MATHEMATICS FOR INFORMATION SCIENCE

NO.2 PRIMITIVE RECURSIVE FUNCTION

Tatsuya Hagino
hagino@sfc.keio.ac.jp

Slides URL

https://vu5.sfc.keio.ac.jp/slide/

So far

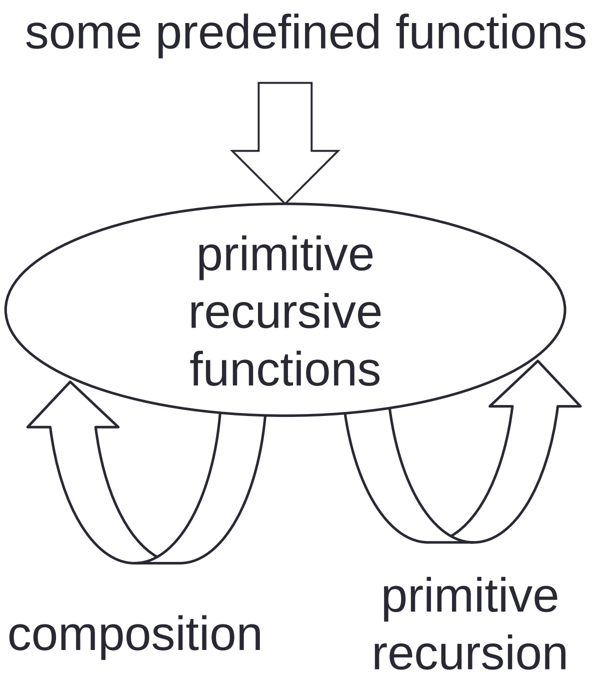
- What is computation?
  - Computation = what computers can calculate
  - Computable functions = mathematical functions which computers can calculate
  - Computability = whether mathematical functions are computable or not
- While program and flow chart are equivalent.
- Any program can be converted into a while program with only one while.

Primitive Recursive Function

Definition：

- Primitive recursive functions consist of the followings:

1. some predefined functions
2. composition of primitive recursive functions
3. functions defined by primitive recursion



Predefined Primitive Recursion Functions

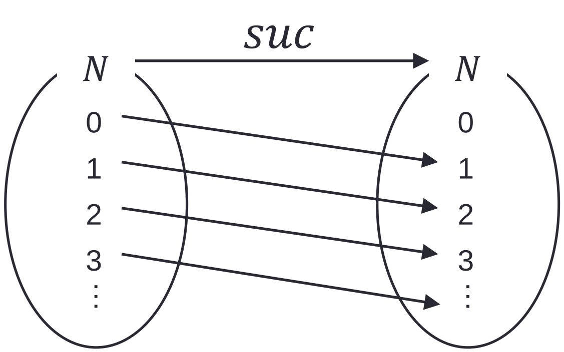
- The following three functions are primitive recursive functions:

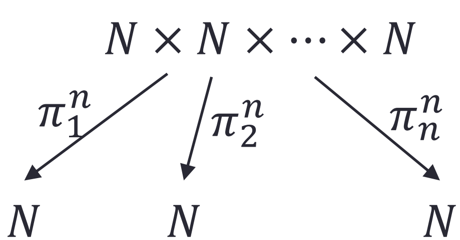
1. $zero : N^0 \rightarrow N$
   - $zero() = 0$
   - Always return 0

2. $suc : N \rightarrow N$
   - $suc(x) = x + 1$
   - next number of x
   - $suc(2) = 3$
   - $suc(100) = 101$

3. $\pi_i^n : N^n \rightarrow N$
   - $\pi_i^n(x_1, \dots, x_n) = x_i$
   - ith element of n given elements
   - projection
   - $\pi_1^2(x, y) = x$
   - $\pi_2^2(x, y) = y$

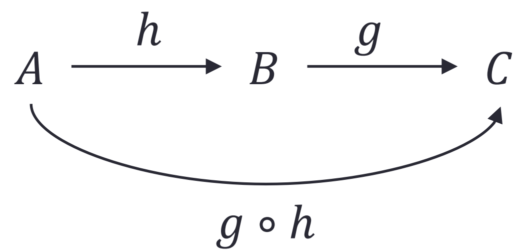
- $N^0 = \{\cdot\}$
- $N^1 = N$
- $N^2 = N \times N = \{(x, y) | x \in N, y \in N\}$
- $N^3 = N \times N \times N$
- $N^n = N \times N \times N \times \dots \times N$



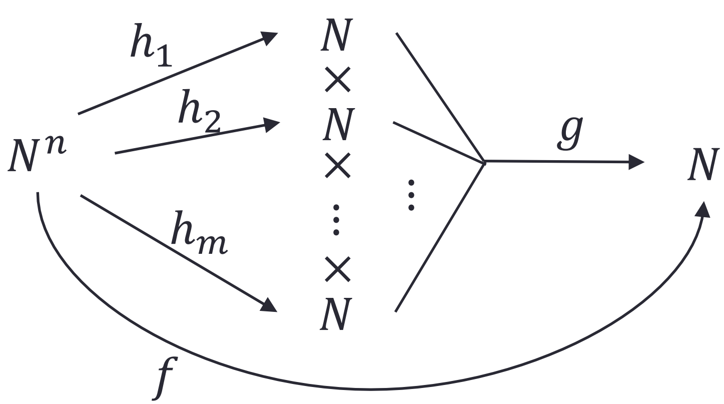


Composition of Primitive Recursive Functions

- Function composition
  - compose two functions
  - $g \circ h(x) = g(h(x))$



- Composition of primitive recursive functions is primitive recursive.
  - Given primitive recursive functions $g : N^m \to N$ and $h_i : N^n \to N$ $(i = 1,2,\dots,m)$, the composition $f : N^n \to N$ is primitive recursive.
  - $f(x_1, x_2, \dots, x_n) = g(h_1(x_1, x_2, \dots, x_n), \dots, h_m(x_1, x_2, \dots, x_n))$



Example 1. Primitive Recursive Function

- $id(x) = x$ $\quad id : N \rightarrow N$
  - The identity function is primitive recursive.
  - $id(x) = \pi_1^1(x)$
- $one() = 1$ $\quad one : N^0 \rightarrow N$
  - Function which always return 1 is primitive recursive.
  - $one() = suc(zero())$
- $two() = 2$ $\quad two : N^0 \rightarrow N$
  - Function which always return a constant is primitive recursive.
- $add2(x) = x + 2$ $\quad add2: N \rightarrow N$
  - Function which adds 2 is primitive recursive.
  - $add2(x) = suc(suc(x))$

Primitive Recursion

- Given primitive recursive functions $g : N^n \to N$ and $h : N^{n+2} \to N$, $f : N^{n+1} \to N$ is also primitive recursion:
  - $f(x_1, \dots, x_n, 0) = g(x_1, \dots, x_n)$
  - $f\big(x_1, \dots, x_n, suc(y)\big) = h(x_1, \dots, x_n, y, f(x_1, \dots, x_n, y))$

- It is called that the function is defined by primitive recursion.

- If $n = 0$, when $g : N^0 \to N$ and $h : N \times N \to N$ are primitive recursive, $f : N \to N$ is primitive recursive:
  - $f(0) = g()$
  - $f\big(suc(y)\big) = h(y, f(y))$
- Define $f(x)$ when x is 0 and when x is not 0:
  - when x is 0, the value is constant $g()$
  - when x is not 0, x can be written as $x = suc(y)$, and $f(x)$ can be calculate from y and $f(y)$ by $h(y, f(y))$
  - If y is 0 then $f(y)$ is $g()$, but if not, repeat the case of $f(suc(y))$ again until y becomes 0

- $f(1) = f\big(suc(0)\big) = h\big(0, f(0)\big) = h(0, g())$
- $f(2) = f\big(suc(1)\big) = h\big(1, f(1)\big) = h\big(1, h(0, g())\big)$
- $f(3) = f\big(suc(2)\big) = h\big(2, f(2)\big) = h\Big(2, h\big(1, h(0, g()\,)\big)\Big)$

Example 2. Primitive Recursive Function

- $pred: N \rightarrow N \qquad pred(x) = x - 1$
  - returns previous number (reverse of suc)

Primitive Recursion ($n = 0$)
- $f(0) = g()$
- $f\big(suc(y)\big) = h(y, f(y))$

  - $pred(0) =$
  - $pred\big(suc(y)\big) =$
  - $pred(1) =$
  - $pred(5) =$

Example 3. Primitive Recursive Function

- $add: N^2 \rightarrow N$ $\quad add(x, y) = x + y$ Addition

Primitive Recursion ($n = 1$)
- $f(x, 0) = g(x)$
- $f(x, suc(y)) = h(x, y, f(x, y))$

  - $add(x, 0) =$
  - $add(x, suc(y)) =$

  - $add(1,1) =$

  - $add(3,2) =$

Example 4. Primitive Recursive Function

- $sub: N^2 \to N$ $\quad sub(x, y) = x - y$ Subtraction
  - $sub(x, 0) =$
  - $sub(x, suc(y)) =$

Primitive Recursion ($n = 1$)
- $f(x, 0) = g(x)$
- $f(x, suc(y)) = h(x, y, f(x, y))$

- $mul: N^2 \to N$ $\quad mul(x, y) = x \times y$ Multiplication
  - $mul(x, 0) =$
  - $mul(x, suc(y)) =$

Primitive Recursive Function

- **Definition:**
  - Primitive recursive functions are:
    - The following three functions are primitive recursive functions:
      - $zero : N^0 \to N$ $\quad zero() = 0$
      - $suc : N \to N$ $\quad suc(x) = x + 1$ (next number of x)
      - $\pi_i^n : N^n \to N$ $\quad \pi_i^n(x_1, \dots, x_n) = x_i$

    For short, we use 0 for $zero()$
    - Composition of primitive recursive functions are primitive recursive:
      - When $g : N^m \to N$ and $h_i : N^n \to N$ ($i = 1,2, \dots, m$) are primitive recursive, $f : N^n \to N$ is also primitive recursive:
        $f(x_1, x_2, \dots, x_n) = g(h_1(x_1, x_2, \dots, x_n), \dots, h_m(x_1, x_2, \dots, x_n))$
    - Define a function using primitive recursion:
      - When $g : N^n \to N$ and $h : N^{n+2} \to N$ are primitive recursive, $f : N^{n+1} \to N$ is also primitive recursive:

$$f(x_1, \dots, x_n, 0) = g(x_1, \dots, x_n)$$
$$f\big(x_1, \dots, x_n, suc(y)\big) = h(x_1, \dots, x_n, y, f(x_1, \dots, x_n, y))$$

Summation and Product

- **Lemma:**
  - If $f(x_1, \dots, x_n, y)$ is primitive recursive, the following functions are also primitive recursive:

$$g(x_1, \dots, x_n, y) = \sum_{z=0}^{y-1} f(x_1, \dots, x_n, z)$$

$$h(x_1, \dots, x_n, y) = \prod_{z=0}^{y-1} f(x_1, \dots, x_n, z)$$

  - If $n = 0$,
    - $g(y) = f(0) + f(1) + \cdots + f(y-1)$
    - $h(y) = f(0) \times f(1) \times \cdots \times f(y-1)$
- Proof:
  - $g(x_1, \dots, x_n, 0) =$
  - $g\big(x_1, \dots, x_n, suc(z)\big) =$

  - $h(x_1, \dots, x_n, 0) =$
  - $h\big(x_1, \dots, x_n, suc(z)\big) =$

Primitive Recursive Predicate

- **Definition:**
  - A predicate $p(x_1, \ldots, x_n): N^n \to \{T, F\}$ is primitive recursive, when its characteristic function $C_p(x_1, \ldots, x_n): N^n \to N$ is primitive recursive.
    - $C_p(x_1, \ldots, x_n) = 1$ when $p(x_1, \ldots, x_n)$ is T
    - $C_p(x_1, \ldots, x_n) = 0$ when $p(x_1, \ldots, x_n)$ is F

- Example:
  - Predicate $x = 0$ is primitive recursive:
    - $C_{=0}(x) =$
  - Predicate $x \leq y$ is primitive recursive:
    - $C_{\leq}(x, y) =$
  - Predicate $x = y$ is primitive recursive:
    - $C_{=}(x, y) =$

Logical Formula

- **Lemma:**
  - When $p(x_1, \dots, x_n)$, $q(x_1, \dots, x_n)$, $r(x_1, \dots, x_n, z)$ are primitive recursive predicates, the following predicates are also primitive recursive:
    - $p(x_1, \dots, x_n) \wedge q(x_1, \dots, x_n)$
    - $p(x_1, \dots, x_n) \vee q(x_1, \dots, x_n)$
    - $\neg p(x_1, \dots, x_n)$
    - $\forall z < y\big(r(x_1, \dots, x_n, z)\big) \equiv \forall z\big(z < y \Rightarrow r(x_1, \dots, x_n, z)\big)$
    - $\exists z < y\big(r(x_1, \dots, x_n, z)\big) \equiv \exists z\big(z < y \wedge r(x_1, \dots, x_n, z)\big)$

Proof:

- $C_{P \wedge Q}(x_1, \dots, x_n) =$
- $C_{\neg P}(x_1, \dots, x_n) =$
- $C_{P \vee Q}(x_1, \dots, x_n) =$
- $C_{\forall R}(x_1, \dots, x_n, y) =$
- $C_{\exists R}(x_1, \dots, x_n, y) =$

Example

- $x > y$ is primitive recursive:
  - $(x > y) \equiv$

- $divisible(x, y) \equiv$ ``x is divisible by y'' is primitive recursive:
  - $divisible(x, y) \equiv$

- $prime(x) \equiv$ ``x is a prime number$''$ is primitive recursive:
  - $prime(x) \equiv$

Minimum

- **Lemma:**
  - If $p(x_1, \dots, x_n, z)$ is a primitive recursive predicate, the following function is primitive recursive:

$$\mu_{z<y} p(x_1, \dots, x_n, z) = \min(\{z \mid p(x_1, \dots, x_n, z)\} \cup \{y\})$$

the minimum number z which makes $p(x_1, \dots, x_n, z)$ hold and less than y.

- Proof:
  - Let $g(x_1, \dots, x_n, z)$ be the characteristic function of $\forall v < z(\neg p(x_1, \dots, x_n, v))$. Then,

$$\mu_{z<y} p(x_1, \dots, x_n, z) = \sum_{z=0}^{y-1} g(x_1, \dots, x_n, z+1)$$

if $n = 1$
$\mu_{z<y} P(x, z)$

z	0	1	2	3	4	5	6	…	$y-1$	y
$P(x,z)$	F	F	F	T	F	T	T	…	F	T
$\neg P(x,z)$	T	T	T	F	T	F	F	…	T	F
$\forall v < z(\neg P(x,v))$	T	T	T	T	F	F	F		F	F
$g(x,z)$	1	1	1	1	0	0	0	…	0	0

$$\mu_{z<y} P(x,z) = \sum_{z=0}^{y-1} g(x, z+1) = \sum_{z=1}^{y} g(x,z) = 3$$

Corollaries

Corollary: $div(x, y) = x \div y$ is primitive recursive.

- $div(x, y) = \mu_{z<x}\left(x < \left(y \times (z+1)\right)\right)$

- $div(5, 2) =$

Theorem: Arithmetic operations (addition, subtraction, multiplication and division) are primitive recursive.

Corollary: $pr(x) =$ ``xth prime number'' is primitive recursive.

- $pr(0) = 2$
- $pr\left(suc(x)\right) = \mu_{y<pr(x)!+2}\left(pr(x) < y \wedge prime(y)\right)$

Summary

- Primitive recursive function
  - $zero$
  - suc
  - π_i^n
  - function composition
  - define by primitive recursion

- Primitive recursive predicate

- Arithmetic operations are primitive recursive.