FUNDAMENTALS OF LOGIC
NO.7 PREDICATE LOGIC

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lecture URL

https://vu5.sfc.keio.ac.jp/slide/

So Far

- Propositional Logic
  - Logical Connectives ($\wedge$, $\vee$, $\rightarrow$, $\neg$)
  - Truth Table
  - Tautology
  - Normal Form
  - Axiom and Proof
  - LK Frame Work (Sequent Calculus)
  - Soundness and Completeness

Limitation of Propositional Logic

- Propositional Logic
  - Each proposition is either true or false.
  - The truth value does not change.
  - The truth value does not depend of objects which are referred in the proposition.

- Socrates problem:
  - Socrates is a man.
  - All men are mortal.
  - Therefore, Socrates is mortal.

- In propositional logic:
  - $p =$ "Socrates is a man"
  - $q =$ "All men are mortal"
  - $r =$ "Socrates is mortal"
  - $p \wedge q \rightarrow r$?

Propositional Logic to Predicate Logic

- Extend logic to handle objects and express properties and relations of objects.

- Set of objects
  - Integer
  - Human

- Variable over a set of objects
  - object variable
  - $x, y, z, \ldots$

- Name of object
  - object constant
  - Socrates, Pythagoras, 123, SFC, Keio, ...

Predicate

- Predicate
  - Object x has property P : $P(x)$
  - Relation R holds between object x and object y : $R(x, y)$

- $Q(x_1, x_2, \ldots, x_n)$
  - Q holds for objects $x_1, x_2, \ldots, x_n$.
  - Q is a predicate with n variables.

- $P(x) =$ "x is a man"
  - $P(\text{Socrates}) =$ "Socrates is a man"
  - $P(\text{Pythagoras}) =$ "Pythagoras is a man"
  - $P(\text{Taro}) =$ "Taro is a man"

- $R(x, y) =$ "x likes y"
  - $R(\text{Taro}, \text{Hanako}) =$ "Taro likes Hanako"
  - $R(\text{Taro}, \text{Momoko}) =$ "Taro likes Momoko"
  - $R(\text{Hanako}, \text{Taro}) =$ "Hanako likes Taro"

Quantifier

- $P(x)$
  - Which x makes P hold?
  - Does it hold for any x ?
  - Does it hold for some x ?

- Quantifier
  - $\forall x\ P(x)$
    - Universal quantifier
    - For any x , $P(x)$ holds.
    - $P(x)$ holds for all x .
  - $\exists x\ P(x)$
    - Existential quantifier
    - For some x , $P(x)$ holds.
    - There exists x which makes $P(x)$ hold.

- $Q(x) =$ "x is mortal"
  - $\forall x\ Q(x) =$ "Everybody is mortal"
  - $\exists x\ Q(x) =$ "Someone is mortal", "There is someone who is mortal"

Predicate Logic

- Predicate Logic
  - Use predicates instead of propositional variables.
  - Four logical connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$
  - Two quantifiers: $\forall x$, $\exists x$

- Socrates example: $P(x) =$ "x is a man", $Q(x) =$ "x is mortal"
  - $P(\text{Socrates}) =$ "Socrates is a man"
  - $\forall x\big(P(x) \rightarrow Q(x)\big) =$ "All men are mortal"
  - $Q(\text{Socrates}) =$ "Socrates is mortal"

- Math example: $P(x) =$ "x is a prime number bigger than 2", $Q(x) =$ "x is an odd number"
  - $P(7) =$ "7 is a prime number bigger than 2"
  - $\forall x\big(P(x) \rightarrow Q(x)\big) =$ "Any prime number bigger than 2 is an odd number"
  - $Q(7) =$ "7 is an odd number"

Example (1)

- Let $S(x)$ and $M(x)$ be as follows:
  - $S(x)$ = "x is an SFC student"
  - $M(x)$ = "x likes math"

- Write the meaning of the following formulae:
  - $\forall x\, S(x)$ = "   "
  - $\exists x\, S(x)$ = "   "
  - $\forall x\big(S(x) \to M(x)\big)$ = "   "
  - $\exists x\big(S(x) \land M(x)\big)$ = "   "
  - $\forall x\big(S(x) \to \neg M(x)\big)$ = "   "
  - $\forall x\big(\neg S(x) \to M(x)\big)$ = "   "
  - $\forall x \neg S(x)$ = "   "
  - $\neg \forall x S(x)$ = "   "
  - $\neg \forall x\big(S(x) \to M(x)\big)$ = "   "
  - $\forall x \neg\big(S(x) \to M(x)\big)$ = "   "
  - $\exists x \neg S(x)$ = "   "
  - $\neg \exists x S(x)$ = "   "

Example (2)

- Let $L(x,y)$ mean "x likes y". Write the meaning of the following formulae:
  - $\forall x\ L(\text{Taro}, x)$ = "   "
  - $\exists x\ L(\text{Taro}, x)$ = "   "
  - $\forall x\ L(x, \text{Taro})$ = "   "
  - $\exists x\ L(x, \text{Taro})$ = "   "
  - $\forall x \forall y\ L(x,y)$ = "   "
  - $\exists x \exists y\ L(x,y)$ = "   "
  - $\forall x \exists y\ L(x,y)$ = "   "
  - $\exists x \forall y\ L(x,y)$ = "   "
  - $\exists y \forall x\ L(x,y)$ = "   "
  - $\forall x \forall y\ \big(S(x) \to L(x,y)\big)$ = "   "
  - $\forall x \forall y\ \big(S(y) \to L(x,y)\big)$ = "   "
  - $\forall x \big(S(x) \to \forall y\ L(x,y)\big)$ = "   "
  - $\forall x \big(\forall y\ L(x,y) \to S(x)\big)$ = "   "

Language for Predicate Logic

- A set of symbols for predicate logic is called language.
  - It is different from linguistic language.
  - It is closer to vocabulary.

- A language L of predicate logic consists of the followings:
  1. Logical connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$
  2. Quantifiers: $\forall$, $\exists$
  3. Object variables: $x, y, z, \ldots$
  4. Object constants: $c, d, \ldots$
  5. Function symbols: $f, g, \ldots$
  6. Predicate symbols: $P, Q, \ldots$

Terms

- A term of a language L is defined as follows:
  1. Object variables and constants of L are terms.
  2. For a function symbol f and m variables (arity m) in L, if $t_1, \ldots, t_m$ are terms, $f(t_1, \ldots, t_m)$ is also a term.

- Example: Natural Number Theory
  - Object constants: $0, 1$, etc.
  - Function symbols: $S(x)$, $+$, $\times$, etc.
  - Predicate symbols: $=, <$, etc.
  - Terms
    - x
    - 0
    - $S(x) + (1 \times S(S(y)))$

Logical Formulae

- Logical Formulae of L are defined as follows:
  1. For a predicate symbol P of n variables in L, if $t_1, \ldots, t_n$ are terms, $P(t_1, \ldots, t_n)$ is a formula (atomic formula).
  2. For formulae A and B, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(\neg A)$ are formulae.
  3. For a formula A and an object variable x, $(\forall x\ A)$ and $(\exists x\ A)$ are formulae.

- Example: Natural Number Theory
  - $\exists x(x \times z = y)$
  - $\forall x \forall y\big((x + S(y)) = S(x + y)\big)$

Bound and Free Variables

- Bound variable
  - In $\exists z(x \times z = y)$, z of $x \times z = y$ is bound by $\exists z$.
  - Bound variables can be renamed without changing the meaning.
  - $\exists w(x \times w = y)$

- Variables which are not bound are free variables.
  - In $\exists z(x \times z = y)$, x and y are free variables.

- Variables may be bound or free depending on their occurrence.
  - $\exists z(x \times z = y) \land \exists y(x + x = y)$

Closed Formulae

- When a logical formula A does not contain free variables, A is called a closed logical formula.
  - $\forall x\left(S(x) \rightarrow \forall y\, L(x,y)\right)$

- If $x_1, \ldots, x_n$ are free variables of a logical formula A,
  - $\forall x_1 \ldots \forall x_n\, A$
  - is calles universal closure of A .

- In mathematics, universal quantifiers are often omitted.
  - Commutative law of addition $x + y = y + x$
  - Its universal closure: $\forall x \forall y(x + y = y + x)$

Assignment of Terms

- For a logical formula A, when all the free occurrence of x are replaced with a term t, it is called an assignment of t to x.
  - $A[t/x]$

- Example
  - Let A be $\exists z(x \times z = y)$.
  - $A[w/y]$ is $\exists z(x \times z = w)$.
  - $A[x/y]$ is $\exists z(x \times z = x)$.
  - $A[(x + w)/x]$ is $\exists z\big((x + w) \times z = y\big)$.

- If bound relationship is affected by an assignment, the bound variable must be changed before the assignment.
  - $A[z/y]$ is not $\exists z(x \times z = z)$, but $\exists w(x \times w = z)$.
  - In general, $(\forall x A)[t/x]$ is $\forall u(A[u/x][t/x])$ where u is a variable which does not occur in A or t.

Sub-formulae

- Define sub-formulae similar to propositional logic.
  1. A is a sub-formula of A.
  2. A and B are sub-formulae of $(A \wedge B)$.
  3. A and B are sub-formulae of $(A \vee B)$.
  4. A and B are sub-formulae of $(A \rightarrow B)$.
  5. A is a sub-formula of$(\neg A)$.
  6. For any term t, $A[t/x]$ is a sub-formula of $\forall x A$.
  7. For any term t, $A[t/x]$ is a sub-formula of $\exists x A$.

- When a formula contains quantifiers, there are in finitely many sub-formulae.
  - Sub-formulae of $\forall x Q(x)$ are: $\forall x Q(x), Q(\text{Socrates}), Q(\text{Taro}), Q(\text{mother}(\text{Taro})), \ldots$

Summary

- Predicate Logic
  - Limitation of propositional logic
  - Description about objects

- Logical Formulae of Predicate Logic
  - Language
  - Term
  - Logical Formulae

- Quantifiers
  - Bound and free variables
  - Closed formulae
  - Universal closure
  - Assingment