MATHEMATICS FOR INFORMATION SCIENCE

NO.8 COMPLETE PARTIAL ORDERED SET

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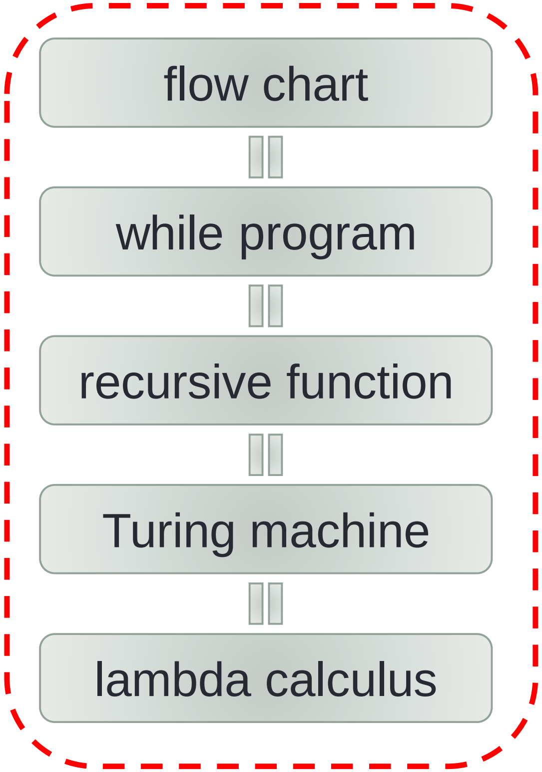
Slides URL

https://vu5.sfc.keio.ac.jp/slide/

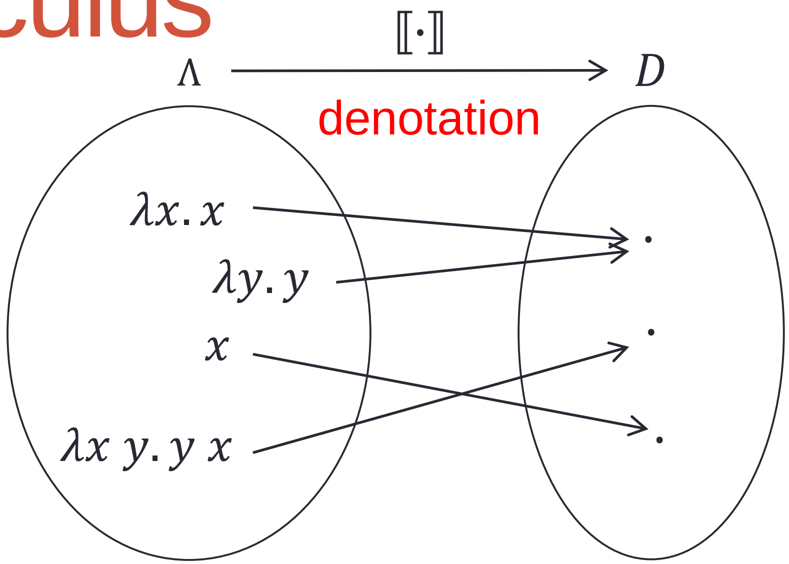
So far

- Computation
  - flow chart program
  - while program
  - recursive function
    - primitive recursive function
    - minimization operator
  - Turing machine
    - undecidable problems
  - Lambda Calculus
    - function abstraction
    - function application
    - λ representation



- λ expression
  - Model function abstraction and application
  - What kind of functions?
  - $\lambda x.xx$
  - x is a function as well as a value.

Model of Lambda Calculus



- Model of λ expression
  - $\Lambda = \{M \mid M \text{ is a } \lambda \text{ expression}\}$
  - M's meaning
    - $[\![M]\!]$
    - M's denotation
  - $[\![\cdot]\!]: \Lambda \to D$
    - Assign a meaning (i.e. a value in D) to each λ expression
    - $[\![\lambda x. M]\!] = \lambda x. [\![M]\!]$
    - $[\![MN]\!] = [\![M]\!]([\![N]\!])$

- Property of D
  - $[\![\lambda x. M]\!] \in D \to D$
    - $D \to D \subseteq D$
  - $[\![MN]\!] = [\![M]\!]([\![N]\!])$
    - $D \subseteq D \to D$

- Therefore,
  - $D \cong D \to D$
  - An element D is always a function.
  - There does not exist a non trivial set which satisfies this.

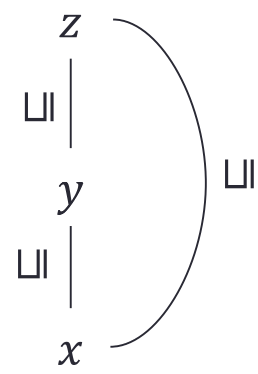
Mathematical Property of Information 1

- Information can be compared.
  - "SFC is in Kanagawa"
  - "SFC is in Kanto"
  - "SFC is in Fujisawa"

- Same as number?
  - "SFC is in Kanagawa"
  - "SFC is a campus of Keio university"

- Not all information can be compared.

Order Relation



- $\sqsubseteq$ is an order relation on a set D when:
  - Reflective: $x \sqsubseteq x$
  - Transitive: $x \sqsubseteq y$ and $y \sqsubseteq z$ then $x \sqsubseteq z$
  - Antisymmetric: $x \sqsubseteq y$ and $y \sqsubseteq x$ then $x = y$

- $(D, \sqsubseteq)$ is a *partially ordered set*.
  - $\sqsubseteq$ is a *partial order*.

- Information is a partial ordered set.
  - It is not a *totally ordered set*.
  - Totality: for any x and y, either $x \sqsubseteq y$ or $y \sqsubseteq x$ holds.

- Which are partially ordered set?
  - Natural number and $x \leq y$
  - Integer and $x \leq y$
  - Natural number and $x < y$
  - Subsets and $A \subseteq B$
  - Friend relation

Mathematical Property of Information 2

- There is the smallest information.
  - No information.
  - Do not know anything.

- $\bot$ is the smallest element when
  - for any x, $\bot \sqsubseteq x$

- What is the smallest element of the following partially ordered sets?
  - Natural number and $x \leq y$
  - Integer and $x \leq y$
  - Subsets and $A \subseteq B$

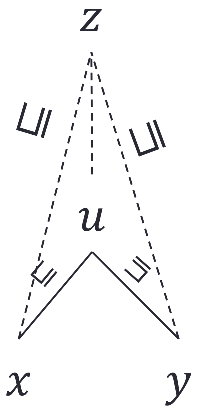
The Largest Element?

- $\top$ is the largest element when
  - for any x, $x \sqsubseteq \top$

- The largest information means:
  - Include all the information
  - Combine all the information
  - Even combine wrong information
  - Contradiction

Mathematical Property of Information 3

- Combine two information.
  (1) "SFC is in Fujisawa"
  (2) "SFC is a campus of Keio university"
  (3) "SFC is a campus of Keio university in Fujisawa"
  $(1) \sqsubseteq (3)$ and $(2) \sqsubseteq (3)$



- (3) is *the least upper bound* of (1) and (2)
- u is *the least upper bound* of x and y when
  - $x \sqsubseteq u$ and $y \sqsubseteq u$
  - for any z which satisfies $x \sqsubseteq z$ and $y \sqsubseteq z$, $u \sqsubseteq z$
  - written as $x \sqcup y$
  - The small element which is larger than x and y
  - The least upper bound is unique if it exists.
- What is the least upper bound?
  - Natural number and $x \leq y$
  - Subsets and $A \subseteq B$

Complete Lattice

- $\sqcup A$ is the least upper bound of a set A when
  - for any $x \in A$, $x \sqsubseteq \sqcup A$
  - if z satisfies $x \sqsubseteq z$ for any $x \in A$, $\sqcup A \sqsubseteq z$

- $\sqcup \{x, y\} = x \sqcup y$

- The least upper bound of A and the largest element of A are different.
  - $\{ 0.9, 0.99, 0.999, 0.9999, \ldots \}$
  - The largest element of A needs to be an element of A .

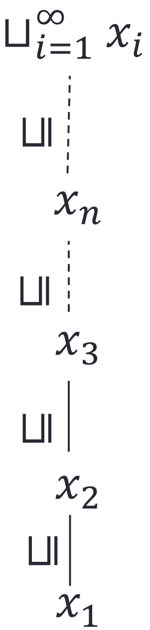
- $(D, \sqsubseteq)$ is a complete lattice when
  - for any set A, $\sqcup A$ exists.

Property of Complete Lattice

- A complete lattice has the smallest element.
  - $\bot = \sqcup \emptyset$
- A complete lattice has the greatest lower bound for any set A.
  - $\sqcap A$ is the greatest lower bound of a set A when
    - for any $x \in A$, $\sqcap A \sqsubseteq x$
    - if z satisfies $z \sqsubseteq x$ for any $x \in A$, $z \sqsubseteq \sqcap A$
  - $\sqcap A = \sqcup \{z \in D \mid z \sqsubseteq x \text{ for any } x \in A\}$
- A complete lattice has the largest element
  - $\top = \sqcap \emptyset$
- Is a set of information a complete lattice?

Complete Partial Ordered Set

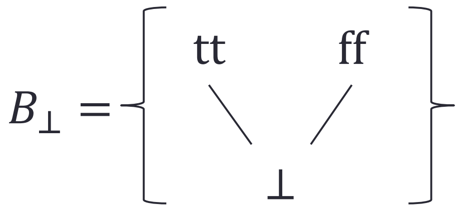
- A partially ordered set $(D, \sqsubseteq)$ is a *complete partial ordered set* (cpo) when
  - it has the small element $\bot$
  - for $x_1 \sqsubseteq x_2 \sqsubseteq x_3 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$, there is the least upper bound $\sqcup_{i=1}^{\infty} x_i$
- There may no exists the largest element.
- A set of information is a complete partial ordered set.



Example

- Subsets and $A \subseteq B$
  - complete lattice

- Flat Domain
  - For a set S, add the smallest element $\perp$
    - for any $x \in S$, $\perp \sqsubseteq x$
  - A flat domain is CPO.
  - For Boolean set $B = \{\mathrm{tt}, \mathrm{ff}\}$, its flat domain is:

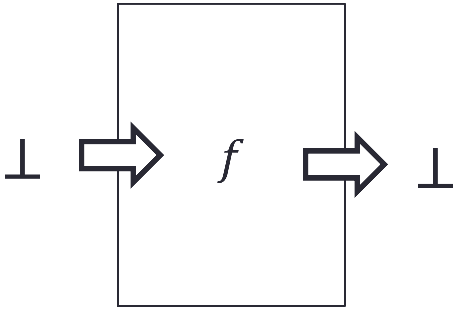


Program as a Function

- A program is a function between complete partial ordered sets.

- The function needs to take care ⊥, ⊑ and ⊔ .

(1) $f(\perp) = \perp$



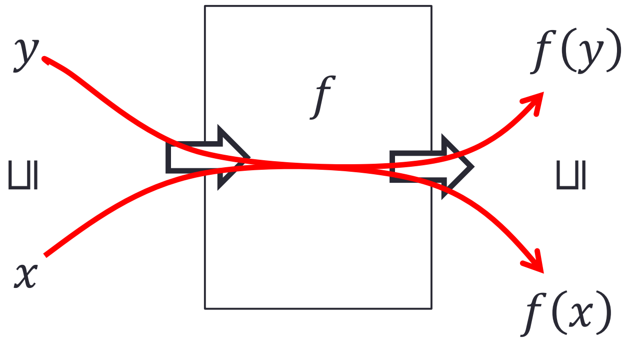
Program as a Function

- A program is a function between complete partial ordered sets.

- The function needs to take care ⊥, ⊑ and ⊔ .

  (1) $f(\bot) = \bot$

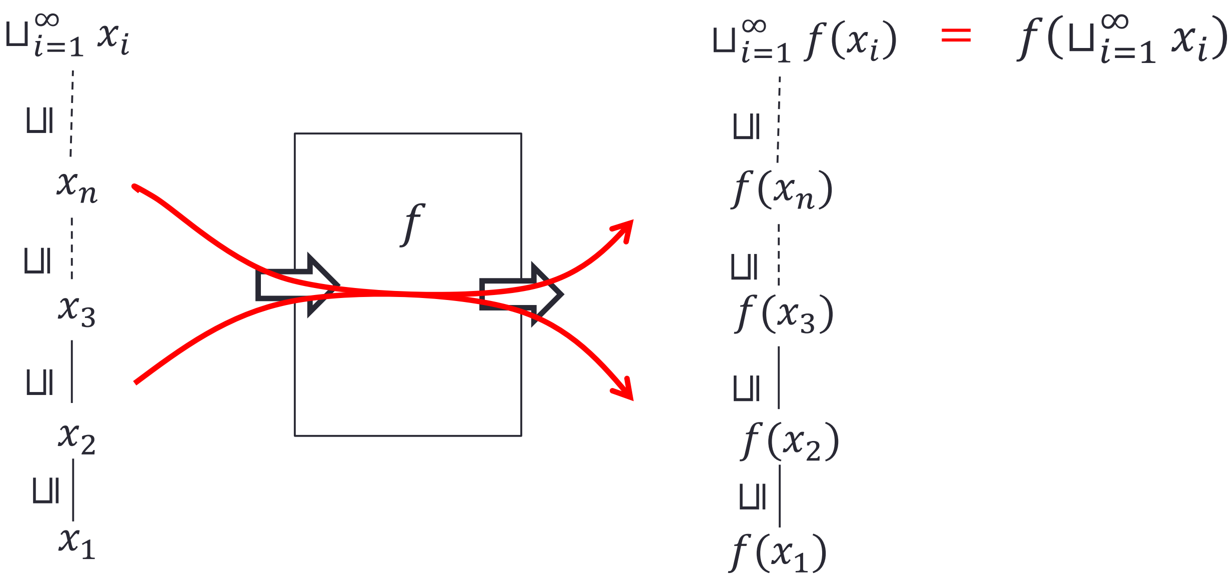
  (2) if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$



Program as a Function

- A program is a function between complete partial ordered sets.

- The function needs to take care $\bot$, $\sqsubseteq$ and $\sqcup$.
  (1) $f(\bot) = \bot$
  (2) if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$
  (3) for $x_1 \sqsubseteq x_2 \sqsubseteq x_3 \sqsubseteq \cdots$, $f(\coprod_{i=1}^{\infty} x_i) = \coprod_{i=1}^{\infty} f(x_i)$



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  (1) $f(\bot) = \bot$

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  (3) for $x_1 \sqsubseteq x_2 \sqsubseteq x_3 \sqsubseteq \cdots$, $f(\bigsqcup_{i=1}^{\infty} x_i) = \bigsqcup_{i=1}^{\infty} f(x_i)$

- If f satisfies (2), f is monotonic.
- If f satisfies (2) and (3), f is continuous.
- If f satisfies (1), f is strict.

- A program is continuous.

What is bottom?

- The meaning of bottom (the smallest element, ⊥):
  - No information
  - Memory is clear.
  - Start of computation
  - Not yet computed
- A program returns ⊥:
  - No answer
  - Computation does not terminate.
  - Undefined
- Pass ⊥ to a program:
  - No information is given
  - Do not use this
- What is a strict function?
  - Is any program strict?

Example of Continuous Function

- $B = \{\mathrm{tt}, \mathrm{ff}\}$
- $\mathrm{not}: B_\perp \to B_\perp$
  - $\mathrm{not}(\mathrm{tt}) = \mathrm{ff}$
  - $\mathrm{not}(\mathrm{ff}) = \mathrm{tt}$
  - $\mathrm{not}(\perp) = \perp$

- Monotonic
  - for $\perp \sqsubseteq \mathrm{ff}$, $\mathrm{not}(\perp) \sqsubseteq \mathrm{not}(\mathrm{ff})$
  - for $\perp \sqsubseteq \mathrm{tt}$, $\mathrm{not}(\perp) \sqsubseteq \mathrm{not}(\mathrm{tt})$
- Continuous
  - for $\perp \sqsubseteq \perp \sqsubseteq \perp \sqsubseteq \cdots \sqsubseteq \perp \sqsubseteq \mathrm{ff} \sqsubseteq \mathrm{ff} \sqsubseteq \cdots$,
    - $\mathrm{not}(\perp) \sqsubseteq \mathrm{not}(\perp) \sqsubseteq \mathrm{not}(\perp) \sqsubseteq \cdots \sqsubseteq \mathrm{not}(\perp) \sqsubseteq \mathrm{not}(\mathrm{ff}) \sqsubseteq \mathrm{not}(\mathrm{ff}) \sqsubseteq \cdots$

- $N = \{0, 1, 2, 3, 4, 5, \ldots\}$
- $\mathrm{add1}: N_\perp \to N_\perp$
  - $\mathrm{add1}(n) = n + 1$
  - $\mathrm{add1}(\perp) = \perp$

Summary

- Complete Partial Order
  - A set of information
  - partial order
  - bottom = the smallest element
  - least upper bound
- Continuous function
  - preserve least upper bounds
  - strictness

Homework 8

- Let $B = \{\mathrm{tt}, \mathrm{ff}\}$. Write all the continuous functions of $B_\perp \to B_\perp$.

- Example:
  - $\mathrm{not}: B_\perp \to B_\perp$ is continuous.
    - $\mathrm{not}(\mathrm{tt}) = \mathrm{ff}$
    - $\mathrm{not}(\mathrm{ff}) = \mathrm{tt}$
    - $\mathrm{not}(\perp) = \perp$

- Write down all the continuous functions including 'not'.
- How many are there?