MATHEMATICS FOR INFORMATION SCIENCE

NO.11 INTRODUCTION TO CATEGORY THEORY

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Set Theory

- Set Theory
  - Foundation of Modern Mathematics
  - a set ≡ a collection of elements with some property
    - $\emptyset$
    - $A \cup B$
    - $A \cap B$
    - A^c
    - $\{x \in A \mid \text{logical formula about } x\}$
  - Description of elements is important: i.e. when $x \in A$

- Limit of set theory
  - Russell's Paradox
  - The collection of all sets is not a set.
  - $R = \{x \mid x \notin x\}$
  - $R \in R$ or $R \notin R$?

Category Theory

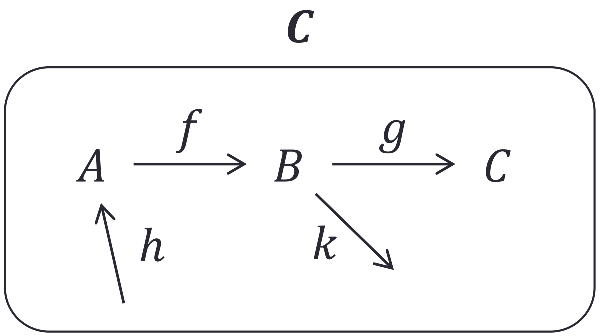
- Alternative foundation of Mathematics
- Some call it Abstract Nonsense.
- Describe things with relationship with others.

Set Theory	Category Theory
$x \in A$	$A \to B$
description of inside	description from outside
contents	actions

- Unify many concepts in one.
- Can see symmetry easily.

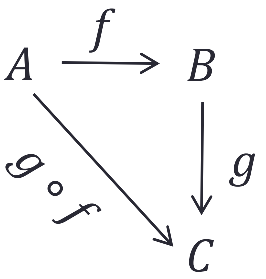
Category

- A category C consists of:
  - A collection of objects: $\boldsymbol{C} = \{A, B, C, \ldots\}$
  - For objects A and B, a collection of arrows (or morphisms): $\hom_{\boldsymbol{C}}(A, B) = \{f, g, h, \ldots\}$
    - If $f \in \hom_{\boldsymbol{C}}(A, B)$, we may write it as: $f: A \to B$
    - A is the domain of f
    - B is the codomain (or range) of f

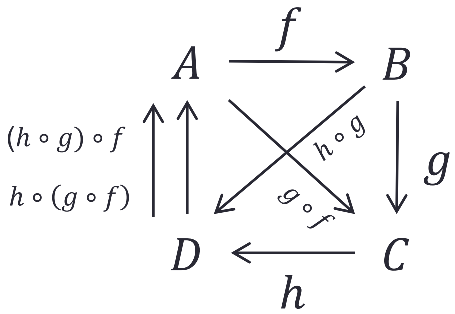


Category (cont)

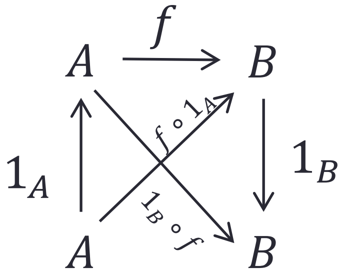
- A category $\boldsymbol{C}$ mush satisfy the following properties:
  - For $f: A \to B$ and $g: B \to C$,
    $g \circ f: A \to C$



  - For $f: A \to B$, $g: B \to C$ and $f: C \to D$,
    $h \circ (g \circ f) = (h \circ g) \circ f$



  - For each object A, there exists an identity arrow $1_A: A \to A$ and for $f: A \to B$,
    $f \circ 1_A = f$ and $1_B \circ f = f$



Example of Category

- ***Set***: the category of sets
  - objects: sets
  - arrows: functions
  - $\circ$ is the function composition
  - 1_A is the identity function of A
- ***Grp***: the category of groups
  - objects: groups $(G, \cdot, e, {}^{-1})$
  - arrows: homomorphisms
  - $\circ$ is the function composition
  - 1_A is the identity function of A

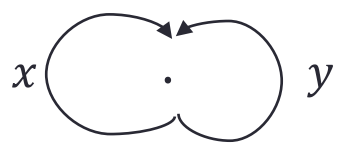
$(G, \cdot, e, {}^{-1})$: group
- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $x \cdot e = e \cdot x = x$
- $x \cdot x^{-1} = x^{-1} \cdot x = e$

homomorphism: $f : G \to H$
- $f(x \cdot y) = f(x) \cdot f(y)$

- ***CPO***: the category of complete partial ordered sets
  - objects: CPO
  - arrows: continuous functions
  - $\circ$ is the function composition.
  - 1_D is the identity function of D

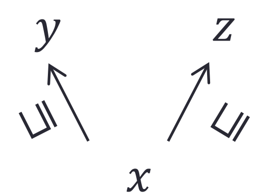
Example of Category (cont.)

- Monoid $(M, \cdot, e)$ as a category
  - object: only one object
  - arrows: M (i.e. elements in M)
  - $\circ$ is $\cdot$
  - 1 is e



$(M, \cdot, e)$: monoid
- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $x \cdot e = e \cdot x = x$

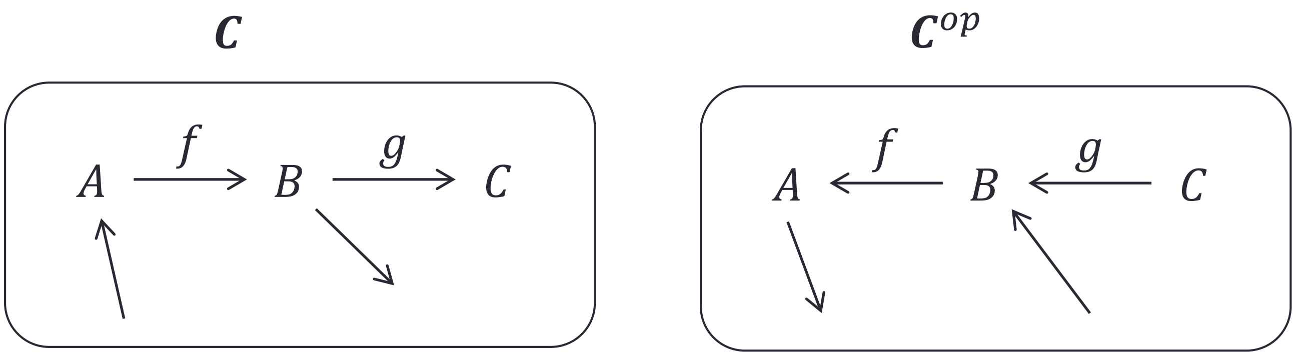
- Partially ordered set $(D, \sqsubseteq)$ as a category
  - objects: D (i.e. elements in D)
  - arrows: $\sqsubseteq$ (i.e. at most one arrow from $x \rightarrow y$)
  - $\circ$ is ``if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$"
  - 1_x is ``$x \sqsubseteq x$"



$(D, \sqsubseteq)$: partially ordered set
- $x \sqsubseteq x$
- if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$
- if $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$

Dual Category

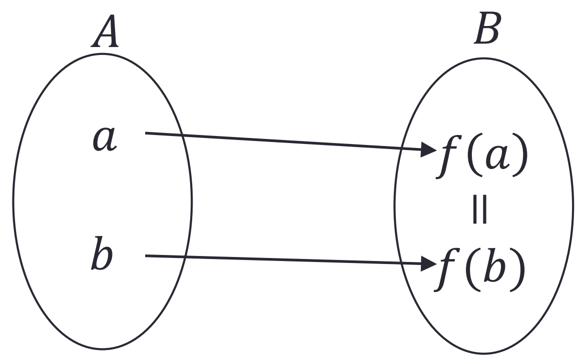
- Dual Category $\boldsymbol{C}^{op}$ of category $\boldsymbol{C}$
  - $\boldsymbol{C}^{op}$ objects = $\boldsymbol{C}$ objects
  - $\boldsymbol{C}^{op}$ arrows: $\hom_{\boldsymbol{C}^{op}}(A, B) = \hom_{\boldsymbol{C}}(B, A)$
  - Reverse the direction of arrows.



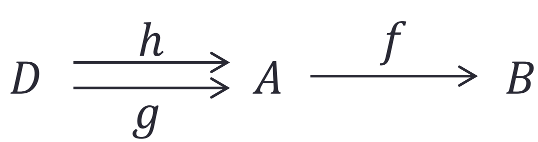
- Any property which is true in category $\boldsymbol{C}$ is also true in its dual category $\boldsymbol{C}^{op}$.
  - $(\boldsymbol{C}^{op})^{op} = \boldsymbol{C}$

Mono Morphism

- One-to-one function
  - A function $f\colon A \to B$ is one-to-one
    $\Leftrightarrow a = b$ if $f(a) = f(b)$



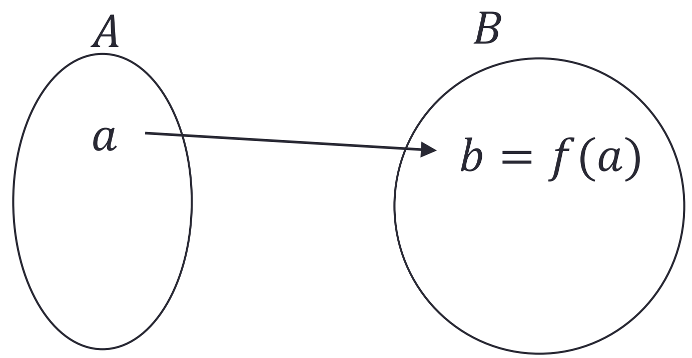
- Mono morphism
  - $f\colon A \to B$ is mono-morphism
    $\Leftrightarrow$ for any object D and any arrows $g\colon D \to A$ and $h\colon D \to A$, if $f \circ g = f \circ h$, then $g = h$.



Epi Morphism

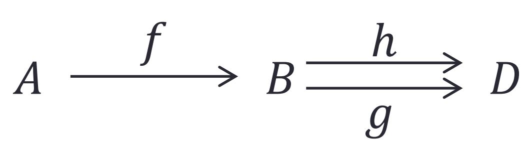
- Onto function
  - A function $f: A \to B$ is onto

    $\Leftrightarrow$ for any $b \in B$, there exists $a \in B$ such that $f(a) = b$.



- Epi morphism
  - $f: A \to B$ is epi-morphism

    $\Leftrightarrow$ for any object D and any arrows $g: B \to D$ and $h: B \to D$, if $g \circ f = h \circ f$, then $g = h$.



Mono and Epi

- Mono
  - $f: A \to B$ is mono-morphism
    $\Leftrightarrow$ for any object D and any arrows $g: D \to A$ and $h: D \to A$, if $f \circ g = f \circ h$, then $g = h$.

$$D \underset{g}{\overset{h}{\rightrightarrows}} A \xrightarrow{f} B$$

Mono in $\boldsymbol{C}^{op}$ is epi in $\boldsymbol{C}$.

- Epi
  - $f: A \to B$ is epi-morphism
    $\Leftrightarrow$ for any object D and any arrows $g: B \to D$ and $h: B \to D$, if $g \circ f = h \circ f$, then $g = h$.

$$A \xrightarrow{f} B \underset{g}{\overset{h}{\rightrightarrows}} D$$

Epi in $\boldsymbol{C}^{op}$ is mono in $\boldsymbol{C}$.

Isomorphic

- Isomorphic (iso)
  - Object A and B is are isomorphic

    $\Longleftrightarrow$ there are $f: A \to B$ and $g: B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

Isomorphic objects play the same role in $\boldsymbol{C}$.

Initial and Final Objects

- Initial object I
  - for any object A, there is a unique arrow from I to A.

$$I \overset{!}{\dashrightarrow} A$$

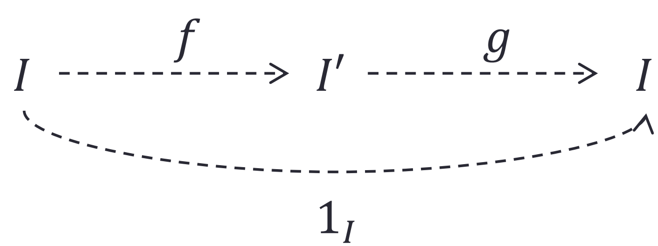
- Final object F
  - for any object A, there is a unique arrow from A to F.

$$A \overset{!}{\dashrightarrow} F$$

	Initial Object	Final Object
Set		
Grp		
CPO		
Partially Ordered Set		

Uniqueness of Initial Object

- **Theorem:** The initial object, if it exists, is unique up to isomorphic.
- **Proof:** Assume there are two initial objects I and I'.
  - Since I is an initial object, there is a unique arrow f from I to I'.
  - Since I' is an initial object, there is a unique arrow g from I' to I.
  - $g \circ f$ is an arrow from I to I.
  - Since I is an initial object, it should be the unique arrow 1_I.
    - $g \circ f = 1_I$

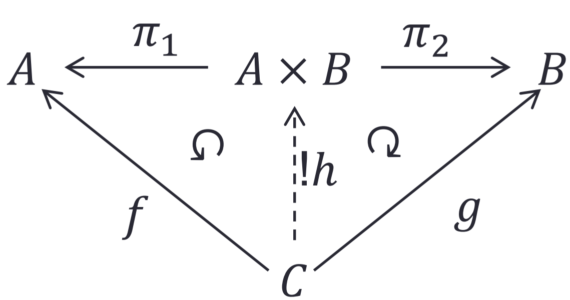


  - Similarly, $f \circ g = 1_{I'}$.
  - Therefore I and I' are isomorphic. QED
- **Dual Theorem:** The final object, if it exists, is unique up to isomorphic.

Product and Co-Product

- $A \times B$ is the product of A and $B \Leftrightarrow$
  - There are two arrows $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$.
  - For any object C and arrows $f : C \to A$ and $g : C \to B$,

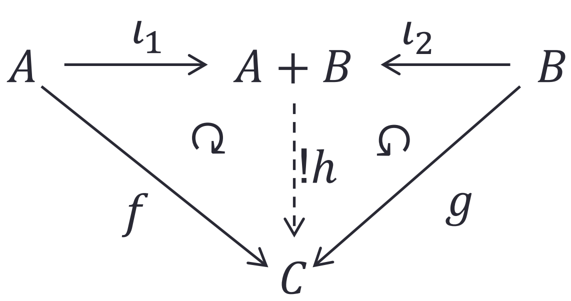
    there exists a unique arrow $h : C \to A \times B$ such that the following diagram commutes:



$\pi_1 \circ h = f$ $\quad\quad$ $\pi_2 \circ h = g$

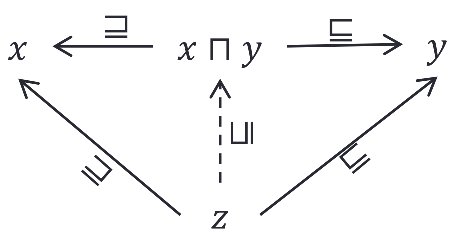
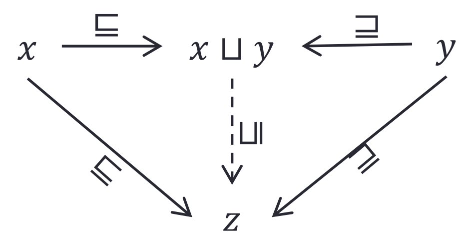
- $A + B$ is the co-product of A and $B \Leftrightarrow$
  - There are two arrows $\iota_1 : A \to A + B$ and $\iota_2 : B \to A + B$. :
  - For any object C and arrows $f : A \to C$ and $g : B \to C$,

    there exists a unique arrow $h : A + B \to C$ such that the following diagram commutes:



$h \circ \iota_1 = f$ $\quad\quad$ $h \circ \iota_2 = g$

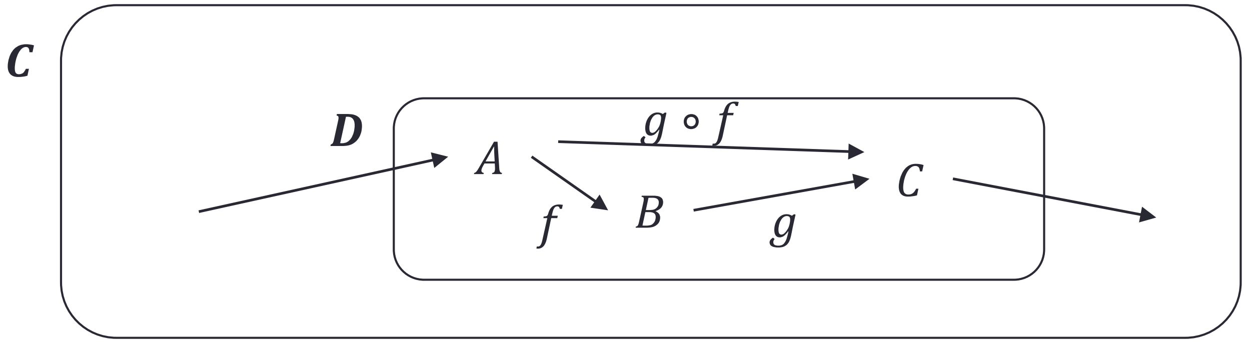
Product and Co-Product

	Product	Co-Product
Set	• $A \times B = \{(x,y) \mid x \in A$ and $y \in B\}$ • $\pi_1\big((x,y)\big) = x$ • $\pi_2\big((x,y)\big) = y$ • For $f: C \to A$ and $g: C \to B$, $h(z) = \big(f(z), g(z)\big)$	• $A + B = \{(x,1) \mid x \in A\} \cup \{(y,2) \mid y \in B\}$ • $\iota_1(x) = (x,1)$ • $\iota_2(y) = (y,2)$ • For $f: A \to C$ and $g: B \to C$, $h\big((x,1)\big) = f(x)$ and $h\big((y,2)\big) = g(y)$
Partially Ordere d Set $(D, \sqsubseteq)$	• $x \times y = x \sqcap y$ • $\pi_1: x \sqcap y \sqsubseteq x$ • $\pi_2: x \sqcap y \sqsubseteq y$ • For $f: z \sqsubseteq x$ and $g: z \sqsubseteq y$, $h: z \sqsubseteq x \sqcap y$ 	• $x + y = x \sqcup y$ • $\iota_1: x \sqsubseteq x \sqcup y$ • $\iota_2: y \sqsubseteq x \sqcup y$ • For $f: x \sqsubseteq z$ and $g: y \sqsubseteq z$, $h: x \sqcup y \sqsubseteq z$ 

- **Theorem:** If product $A \times B$ exists, it is unique up to isomorphic.
- **Dual Theorem:** If co-product $A + B$ exists, it is unique up to isomorphic.

Subcategory

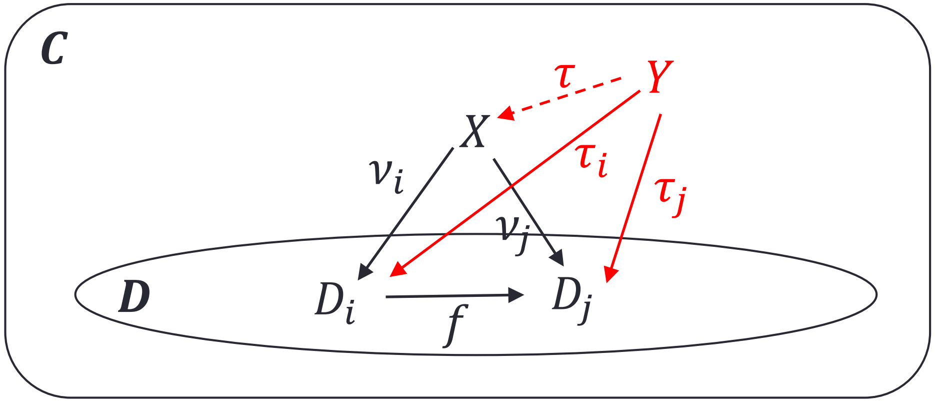
- $\boldsymbol{D}$ is a subcategory of category $\boldsymbol{C}$ if the following conditions are hold:
  - For any object $A \in \boldsymbol{D}$, $A \in \boldsymbol{C}$
  - For any arrow $f: A \to B \in \boldsymbol{D}$, $A, B \in \boldsymbol{D}$
  - For any object $A \in \boldsymbol{D}$, $1_A \in \boldsymbol{D}$
  - For any arrows $f, g \in \boldsymbol{D}$, $g \circ f \in \boldsymbol{D}$



- A subcategory $\boldsymbol{D}$ is a category.
- Most of the diagrams we saw are subcategories.

Limit

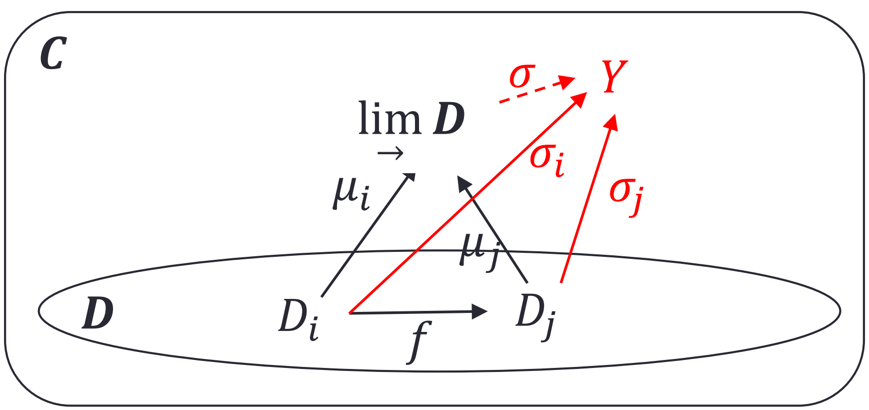
- The limit of a subcategory $\boldsymbol{D}$ of category $\boldsymbol{C}$ is the object X in $\boldsymbol{C}$ which satisfies the following conditions:
  - For any object $D_i \in \boldsymbol{D}$, there exists an arrow $\nu_i: X \to D_i$.
  - For any arrow $f: D_i \to D_j \in \boldsymbol{D}$, $f \circ \nu_i = \nu_j$.
  - For any object Y in $\boldsymbol{C}$ and arrows $\tau_i: Y \to D_i$ which satisfies for any arrow $f: D_i \to D_j \in \boldsymbol{D}$ of $f \circ \tau_i = \tau_j$, there exists a unique arrow $\tau: Y \to X$ such that $\tau_i = \nu_i \circ \tau$.

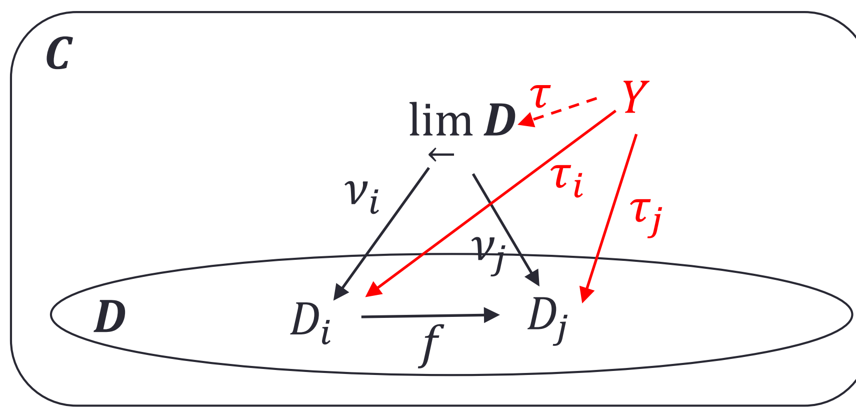


- The limit X of $\boldsymbol{D}$ is written as $\varprojlim \boldsymbol{D}$.

Colimit

- The colimit of a subcategory $\boldsymbol{D}$ of category $\boldsymbol{C}$ is the object X in $\boldsymbol{C}$ which satisfies the following conditions:
  - For any object $D_i \in \boldsymbol{D}$, there exists an arrow $\mu_i : D_i \to X$.
  - For any arrow $f : D_i \to D_j \in \boldsymbol{D}$, $\mu_j \circ f = \mu_i$.
  - For any object Y in $\boldsymbol{C}$ and arrows $\sigma_i : D_i \to Y$ which satisfies for any arrow $f : D_i \to D_j \in \boldsymbol{D}$ of $\sigma_j \circ f = \tau_i$, , there exists a unique arrow $\sigma : X \to Y$ such that $\sigma_i = \sigma \circ \mu_i$.



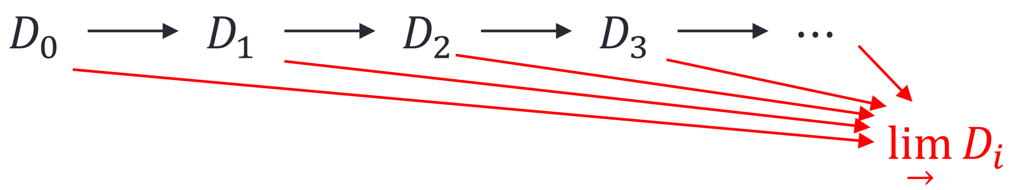


- The colimit X of $\boldsymbol{D}$ is written as $\varinjlim \boldsymbol{D}$.

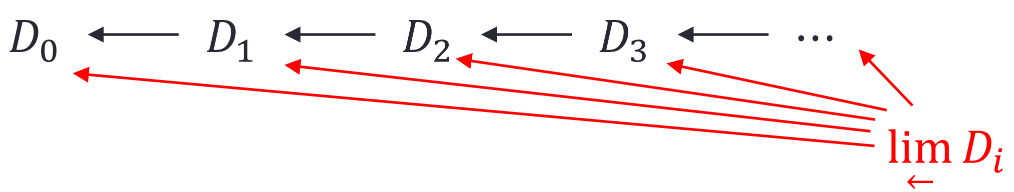
$\varinjlim \boldsymbol{D}$ and $\varprojlim \boldsymbol{D}$ are dual

Inductive Limit and Projective Limit

- Inductive limit
  - The colimit of the following diagram:

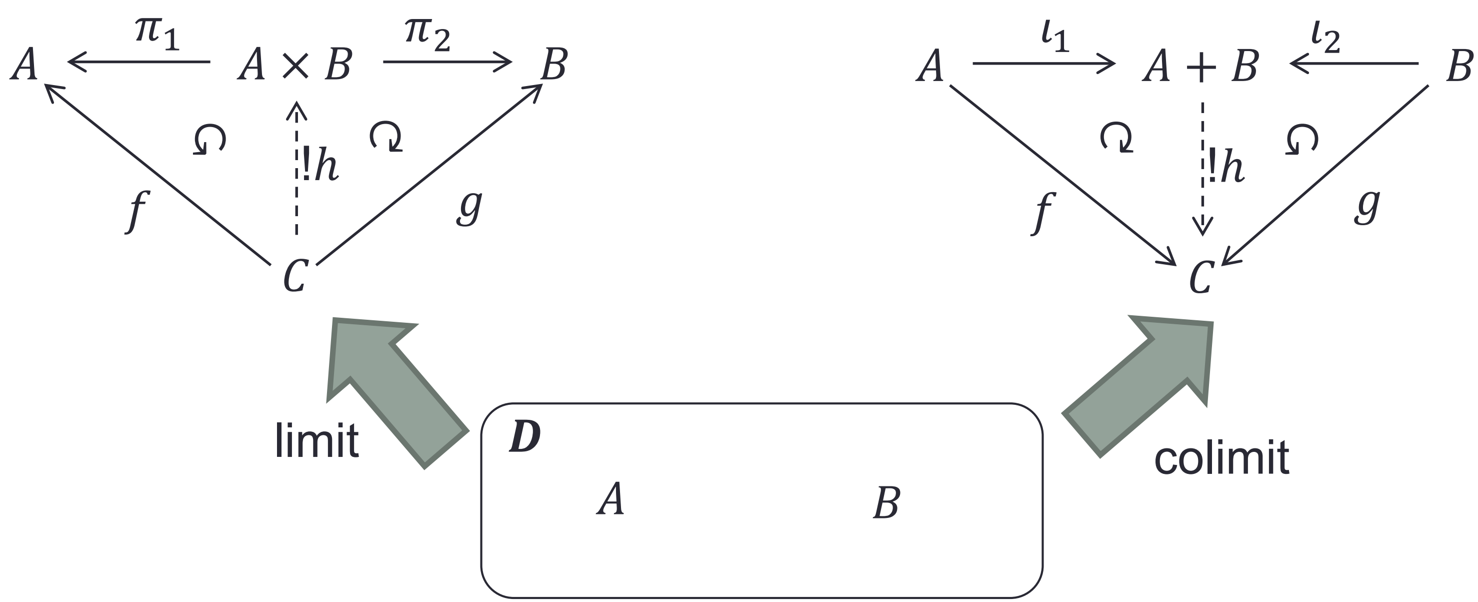


- Projective limit
  - The limit of the following diagram:

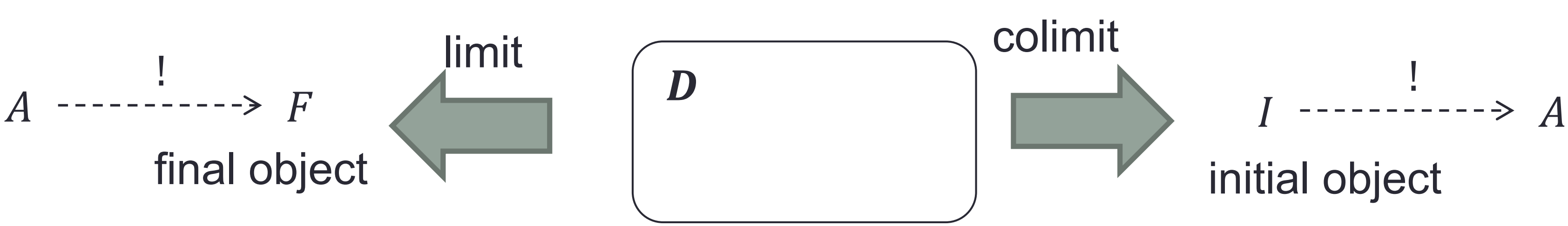


Everything is Limit and Colimit

- product and co-product

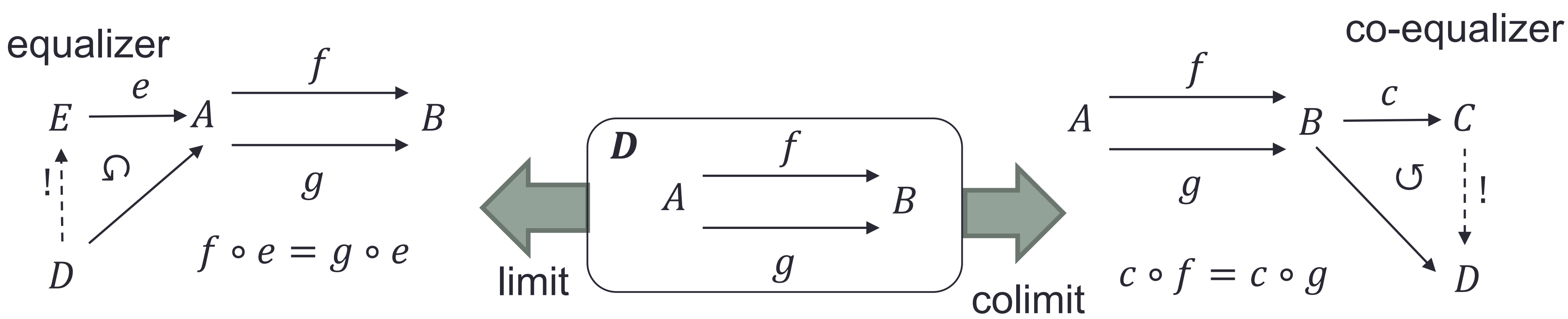


- final object and initial object

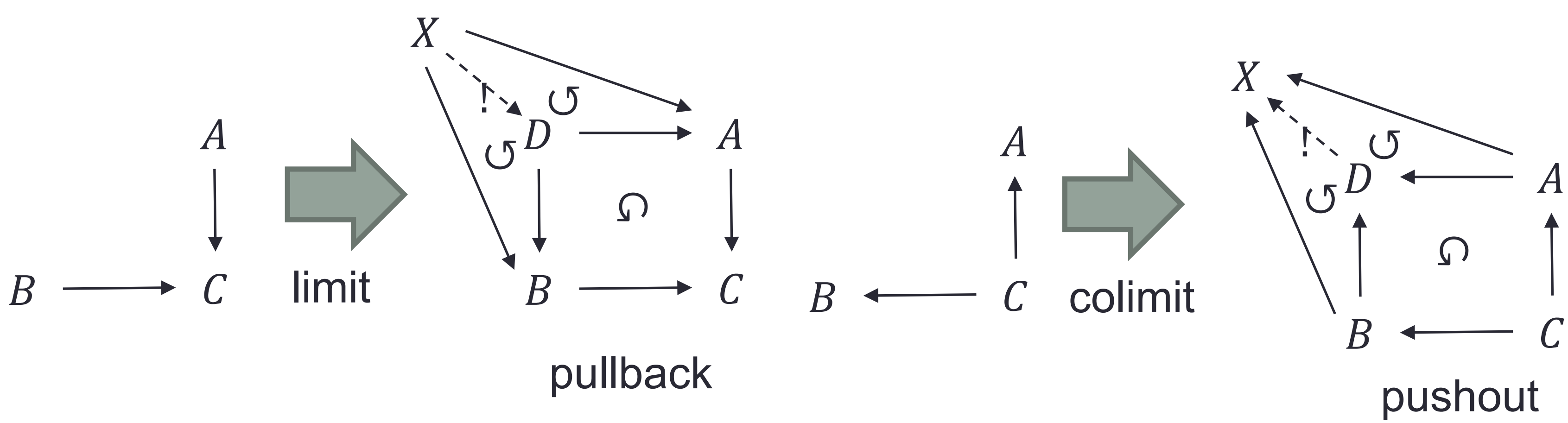


Other Limits and Colimits

- equalizer and co-equalizer



- pullback and pushout

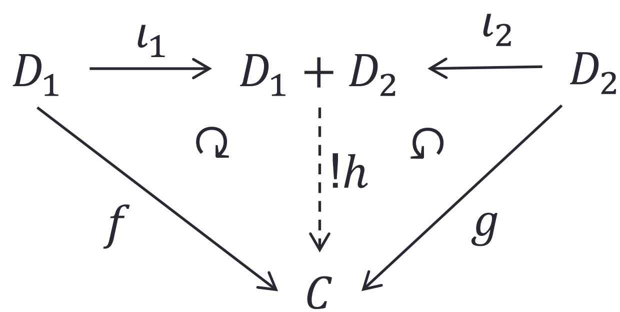


Summary

- Category Theory
  - Alternative foundation of Mathematics
- Category
  - Objects and Arrows
- Special objects
  - Initial and final objects
  - Product and co-product
- Limit and Colimit

Homework 11

- In CPO, we defined $D_1 + D_2$ as:
  - $D_1 + D_2 = \{\langle x, 1\rangle \mid x \in D_1\} \cup \{\langle y, 2\rangle \mid y \in D_2\} \cup \{\bot_{D_1+D_2}\}$
    - $\langle x, 1\rangle \sqsubseteq \langle x', 1\rangle \Leftrightarrow x \sqsubseteq x'$
    - $\langle y, 2\rangle \sqsubseteq \langle y', 2\rangle \Leftrightarrow y \sqsubseteq y'$
    - $\bot_{D_1+D_2} \sqsubseteq \langle x, 1\rangle$
    - $\bot_{D_1+D_2} \sqsubseteq \langle y, 2\rangle$
    - $\iota_1(x) = \langle x, 1\rangle$
    - $\iota_2(y) = \langle y, 1\rangle$



- Show that it is not co-product in the sense of category theory.
  - Hint: Create a counter example for $B_\bot + B_\bot$ such that h is not unique.