarrow$

Proof

$$\frac{}{\vdash \big((p \to q) \to p\big) \to p}$$

Cut Elimination Theorem

Theorem: If $\Gamma \vdash \Delta$ is provable in LK, there exists a proof figure of $\Gamma \vdash \Delta$ without cut.

Proof: Since any provable sequent is a tautology, it is provable without cut. QED

A proof figure with cut

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\UnaryInfC{$\vdash A, \neg A$}
\UnaryInfC{$\vdash A, A \vee \neg A$}
\UnaryInfC{$\vdash A \vee \neg A, A$}
\UnaryInfC{$\vdash A \vee \neg A, A \vee \neg A$}
\UnaryInfC{$\vdash A \vee \neg A$}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\UnaryInfC{$B, A \vdash A$}
\UnaryInfC{$A \vdash B \to A$}
\UnaryInfC{$A \vdash (A \to B) \vee (B \to A)$}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\UnaryInfC{$\neg A, A \vdash$}
\UnaryInfC{$A, \neg A \vdash$}
\UnaryInfC{$A, \neg A \vdash B$}
\UnaryInfC{$\neg A \vdash A \to B$}
\UnaryInfC{$\neg A \vdash (A \to B) \vee (B \to A)$}
\BinaryInfC{$A \vee \neg A \vdash (A \to B) \vee (B \to A)$}
\BinaryInfC{$\vdash (A \to B) \vee (B \to A)$}
\end{prooftree}
$$

Theorem: Any formula in a cut free proof figure of $\Gamma \vdash \Delta$ is a sub formula of $\Gamma \vdash \Delta$.

Exercise

- Find a proof figure of $\vdash (A \to B) \lor (B \to A)$ without cut.

Duality

- Duality in inference rules:
  - (¬L) and (¬R) is symmetrical by exchanging left and right of $\Gamma \vdash \Delta$.

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ (\neg\mathrm{L}) \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ (\neg\mathrm{R})$$

  - Inference rules for ∨ and ∧ are symmetrical by replacing left and right and ∨ and ∧.

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge\mathrm{L}_1) \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ (\vee\mathrm{R}_1)$$

$$\frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge\mathrm{L}_2) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ (\vee\mathrm{R}_2)$$

$$\frac{A, \Gamma_1 \vdash \Delta_1 \qquad B, \Gamma_2 \vdash \Delta_2}{A \vee B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\vee\mathrm{L}) \qquad \frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B}\ (\wedge\mathrm{R})$$

Duality Theorem:

Let A and B be formulae without →. Let $\tilde{A}$ and $\tilde{B}$ be formulae A and B but replacing ∧ and ∨. If $A \vdash B$ is provable, $\tilde{B} \vdash \tilde{A}$ is also provable.

Summary

- Soundness of LK framework
  - Any sequent which is provable in LK is a tautology.

- Completeness of LK framework
  - A tautology is provable in LK.
  - There is an algorithm of checking whether a formula is provable or not.

- Properties of LK framework
  - cut elimination theorem
  - Duality theorem