FUNDAMENTALS OF LOGIC
NO. 14 OTHER FRAMEWORKS

Tatsuya Hagino
hagino@sfc.keio.ac.jp

lecture URL

https://vu5.sfc.keio.ac.jp/slide/

Intuitionistic Logic

- 20th Century Mathematics
  - Set Theory by George Cantor
    - ``The essence of mathematics is its freedom.''
  - Mathematical Objects exist so long as they do not cause any contradiction.
  - In 19th century, mathematics only handle real objects: geometry, integers, fractions, real numbers, and so on.
  - In 20th century, mathematics handles abstract objects: group, field, topological space, category theory, and so on.

- Overcoming contradiction
  - Cantor's set theory contains contradictions like Russel's paradox.
  - From formalism point of view, set theory introduced some restriction to its axioms in order to overcome the contradiction.
  - All the mathematical results have been safely formalized on top of set theory.

- Intuitionism
  - Mathematics is considered to be purely the result of the constructive mental activity of humans rather than the discovery of fundamental principles claimed to exist in an objective reality.
  - The truth of a mathematical statement is a subjective claim: a mathematical statement corresponds to a mental construction, and a mathematician can assert the truth of a statement only by verifying the validity of that construction by intuition.

Meaning of Existence

- In classical logic, refutation can be used to show the existence.
  - Assume there does not exist an object of property P, and if it contradicts, then there must exist an object having property of P.
- Intuitionism does not accept the existence proof by refutation.
  - In order to show an existence of an object of property P, one need to find a concrete finite procedure to fine the object.
- Example: A proof in classical logic of ``there exists irrational number x and y and a rational number z where $x^y = z$''.
  - $\sqrt{2}^{\sqrt{2}}$ is either a rational number or an irrational number.
  - If $\sqrt{2}^{\sqrt{2}}$ is a rational number, taking x and y as $\sqrt{2}$ makes $\mathrm{z} = x^y$ a rational number.
  - If $\sqrt{2}^{\sqrt{2}}$ is an irrational number, taking x as $\sqrt{2}^{\sqrt{2}}$ and y as $\sqrt{2}$ makes $z = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$ a rational number.
- This proof does not show what are x, y, z concretely, so intuitionism does accept this proof.

Difference with Classical Logic

- In classical logic, $\exists x \forall y \big(P(y) \to P(x)\big)$ is valid.
  - If $P(a_0)$ holds for some a_0 , $P(b) \to P(a_0)$ holds for any b . Therefore, $\forall y \big(P(b) \supset P(a_0)\big)$ holds.
  - If $P(c)$ does not hold for any c , $P(c) \to P(d)$ holds for an arbitrary chosen d . Therefore, $\forall y \big(P(y) \to P(d)\big)$ holds.
  - In both cases, $\exists x \forall y \big(P(y) \to P(x)\big)$ holds.
  - Intuitionism does not accept this proof because it does not show concretely what x should be chosen to make $\forall y (P(y) \to P(x))$ true.

- Double Negation
  - $\neg\neg A \to A$ is valid in classical logic.
  - Let A be $\exists x\, P(x)$, then $\neg\neg\exists x\, P(x) \to \exists x\, P(x)$.
  - $\neg\neg\exists x\, P(x)$ means ``It contradicts if x does not exist to make $P(x)$ true."
  - In intuitionisum, $\exists x P(x)$ means ``There is a concrete way to find x to make $P(x)$ true."
  - $\neg\neg\exists x\, P(x) \to \exists x\, P(x)$ means in intuitionism ``If it contradicts that non existence of x to make $P(x)$ true, then there exists a concrete way to find x to make $P(x)$ true." Therefore, this is not valid in intuitionism.

Law of Excluded Middle

- $A \vee B$
  - In intuitionism, this is true when A is shown true concretely or B is shown true concretely.

- Law of Excluded Middle
  - $A \vee \neg A$
  - This is valid in classical logic.
  - In general, this is not valid in intuitionism.

- Goldbach's conjecture
  - ``Every even integer greater than 2 can be expressed as the sum of two primes."
  - The conjecture has been shown to hold for all integers less than 4×10^{18}, but remains unproven despite considerable effort.
  - $P(n) =$``$2(n + 2)$ cannot be expressed as the sum of two primes"
  - $\exists x\, P(x) \vee \neg \exists x\, P(x)$ is not proven true in intuitionism.

Constructive Mathematics and Programs

- Constructive mathematics
  - mathematics based on intuitionism
  - do not use refutation
  - constructive concepts and constructive proofs
- Program
  - calculates a solution to a problem
  - gives a procedure to find a solution.
  - The program itself is a proof of the problem.
  - Example:
    - ``There exists a greatest common divisor for any two natural numbers.''
    - Euclidean algorithm is an existence proof of this problem.
- Program = Proof
  - A program can be extracted from a constructive proof.

Intuitionistic Logic Framework

- Intuitionistic Logic Framework LJ
  - Same axioms and rules with classical logic framework LK,
  - But, LJ restricts a sequent as $A_1, \ldots, A_m \vdash B$ (where m can be 0 and B may be empty)

- Axioms and inference rules for structure:

$$\frac{}{A \vdash A}\ (\mathrm{I}) \qquad \frac{}{\vdash \top}\ (\top) \qquad \frac{}{\bot \vdash}\ (\bot)$$

$$\frac{\Gamma \vdash B}{A, \Gamma \vdash B}\ (\mathrm{WL}) \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\mathrm{WR}) \qquad \frac{A, A, \Gamma \vdash B}{A, \Gamma \vdash B}\ (\mathrm{CL})$$

$$\frac{\Gamma_1, A, A', \Gamma_2 \vdash B}{\Gamma_1, A', A, \Gamma_2 \vdash B}\ (\mathrm{EL}) \qquad \frac{\Gamma_1 \vdash A \qquad A, \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash B}\ (\mathrm{Cut})$$

(where Γ is a sequence of formulae and B is empty of a single formua)

Inference Rules of Intuitionistic Logic

- Inference rules for logical connectives:

$$\frac{A, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C}\ (\wedge L_1) \qquad \frac{B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C}\ (\wedge L_2) \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}\ (\wedge R)$$

$$\frac{A, \Gamma_1 \vdash C \qquad B, \Gamma_2 \vdash C}{A \vee B, \Gamma_1, \Gamma_2 \vdash C}\ (\vee L) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee R_1) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee R_2)$$

$$\frac{\Gamma_1 \vdash A \qquad B, \Gamma_2 \vdash C}{A \to B, \Gamma_1, \Gamma_2 \vdash C}\ (\to L) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}\ (\to R)$$

$$\frac{\Gamma \vdash A}{\neg A, \Gamma \vdash}\ (\neg L) \qquad \frac{A[t/x], \Gamma \vdash B}{\forall x\, A, \Gamma \vdash B}\ (\forall L) \qquad \frac{A[z/x], \Gamma \vdash B}{\exists x\, A, \Gamma \vdash B}\ (\exists L)$$

$$\frac{A, \Gamma \vdash}{\Gamma \vdash \neg A}\ (\neg R) \qquad \frac{\Gamma \vdash A[z/x]}{\Gamma \vdash \forall x\, A}\ (\forall R) \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x\, A}\ (\exists R)$$

Theorem about Intuitionistic Logic

- Cut elimination theorem for intuitionistic logic
  - If $\Gamma \vdash A$ is provable in LJ, there exists a proof figure of $\Gamma \vdash A$ which does not contain cut rule.
- Decision problem of intuitionistic propositional logic
  - There is an algorithm to check whether a given propositional formula is provable in LJ or not.
- Decision problem of intuitionistic predicate logic
  - There does not exist an algorithm to check whether a given predicate formula is provable in LJ or not.
- Disjunction property of intuitionistic logic
  - If $A \vee B$ is provable in LJ, A or B is provable in LJ.
- Existence property of intuitionistic logic
  - If $\exists x\, A$ is provable in LJ, there is a term t and $A[t/x]$ is provable in LJ.

Modal Logic

- Modal logic
  - Everyday things may change its truth depending on state or situation.
  - ``The sum of inner angles of a triangle is $180°$'' is always true (i.e. does not depend on state or situation).
  - ``The elevator is on the third floor'' is true or false depending on where the elevator is.
  - Separate the truth of a statement from its necessity or possibility.

- Formula in modal logic
  - $\Box A$
    - it is necessarily A .
  - $\neg\Box A$
    - It is not necessarily A .
  - $\Box\neg A$
    - It is necessarily true that A is not true.
    - There is no possibility of A .
  - $\neg\Box\neg A$
    - It is possibly A .
    - $\Diamond A$

Modal Logic Framework

- Framework K
  - Add the following rule to the sequent calculus framework LK of classical logic:

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

- The following formulae can be proved in K:
  - $\Box A \land \Box B \vdash \Box(A \land B)$
  - $\Box(A \land B) \vdash \Box A \land \Box B$
  - $\Box A \lor \Box B \vdash \Box(A \lor B)$
  - $\Box(A \to B) \vdash \Box A \to \Box B$

Axioms in Modal Logic

- Several axioms may be add:
  - D: $\Box A \rightarrow \Diamond A$
  - T: $\Box A \rightarrow A$
  - 4: $\Box A \rightarrow \Box\Box A$
  - B: $A \rightarrow \Box\Diamond A$
  - 5: $\Diamond A \rightarrow \Box\Diamond A$

- Modal logic S4
  - Framework K with axioms T and 4.

- Modal logic S5
  - Framework K with axioms T and 5.

- Relation among axioms.
  - KB4 and KB5 are the same modal logic.
  - KDB4, KDB5 and S5 are the same modal logic.

Meaning of Modal Operators

- Interpretation of $\Box A$ in the time flow:
  - If A is always true, $\Box A$ is true.

- If `always' means `always in the future',
  - Axiom T : $\Box A \to A$ does not hold (A may not be true now).

- If `always' means `now and in the future',
  - Axiom T : $\Box A \to A$ holds.
  - Axiom 4 : $\Box A \to \Box\Box A$ holds.
  - Axiom 5 : $\Diamond A \to \Box\Diamond A$ does not hold.

- If `always' means `now and in the past and future',
  - Axiom 5 : $\Diamond A \to \Box\Diamond A$ holds.

- Kripke Semantics
  - Using possible worlds and reachability of worlds, the truth is determined.
  - $\Box A$ is true in a possible world a if and only if A is true in any world b which is reachable from a.

Intuitionistic Logic and Modal Logic

- Gödel translation or McKinsey-Tarski translation T
  - $T(p) = \Box p$
  - $T(A \wedge B) = T(A) \wedge T(B)$
  - $T(A \vee B) = T(A) \vee T(B)$
  - $T(A \rightarrow B) = \Box(T(A) \rightarrow T(B))$
  - $T(\neg A) = \Box\neg T(A)$

- Gödel-McKinsey-Tarski Theorem:
  - A sequent $\Gamma \vdash A$ consisting of propositional logical formulae is provable in intuitionistic logic framework LJ, if and only if $T(\Gamma) \rightarrow T(A)$ is provable in modal logic S4.

Tense Logic

- Tense logic or Temporal logic
  - Refine modal logic for tense.
  - $[P]A = $ ``A is always true in the past"
  - $[F]A = $ ``A is always true in the future"
  - $<P>A = \neg[P]\neg A$ $\quad$ $<F>A = \neg[F]\neg A$
  - $\Box A = [P]A \wedge A \wedge [F]A$ $\quad$ $\Box A = <P>A \vee A \vee <F>A$

- Framework Kt
  - $A \rightarrow [P]<F>A$ $\quad$ $A \rightarrow [F]<P>A$
  - $[P]A \rightarrow [P][P]A$ $\quad$ $[F]A \rightarrow [F][F]A$
  - $\dfrac{\Gamma \vdash A}{[P]\Gamma \vdash [P]A}$ $\quad$ $\dfrac{\Gamma \vdash A}{[F]\Gamma \vdash [F]A}$

- More modal operators:
  - $\bigcirc A = $ ``A is true next"
  - $A \mathcal{U} B = $ ``A is true until B"

Intentional Logic

- Intentional logic
  - $\Box A$ has been interpreted as ``necessarily A'', but can be interpreted differently.

- Deontic logic
  - $\Box A$ is interpreted as ``A is required''.
  - $\Diamond A$ is interpreted as ``A is permitted''.
  - KD axiom $\Box A \to \Diamond A$ means ``If A is required, A is permitted'', so this holds.
  - T axiom $\Box A \to A$ means ``If A is required, A is true'', so this does not hold in general.

- Logic of Knowledge or Logic of Belief
  - $\Box A$ is interpreted as ``A is know.''

- Provability logic
  - $\Box A$ is interpreted as ``A is provable.''
  - Gödel second incompleteness theorem ``If PA is consistent it cannot prove its consistency'' is:
    - $\neg\Box \bot \to \neg\Box\neg\Box \bot$

Dynamic Logic

- Dynamic logic
  - Used in program specification and verification.

  - For a program Π, $[\Pi]A$ means ``After the execution of program Π, A holds."

  - Let $\Pi_1; \Pi_2$ be execute Π_2 after the execution of Π_1, and Π^* be the finite repetitive execution of Π:
    - $[\Pi_1; \Pi_2]A \equiv [\Pi_1][\Pi_2]A$
    - $[\Pi^*]A \rightarrow (A \wedge [\Pi][\Pi^*]A)$
    - $(A \rightarrow [\Pi]A) \rightarrow (A \rightarrow [\Pi^*]A)$

Resource Logic

- Resource logic
  - Do not allow weakening and contraction of intuitionistic logic framework LJ

$$\frac{\Gamma \vdash B}{A, \Gamma \vdash B}\ \text{(WL)} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ \text{(WR)} \qquad \frac{A, A, \Gamma \vdash B}{A, \Gamma \vdash B}\ \text{(CL)}$$

  - Introduce new logical connective $A \otimes B$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \otimes B, \Gamma_2 \vdash C}\ \text{(WL)} \qquad \frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \text{(WR)}$$

- Example: $A =$ ``Pay 500 yen'', $B =$ ``Can buy one book'', $C =$ ``Can drink a cup of coffee''
  - From $A \vdash B$ and $A \vdash C$, $A \otimes A \vdash B \otimes C$ can be proved.
  - $A \vdash B \otimes C$ cannot be proved.
  - $A \vdash B \land C$ can be proved.

Other Topics about Logic

- Second order predicate logic and higher order predicate logic
  - In the first order predicate logic, variables over objects can be quantified.
  - In the second order predicate logic, variables over subsets of objects (i.e. predicates) can also be quantified.
  - In the third order predicate logic, variables over subsets of subsets of objects can also be quantified.
  - In general, nth order predicate logic can be defined.
  - In the higher order predicate logic include all the nth order predicate logic.

- Three value logic and fuzzy logic
  - In three value logic, there is a value in between true and false.
  - In fuzzy logic, the assignment is a continual function to [0,1].

- Non-monotonic logic
  - Monotonicity: new knowledge does not interfere with already existing knowledge.
  - ``A bird can fly'' but ``Penguins cannot fly''.

Summary

- Mathematical Logic
  - Propositional Logic
    - Logical Connectives
  - Predicate Logic
    - Quantifiers

- Logical Framework
  - Proof and Theorem
  - LK Framework, NK Framework
  - Soundness and Completeness

- Resolution Principle
  - Normal Form
  - Clause