FUNDAMENTALS OF LOGIC
NO.12 INCOMPLETENESS THEOREM

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lecture URL

https://vu5.sfc.keio.ac.jp/slide/

So Far

- Propositional Logic
  - Logical connectives ($\wedge$, $\vee$, $\rightarrow$, $\neg$)
  - Truth table
  - Tautology
  - Normal form
  - Axiom and theorem
  - LK framework
  - Soundness and completeness

- Predicate Logic
  - Logical Formulas (language, term)
  - Quantifiers ($\forall x\ P(x)$, $\exists x\ P(x)$)
  - Closed formulae (bound and free variables)
  - Semantics of predicate logic (domain, interpretation, structure)
  - Valid formulae
  - Prenex formulae
  - LK framework for predicate logic
  - Soundness and completeness
  - Skolemization
  - Herbrand Theorem
  - Resolution Principle

What is number?

- Various kinds of number:
  - natural number
  - integer
  - rational number
  - real number
  - complex number

- Taught from elementary school.
  - What is the definition?
  - Does it really exist?
  - Why $1 + 1 = 2$? Definition? Prove?
  - Why $(-1) \times (-1) = 1$?

Peano Axioms for Natural Numbers

- PA: Peano Axioms for natural numbers

- Language L_{PA}
  - constant: 0
  - unary function symbol: S
  - binary function symbol: +, ×
  - axioms:

    (1) $\forall x \neg(S(x) = 0)$

    (2) $\forall x \forall y(S(x) = S(y) \to x = y)$

    (3) $\forall x(x + 0 = x)$

    (4) $\forall x \forall y(x + S(y) = S(x + y))$

    (5) $\forall x(x \times 0 = 0)$

    (6) $\forall x \forall y(x \times S(y) = (x \times y) + x)$

    (7) for any formula A, $\big(A[0/x] \wedge \forall x(A \to A[S(x)/x])\big) \to \forall x\, A$
  - (7) is the mathematical induction.

Standard Model

- Language L_{PA} structure $N = \langle N, I \rangle$
  - N is the set of natural numbers.
  - 0^I is natural number 0
  - $S^I(n) = n + 1$ (next number)
  - $+^I(n, m) = n + m$ (addition)
  - $\times^I(n, m) = n \times m$ (multiplication)

- N is a model of PA theory.
  - PA is the theory to capture the natural number N.
  - N is the *standard model* of PA theory.

- $\bar{n}$
  - apply S to 0 for n times.
  - $\bar{n} = S(\cdots(S(0))\cdots)$
  - $\bar{1} = S(0), \bar{2} = S(S(0)), \bar{3} = S\left(S(S(0))\right)$

Existence of Natural Number

- The natural number N satisfies PA theory, but does it really exist?

- Assuming the set theory:
  - 0 is the empty set $\emptyset$.
  - For a set A, $S(A) = A \cup \{A\}$.
  - Let N be the intersection of sets which are closed under S .

- From the infinite set axiom, N exists.
  - $0 = \emptyset$
  - $\bar{1} = S(0) = \emptyset \cup \{\emptyset\} = \{\emptyset\}$
  - $\bar{2} = S(\bar{1}) = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$
  - $\bar{3} = S(\bar{2}) = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
  - …
  - Define $+$ and $\times$ by PA axioms.

Properties of Addition

- The addition is defined by the following axioms of PA theory.

  (3) $\forall x(x + 0 = x)$

  (4) $\forall x \forall y(x + S(y) = S(x + y))$

- $\overline{1} + \overline{1} = ?$
  - $\overline{1} + \overline{1} = S(0) + S(0) = S(S(0) + 0) = S\big(S(0)\big) = \overline{2}$

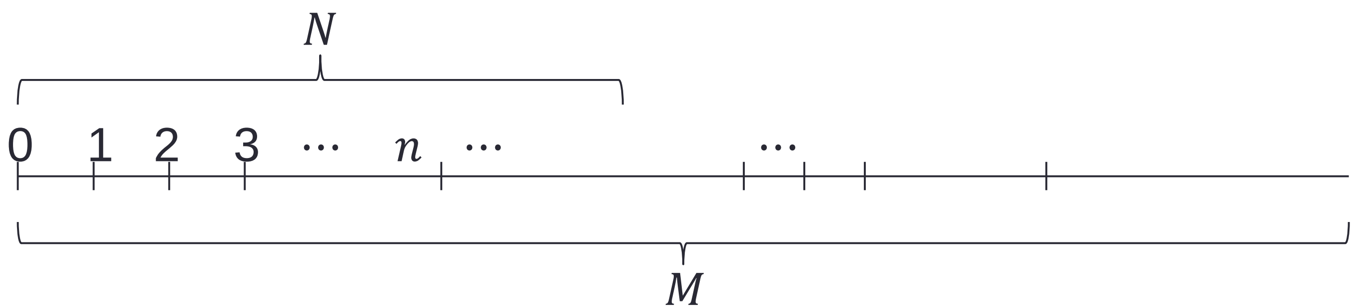
- Theorems about addition:
  - $\forall x \forall y(x + y = y + x)$
  - $\forall x \forall y \forall z((x + y) + z = x + (y + z))$

Non Standard Model

- $x < y \ \equiv\ \exists z(x + S(z) = y)$
  - $0 < \bar{1} < \bar{2} < \bar{3} < \cdots < \bar{n} < \cdots$
  - $\forall x(x < \overline{n+1} \rightarrow (x = 0 \vee x = \bar{1} \vee \cdots \vee x = \bar{n}))$

- $x \leq y \ \equiv\ \ x = y \vee x < y$

- non-standard model of PA theory.
  - M has the same structure with N.



- $\leq$ of M is not well-ordered.
  - well-ordered set: any subset has the smallest element.
  - $\leq$ of N is well-ordered.
  - $\forall x(x \neq 0 \rightarrow \exists y(x = S(y)))$

Soundness and Completeness

- Soundness
  - What has been proved is always valid.
  - Logical framework and axiomatic theory need to be sound.

- Completeness
  - Valid thing can be proved.
  - The predicate logic is sound and complete.
  - A closed formula is either valid or not valid, and a valid formula can be proved.

- For axiomatic theory, it does not need to be complete.
  - Axiomatic theory of groups:
    1. $\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$
    2. $\forall x(e \cdot x = x \land x \cdot e = x)$
    3. $\forall x(x \cdot x^{-1} = e \land x^{-1} \cdot x = e)$
  - $\forall x \forall y(x \cdot y = y \cdot x)$ or its negation cannot be proved.

Incompleteness of Peano Arithmetic

- Peano arithmetic for natural number: PA
  - Axiomatic theory for natural number structure $N = \langle N, I \rangle$.
  - List all the axioms which are necessary to prove mathematical theorems of natural number.
  - Its completeness is preferable.

- *Gödel First Incompleteness Theorem*
  - If PA is consistent (i.e. cannot prove contradiction), it is incomplete.

- PA is *incomplete*!
  - There is a theorem which is valid for natural number but cannot be proved from PA axioms.

- How about add more axioms to PA ?
  - No matter how axioms are added to PA, if the set of axioms are decidable (i.e. there is a procedure to check whether give formula is an axiom or not), it is incomplete.

- *Gödel Second Incompleteness Theorem*
  - PA cannot prove that PA is consistent.

Gödel Statement

- The incompleteness of PA can be shown by creating the following Gödel statement,
  - G is the formula that `G cannot be proved'.

- If PA is complete, either G or its negation is provable.
  - If G is provable, G cannot be proved.
  - If G is not provable, its negation is provable, but it is `G is provable'.
  - Either way, contradicts. Therefore, if PA is complete, PA is inconsistent.
  - Taking its contraposition, if PA is consistent, PA is incomplete.

Gödel Number

- Gödel statement $G =$ `G cannot be proved' is the self reference statement (like I am a liar).
  - Peano arithmetic is an axiomatic theory for natural number.
  - Code formulae as numbers.

- *Gödel number*
  - Assign unique number to each symbol of the language.
  - Formula P is a sequence of symbols and let $x_1, x_2, \ldots, x_n$ be numbers associated with them.
  - The Gödel number of P is: $\#P = 2^{x_1} \times 3^{x_2} \times \cdots \times p_n^{x_n}$

    where p_n it the nth prime number.
  - Given a Gödel number, by using prime number decomposition, the original formula can be recovered.

Construction of Gödel Statement

- Let PA be an axiomatic theory which contains the natural number axioms and modus ponens as the inference rule.

- The following formulae can be constructed:
  - $A(x) =$ `x is a Gödel number of an PA axiom'
  - $M(x, y, z) =$ `formula z can be inferenced from formula x and formula y using modus ponens'
  - $P(x, y) =$ `sequence y of formulae is the proof of formula x'
  - $T(x) = \exists y\ P(x, y) =$ `formula x is provable in PA'
  - $a(y, z)\ =$ `the Gödel number of a formula y assign z to the variable x'

- Gödel statement:
  - Let g be the Gödel number of $\neg T\big(a(x, x)\big)$.
  - Gödel statement is $G\ =\ \neg T(a(g, g))$.

Proof of Incompleteness Theorem

- Proof of the first incompleteness theorem:
  - If G is provable, $a(g,g)$ represents the Gödel number of a formula which cannot be proved, but it is G and contradicts.
  - If $\neg G$ is provable, $a(g,g)$ represents the Gödel number of a formula which can be proved, but it is G and contradicts.
  - Therefore, neither G or $\neg G$ can be proved.

- Proof of the second incompleteness theorem:
  - Let b be the Gödel number of contradiction.
  - $\neg T(b) =$ `PA is consistent'
  - If G of the first incompleteness theorem is provable, it contradicts, therefore, $G \to T(b)$
  - Taking the contraposition, $\neg T(b) \to \neg G$
  - Similarly, if $\neg G$ is provable, it contradicts, therefore, $\neg G \to T(b)$
  - Taking the contraposition, $\neg T(b) \to G$
  - Therefore, if PA is consistent, it contradicts.
  - PA consistency cannot be proved in PA.

Incompleteness Theorem and Invention of Computer

- Why PA is incomplete?
  - By coding formula as a number, construct a self reference formula.
  - PA gives a power of encoding anything as a number.

- John von Neumann
  - Tried to prove that PA is complete.
  - Shocked by Gödel incompleteness theorem.
  - Use idea of Gödel number to invent von Neumann computer.
  - Digital computer handle numbers.
  - Programs are represented as numbers and stored in the memory (stored program).
  - Programs are data.

Summary

- Peano Arithmetic
  - mathematical induction
  - standard model

- Incompleteness theorem
  - If PA is consistent, it is incomplete.
  - Consistency of PA cannot be proved in PA.