FUNDAMENTALS OF LOGIC NO.7 OTHER FRAMEWORKS

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lecture URL

https://vu5.sfc.keio.ac.jp/slide/

So Far

- Propositional Logic
  - Logical Connectives（$\wedge$, $\vee$, $\rightarrow$, $\neg$）
  - Truth Table
  - Tautology
- Normal From
  - Disjunctive Normal Form
  - Conjunctive Normal Form
- Proof
  - Axiom and Theorem
  - LK Framework
- Soundness and Completeness
  - LK Framework is both sound and complete.

LK Framework

- Use sequent

$$A_1, \ldots, A_m \vdash B_1, \ldots, B_n$$

- Meaning: Assume $A_1, \ldots, A_m$ all hold, deduce one of $B_1, \ldots, B_n$

- Axioms: initial sequent and for constants (⊤ and ⊥)

$$\frac{}{A \vdash A}\,(\mathrm{I}) \qquad \frac{}{\vdash \top}\,(\top) \qquad \frac{}{\bot \vdash}\,(\bot)$$

- Two kinds of inference rules:
  - inference rules for structure
  - inference rules for logical connectives

LK Inference Rules

$$\frac{}{A \vdash A}\ (\mathrm{I}) \qquad \frac{}{\vdash \top}\ (\top) \qquad \frac{}{\bot \vdash}\ (\bot)$$

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ (\mathrm{WL}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ (\mathrm{WR}) \qquad \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ (\mathrm{CL}) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ (\mathrm{CR})$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ (\mathrm{EL}) \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ (\mathrm{ER}) \qquad \frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\mathrm{Cut})$$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge \mathrm{L}_1) \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge \mathrm{L}_2) \qquad \frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B}\ (\wedge \mathrm{R})$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ (\vee \mathrm{R}_1) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ (\vee \mathrm{R}_2) \qquad \frac{A, \Gamma_1 \vdash \Delta_1 \qquad B, \Gamma_2 \vdash \Delta_2}{A \vee B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\vee \mathrm{L})$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad B, \Gamma_2 \vdash \Delta_2}{A \to B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\to \mathrm{L}) \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ (\to \mathrm{R})$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ (\neg \mathrm{L}) \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ (\neg \mathrm{R})$$

Natural Deduction

- Natural deduction
  - NK Framework
  - Introduced by Gentzen (same as LK framework)
  - LK framework is too formal.
  - NK inference is closer to natural inference.
  - No inference rules for structure
  - Inference rules for logical connectives only
  - Each logical connective has two kinds of inference rules:
    - Introduction rule and elimination rule
    - correspond to right and left rules of LK framework

$\wedge$ Elimination Rule

- $A \wedge B$ means A as well as B :
  - If $A \wedge B$, then A and B can be used.

- corresponds to the left rule of $A \wedge B$ in LK framework.

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} (\wedge L_1) \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} (\wedge L_2)$$

  - $A \wedge B, \Gamma \vdash \Delta$ has $A \wedge B$ in the antecedent, it can be replaced with A.

- The elimination rule of $A \wedge B$ in NK framework:

$$\frac{A \wedge B}{A} (\wedge E_1) \qquad \frac{A \wedge B}{B} (\wedge E_2)$$

  - $\wedge$ in the above of $A \wedge B$ is eliminated in the below.

∧ Introduction Rule

- In order to show $A \wedge B$, it is necessary to show both A and B.
- corresponds to the right rule of $A \wedge B$ in LK framework:

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B} \ (\wedge R)$$

  - If A and B are shown, $A \wedge B$ is shown.
- The introduction rule of $A \wedge B$ in NK framework:

$$\frac{A \qquad B}{A \wedge B} \ (\wedge I)$$

  - $\wedge$ of $A \wedge B$ is introduced.

$\vee$ Introduction Rule

- To show $A \vee B$, it is enough to show A or B.
  - If A, then $A \vee B$.
  - If B, then $A \vee B$.

- corresponds to the right rule of $A \vee B$ in LK framework:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ (\vee R_1) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ (\vee R_2)$$

- The introduction rule of $A \vee B$ in NK framework:

$$\frac{A}{A \vee B}\ (\vee I_1) \qquad \frac{B}{A \vee B}\ (\vee I_2)$$

$\vee$ Elimination Rule

- When $A \vee B$ is shown, in order to show C the following two need to be shown:
  - In case A is shown, C can be shown, and
  - In case B is shown, C can be shown too.

- corresponds to the left rule of $A \vee B$ in LK framework.

$$\frac{A, \Gamma_1 \vdash \Delta_1 \qquad B, \Gamma_2 \vdash \Delta_2}{A \vee B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\vee L)$$

- The elimination rule of $A \vee B$ in NK framework:

$$\frac{A \vee B \qquad \begin{matrix}[A]_i \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]_i \\ \vdots \\ C\end{matrix}}{C}\,i\ (\vee E)$$

$$\begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}$$

Assuming A, show C — Assuming B, show C

C is shwon

- The index i indicates that this rule discharged some assumptions.

$\rightarrow$ Introduction and Elimination Rules

- LK rules for $\rightarrow$:

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad B, \Gamma_2 \vdash \Delta_2}{A \rightarrow B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \ (\rightarrow\mathrm{L}) \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B} \ (\rightarrow\mathrm{R})$$

- NK rules for $\rightarrow$:

$$\frac{A \rightarrow B \qquad A}{B} \ (\rightarrow\mathrm{E}) \qquad \frac{\begin{matrix}[A]_i \\ \vdots \\ B\end{matrix}}{A \rightarrow B}\, i \ \ (\rightarrow\mathrm{I})$$

- The elimination rule of $\rightarrow$ is the modus ponens.

¬ Introduction and Elimination Rules

- LK rules for ¬:

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ (\neg L) \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ (\neg R)$$

- NK rules for ¬:

$$\frac{\neg A \quad A}{\bot}\ (\neg E) \qquad \frac{\begin{matrix}[A]_i \\ \vdots \\ \bot\end{matrix}}{\neg A}\, i\ (\neg I) \qquad \frac{\neg\neg A}{A}\ (\neg\neg)$$

- From ⊥, anything can be shown:

$$\frac{\bot}{A}\ (\bot E)$$

NK Inference Rules

$\dfrac{A \land B}{A}\,(\land E_1)$ $\dfrac{A \land B}{B}\,(\land E_2)$	$\dfrac{A \qquad B}{A \land B}\,(\land I)$
$\dfrac{A \lor B \qquad \begin{matrix}[A]_i \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]_i \\ \vdots \\ C\end{matrix}}{C}\,i\ \ (\lor E)$	$\dfrac{A}{A \lor B}\,(\lor I_1)$ $\dfrac{B}{A \lor B}\,(\lor I_2)$
$\dfrac{A \to B \qquad A}{B}\,(\to E)$	$\dfrac{\begin{matrix}[A]_i \\ \vdots \\ B\end{matrix}}{A \to B}\,i\ \ (\to I)$
$\dfrac{\neg A \qquad A}{\bot}\,(\neg E)$	$\dfrac{\begin{matrix}[A]_i \\ \vdots \\ \bot\end{matrix}}{\neg A}\,i\ \ (\neg I)$
$\dfrac{\bot}{A}\,(\bot E)$	$\dfrac{\neg\neg A}{A}\,(\neg\neg)$

Proof in NK Framework

- Proof in NK framework:
  - Combine inference rules.
  - The bottom (or end) formula is proved.
  - All the assumptions need to be discharged.

- Example: $A \to (B \to A)$

LK proof figure

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I})}{B, A \vdash A}\,(\mathrm{WL})}{A \vdash B \to A}\,(\to\mathrm{R})}{\vdash A \to (B \to A)}\,(\to\mathrm{R})$$

NK proof figure

$$\dfrac{\dfrac{\begin{matrix}[B]_2 \\ [A]_1\end{matrix}}{B \to A}\,2\ (\to\mathrm{I})}{A \to (B \to A)}\,1\ (\to\mathrm{I})$$

Example (1)

- Prove $A \wedge B \to B \wedge A$

LK proof figure

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(\mathrm{I})}{A \wedge B \vdash B}\,(\wedge\mathrm{L}_2) \qquad \dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I})}{A \wedge B \vdash A}\,(\wedge\mathrm{L}_1)}{A \wedge B \vdash B \wedge A}\,(\wedge\mathrm{R})}{\vdash A \wedge B \to B \wedge A}\,(\to\mathrm{R})$$

NK proof figure

$$\dfrac{\dfrac{\dfrac{[A \wedge B]_1}{B}\,(\wedge\mathrm{E}_2) \qquad \dfrac{[A \wedge B]_1}{A}\,(\wedge\mathrm{E}_1)}{B \wedge A}\,(\wedge\mathrm{I})}{A \wedge B \to B \wedge A}\,1\ (\to\mathrm{I})$$

Multiple $[A \wedge B]_1$ are discharged together.

Example (2)

- Prove $A \vee B \to B \vee A$

LK proof figure

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(\mathrm{I})}{A \vdash B \vee A}\,(\vee \mathrm{R}_2) \qquad \dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I})}{B \vdash B \vee A}\,(\vee \mathrm{R}_1)}{A \vee B \vdash B \vee A}\,(\vee \mathrm{L})}{\vdash A \vee B \to B \vee A}\,(\to \mathrm{R})$$

NK proof figure

$$\dfrac{}{A \vee B \to B \vee A}$$

Exercise (1)

- Prove $\neg A \land \neg B \rightarrow \neg(A \lor B)$

$$\frac{}{\neg A \land \neg B \rightarrow \neg(A \lor B)}$$

Exercise (2)

- Prove $\neg(A \lor B) \rightarrow \neg A \land \neg B$

$$\frac{}{\neg(A \lor B) \rightarrow \neg A \land \neg B}$$

Proof of Double Negation

- Proof of $A \to \neg\neg A$

LK proof figure

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I})}{\neg A, A \vdash}\,(\neg\mathrm{L})}{A \vdash \neg\neg A}\,(\neg\mathrm{R})}{\vdash A \to \neg\neg A}\,(\to\mathrm{R})$$

NK proof figure

$$\dfrac{\dfrac{\dfrac{[\neg A]_2 \qquad [A]_1}{\bot}\,(\neg\mathrm{E})}{\neg\neg A}\,2\ (\neg\mathrm{I})}{A \to \neg\neg A}\,1\ (\to\mathrm{I})$$

- Proof of $\neg\neg A \to A$

LK proof figure

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I})}{\vdash A, \neg A}\,(\neg\mathrm{R})}{\neg\neg A \vdash A}\,(\neg\mathrm{L})}{\vdash \neg\neg A \to A}\,(\to\mathrm{R})$$

NK proof figure

$$\dfrac{\dfrac{[\neg\neg A]_1}{A}\,(\neg\neg)}{\neg\neg A \to A}\,1\ (\to\mathrm{I})$$

Law of Excluded Middle

- Proof of $A \lor \neg A$

LK proof figure

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{I})}{A \vdash A \lor \neg A}\,(\lor \mathrm{R}_1)}{\vdash A \lor \neg A, \neg A}\,(\neg \mathrm{R})}{\vdash A \lor \neg A, A \lor \neg A}\,(\lor \mathrm{R}_2)}{\vdash A \lor \neg A}\,(\mathrm{CR})$$

NK proof figure

$$\dfrac{\dfrac{\dfrac{[\neg(A \lor \neg A)]_1 \quad \dfrac{\dfrac{\dfrac{[\neg(A \lor \neg A)]_1 \quad \dfrac{[A]_2}{A \lor \neg A}\,(\lor \mathrm{I}_1)}{\bot}\,(\neg \mathrm{E})}{\neg A}\,2\ (\neg \mathrm{I})}{A \lor \neg A}\,(\lor \mathrm{I}_2)}{\bot}\,(\neg \mathrm{E})}{\neg\neg(A \lor \neg A)}\,1\ (\neg \mathrm{I})}{A \lor \neg A}\,(\neg\neg)$$

Hilbert Logical Framework

- Both LK and NK frameworks are by Gentzen.

- Hilbert Logical Framework
  - Axioms oriented.
  - The only inference rule: modus ponens

$$\frac{A \rightarrow B \qquad A}{B}\ \text{(MP)}$$

- Axioms:

A1. $A \rightarrow (B \rightarrow A)$
A2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
A3. $A \rightarrow (B \rightarrow A \wedge B)$
A4. $A \wedge B \rightarrow A$
A5. $A \wedge B \rightarrow B$
A6. $A \rightarrow A \vee B$
A7. $B \rightarrow A \vee B$
A8. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
A9. $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$
A10. $\neg\neg A \rightarrow A$

Proofs in Hilbert Logical Framework (1)

T1. $A \to A$

[1] $(A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$ (∵A2)
[2] $A \to \big((A \to A) \to A\big)$ (∵A1)
[3] $(A \to (A \to A)) \to (A \to A)$ (∵1,2,MP)
[4] $A \to (A \to A)$ (∵A2)
[5] $A \to A$ (∵3,4,MP)

T2. $(D \to \big(A \to (B \to C)\big)) \to (D \to ((A \to B) \to (A \to C)))$

[1] $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ (∵A2)
[2] $[1] \to (D \to [1])$ (∵A1)
[3] $D \to [1]$ (∵1,2,MP)
[4] $(D \to [1]) \to (\big(D \to (A \to (B \to C))\big) \to (D \to ((A \to B) \to (A \to C))))$ (∵A2)
[5] $\big(D \to (A \to (B \to C))\big) \to (D \to ((A \to B) \to (A \to C)))$ (∵3,4,MP)

T3. $(B \to C) \to ((A \to B) \to (A \to C))$

[1] $((B \to C) \to \big(A \to (B \to C)\big)) \to ((B \to C) \to ((A \to B) \to (A \to C)))$ (∵T2)
[2] $((B \to C) \to \big(A \to (B \to C)\big)$ (∵A1)
[3] $((B \to C) \to ((A \to B) \to (A \to C))$ (∵1,2,MP)

T4. $(A \to B) \to ((B \to C) \to (A \to C))$

[1] $(B \to C) \to ((A \to B) \to (A \to C))$ (∵T3)
[2] $[1] \to \big(((B \to C) \to (A \to B)) \to ((B \to C) \to (A \to C))\big)$ (∵A2)
[3] $\big((B \to C) \to (A \to B)\big) \to ((B \to C) \to (A \to C))$ (∵1,2,MP)
[4] $[3] \to ((A \to B) \to [3])$ (∵A1)
[5] $(A \to B) \to [3]$ (∵3,4,MP)
[6] $\big((A \to B) \to [3]\big) \to \big(((A \to B) \to ((B \to C) \to (A \to B))\big) \to \mathrm{T4})$ (∵A2)
[7] $\big((A \to B) \to ((B \to C) \to (A \to B))\big) \to T4$ (∵6,7,MP)
[8] $(A \to B) \to ((B \to C) \to (A \to B))$ (∵A1)
[9] $(A \to B) \to ((B \to C) \to (A \to C))$ (∵7,8,MP)

Proofs in Hilbert Logical Framework (2)

T5. $(C \to A) \to ((C \to B) \to (C \to A \land B))$

[1] $A \to (B \to A \land B)$ (∵A3)
[2] $[1] \to (C \to [1])$ (∵A1)
[3] $C \to (A \to (B \to A \land B))$ (∵1,2,MP)
[4] $(C \to (A \to (B \to A \land B))) \to ((C \to A) \to (C \to (B \to A \land B))$ (∵A2)
[5] $(C \to A) \to (C \to (B \to A \land B))$ (∵3,4,MP)
[6] $(C \to (B \to A \land B)) \to ((C \to B) \to (C \to A \land B))$ (∵A2)
[7] $[6] \to ((C \to A) \to [6])$ (∵A1)
[8] $(C \to A) \to [6]$ (∵6,7,MP)
[9] $((C \to A) \to [6]) \to (((C \to A) \to (C \to (B \to A \land B))) \to \mathrm{T5})$ (∵A2)
[10] $((C \to A) \to (C \to (B \to A \land B))) \to \mathrm{T5})$ (∵8,9,MP)
[11] $(C \to A) \to ((C \to B) \to (C \to A \land B))$ (∵5,10,MP)

T6. $A \land B \to B \land A$

[1] $(A \land B \to B) \to ((A \land B \to A) \to (A \land B \to B \land A))$ (∵T5)
[2] $A \land B \to B$ (∵A5)
[3] $(A \land B \to A) \to (A \land B \to B \land A)$ (∵1,2,MP)
[4] $A \land B \to A$ (∵A4)
[5] $A \land B \to B \land A$ (∵3,4,MP)

Lambda Calculus and NK Framework

- Lambda Calculus
  - A model of computation
  - Consist of two expressions:
    - function abstraction: $(\lambda x. M)$
    - function application: (MN)

- Type of lambda expressions:

$$\frac{\begin{array}{c} x : A \\ \vdots \\ M : B \end{array}}{(\lambda x. M) : A \to B}$$

$\to$ introduction rule

$$\frac{M : A \to B \qquad N : A}{(MN) : B}$$

$\to$ elimination rule

- The type determination of a lambda expression corresponds to a proof in NK framework.

Hilbert Logical Framework and Combinators

- Combinators:
  - Special lambda expressions:
    - $K \equiv \lambda xy.x$
    - $S \equiv \lambda xyz.xz(yz)$

- Theorem: For any lambda expression, there exists an SK expression which is $\alpha\beta$ equivalent to the lambda expression.

- The axioms for → in the Hilbert logical framework correspond to types of S and K:
  - $K \equiv \lambda xy.x : \big(A \to (B \to A)\big)$
  - $S \equiv \lambda xyz.xz(yz) : \big(A \to (B \to C)\big) \to \big((A \to B) \to (A \to C)\big)$

- The inference rule of the Hilbert logical framework is the modus ponens and it corresponds to the function application.

Summary

- Logical Framework
  - Axioms and inference rules

- LK Framework
  - Sequent $A_1, \ldots, A_m \vdash B_1, \ldots, B_n$
  - Axioms: initial sequent and for constants
  - Inference rules for structure: weakening, contraction, exchange, cut
  - Inference rules for logical connectives: left and right

- NK Framework
  - more natural
  - Introduction and elimination rules
  - Discharge assumptions

- Hilbert Logical Framework
  - The modus ponens is the only inference rule
  - Axioms oriented

- Lambda expressions and logical frameworks