MATHEMATICS FOR INFORMATION SCIENCE

NO.5 TURING MACHINE AND COMPUTABILITY

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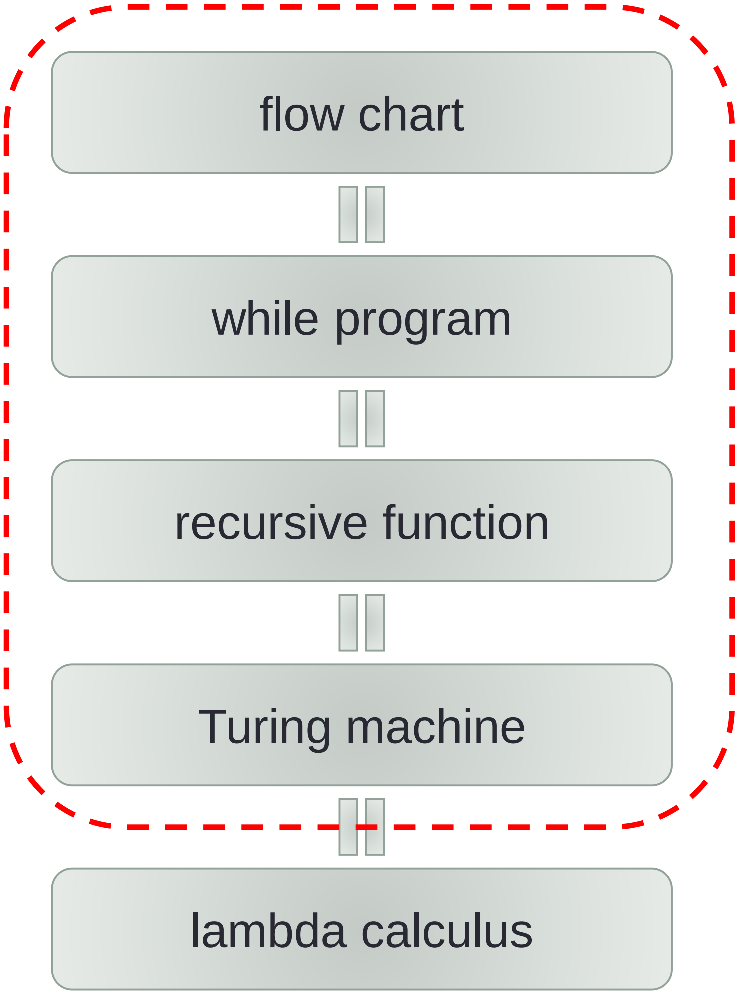
Slides URL

https://vu5.sfc.keio.ac.jp/slide/

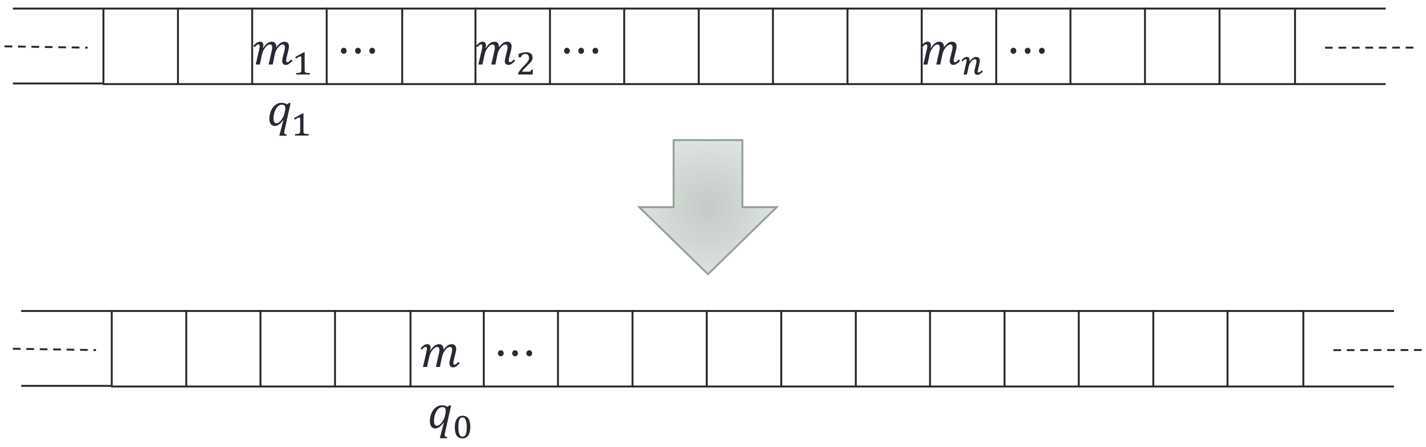
So far

- Computation
  - flow chart program
  - while program
  - recursive function
    - primitive recursive function
    - minimization operator
  - Turing machine



Computation

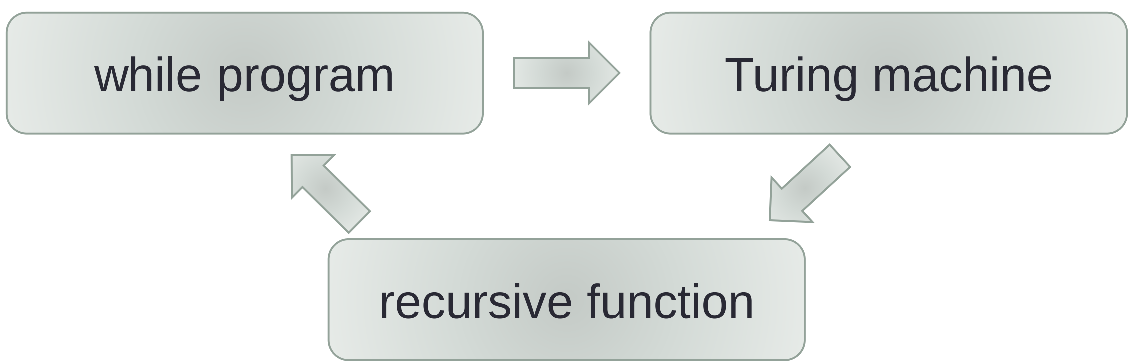
- A Turing machine M computes $f: N^n \to N$ when:
  - Place $m_1, m_2, \cdots, m_n$ on the tape with decimal numbers separated with a blank
  - Start M with the head at the leftmost number position.
  - When M terminates, the number at the head is the decimal number of $f(m_1, m_2, \cdots, m_n)$.



Computation and Program

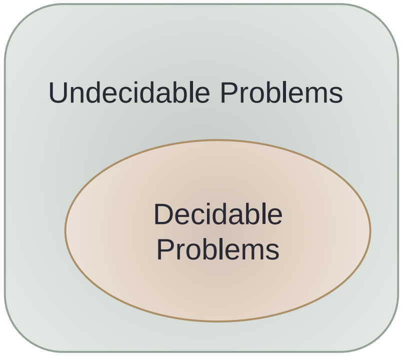
- A Turing machine may not terminate.
  - The function it computes is not total, but partial.

- **Theorem**
  - If a Turing machine can compute $f: N^n \rightarrow N$, it can be computed by a while program.
  - If $f: N^n \rightarrow N$ is a recursive function, there is a Turing machine which can compute the same function.



Decidable vs Undecidable Problems

- Decidable Problem
  - A problem for which a program can say yes or no.
  - The program needs to terminate.
  - The corresponding recursive function needs to be total.



- Undecidable Problem
  - A problem which is not decidable.
  - There might be a program which may say yes, but it does not termination if the answer is no.
  - The corresponding function is not recursive, or it is recursive but not total.

- **Halting Problem:**
  - Is there a program which tells whether a given program P for a given input $a_1, \dots, a_n$ will eventually terminate and return a value or will run forever?

Encoding Programs

- In order to make a program as an input to another program, we need to represent a program as a number (i.e. encoding)
- Encoding flow chart programs:
  - Boxes are connected by arrows
  - Put a number to each box
  - Each box is one of the following:
    - $\text{input}(x_1, x_2, \cdots, x_n)$
    - $x_i := m$
    - $x_i := x_j + x_k$
    - $x_i := x_j - x_k$
    - $x_i := x_j \times x_k$
    - $x_i := x_j \div x_k$
    - $\text{if}\left(x_i = x_j\right)$
    - $\text{output}(x_i)$

Encoding

- Let $x_1, \dots, x_n$ be input variables and $x_{n+1}, x_{n+2}, \dots, x_t$ be other variables.

- Let $A_1, A_2, \dots, A_l$ be boxes of program P where A_1 is the input box and A_l is the output box.

- Using Gödel number, encode each box as $\#A$:

A_a	$\#A_a$
$\text{input}(x_1, x_2, \cdots, x_n)$	$\langle 1, n, a' \rangle$
$x_i := m$	$\langle 2, i, m, a' \rangle$
$x_i := x_j + x_k$	$\langle 3, i, j, k, a' \rangle$
$x_i := x_j - x_k$	$\langle 4, i, j, k, a' \rangle$
$x_i := x_j \times x_k$	$\langle 5, i, j, k, a' \rangle$
$x_i := x_j \div x_k$	$\langle 6, i, j, k, a' \rangle$
$\text{if } (x_i = x_j)$	$\langle 7, i, j, a', a'' \rangle$
$\text{output}(x_i)$	$\langle 8, i \rangle$

- The program can be encoded as:
  - $\#P = \langle \#A_1, \#A_2, \dots, \#A_l \rangle$

Interpreter for P

Theorem:

- The following partial function $\mathrm{comp}_n : N^{n+1} \to N$ is computable.

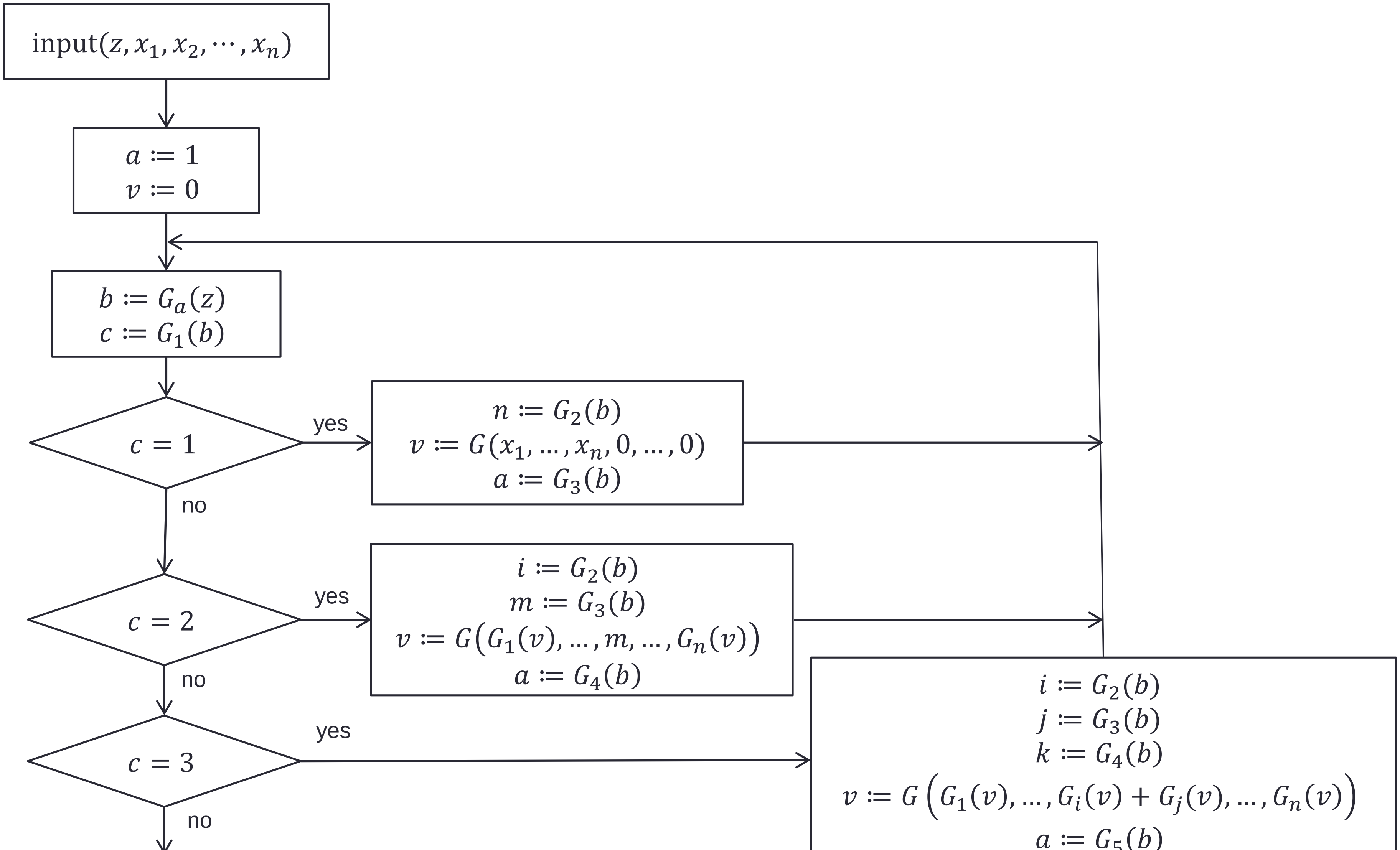
$$\mathrm{comp}_n(z, x_1, \dots, x_n) = \begin{cases} y & \text{when } z = \#P \text{ and } y = f_P(x_1, \dots x_n) \\ \text{undefined} & \text{otherwise} \end{cases}$$

where f_P is the recursive function for program P.

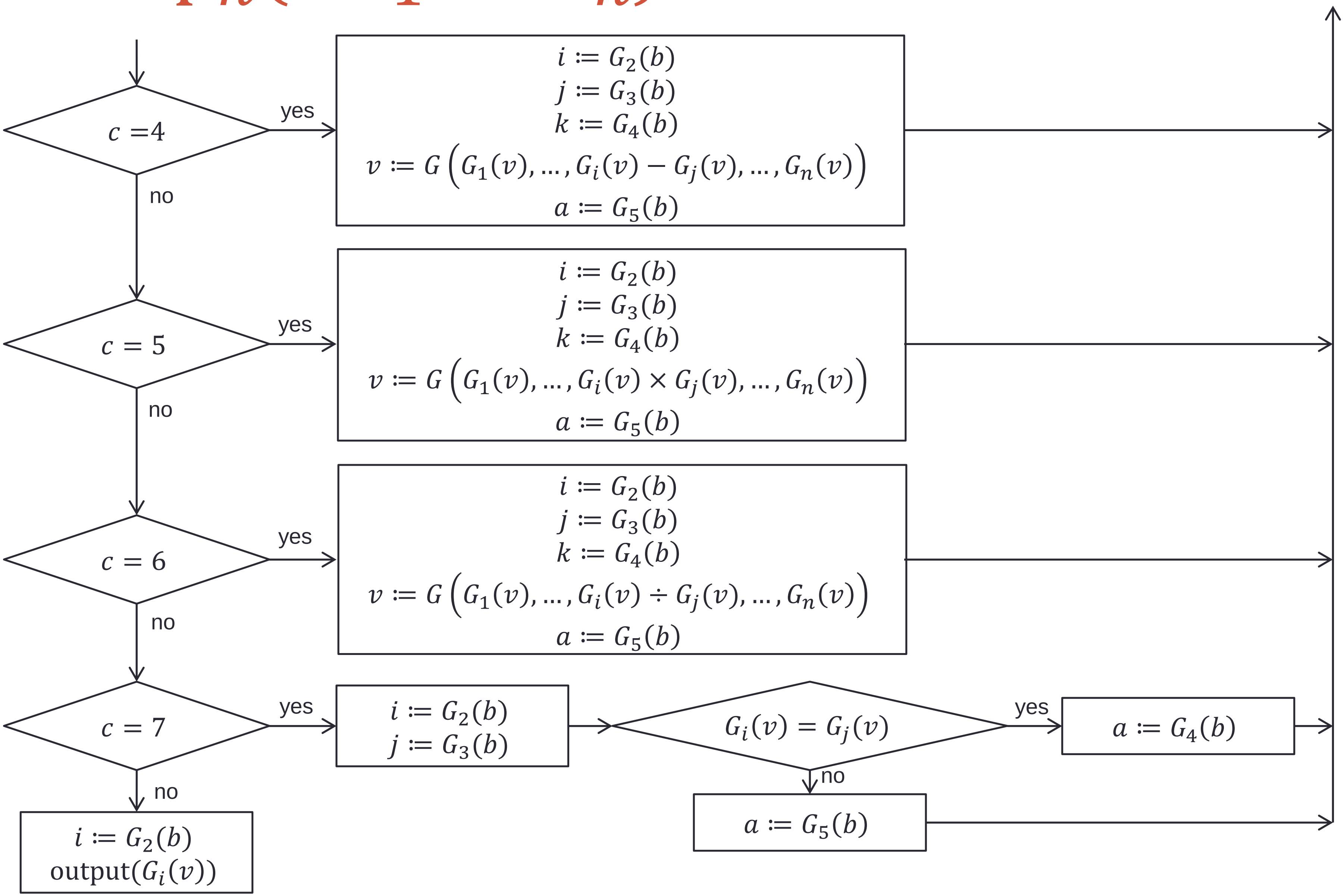
Proof:

- Write a program which computes comp_n by simulating the flow chart program represented by $\#P$.

$\mathrm{comp}_n(z, x_1, \ldots, x_n)$



$\mathrm{comp}_n(z, x_1, \ldots, x_n)$ cont.



Is comp Total?

Theorem: If $\mathrm{comp}_n: N^{n+1} \to N$ is extended to a total function

$$g: N^{n+1} \to N$$

g is not recursive.

Proof:

- Show the case for $n = 1$:
- Proof by contradiction and use Cantor's diagonal argument.
- Assume $\mathrm{comp}_1(z, x) = g(z, x)$ and $g: N^2 \to N$ is a total recursive function.
- Let $h(x) = g(x, x) + 1$. Then, h is also a total recursive function.
- There is a program which calculates h.
- Let c be the code.
- Then, from the definition of comp_1, $h(x) = \mathrm{comp}_1(c, x)$.
- Give h an input c.

$$h(c) = \mathrm{comp}_1(c, c) = g(c, c)$$

- This contradicts with $h(c) = g(c, c) + 1$.
- Therefore, a recursive total function g does not exist. (QED)

Recursive Predicate

Definition: Predicate $p: N^n \rightarrow \{T, F\}$ is a recursive predicate if its characteristic function $C_p: N^n \rightarrow N$ is recursive.

- C_p is total.
- p is decidable.

- If $p(x_1, \dots, x_n), q(x_1, \dots, x_n)$ and $r(x_1, \dots, x_n, y)$ are recursive, the following predicates are also recursive:
  - $p(x_1, \dots, x_n) \wedge q(x_1, \dots, x_n)$
  - $p(x_1, \dots, x_n) \vee q(x_1, \dots, x_n)$
  - $\neg p(x_1, \dots, x_n)$
  - $\forall z < y\big(r(x_1, \dots, x_n, z)\big)$
  - $\exists z < y\big(r(x_1, \dots, x_n, z)\big)$

Halting Problem is Undecidable

- Define predicate $\mathrm{halt}_\mathrm{n}(z, x_1, \ldots, x_n) : N^{n+1} \rightarrow \{T, F\}$ as follows:

$$\mathrm{halt}_n(z, x_1, \ldots, x_n) = \begin{cases} T & \text{when } \mathrm{comp}_n(z, x_1, \ldots, x_n) \text{ is defined} \\ F & \text{when } \mathrm{comp}_n(z, x_1, \ldots, x_n) \text{ is undefined} \end{cases}$$

Theorem: $\mathrm{halt}_\mathrm{n}(z, x_1, \ldots, x_n)$ is not recursive (i.e. undecidable).

Proof:

- If $\mathrm{halt}_n(z, x_1, \ldots, x_n)$ is a recursive predicate, its characteristic function C_{halt_n} is recursive and total. Then,

$$g(z, x_1, \ldots, x_n) = C_{\mathrm{halt}_n}(z, x_1, \ldots, x_n) \times \mathrm{comp}_n(z, x_1, \ldots, x_n)$$

is a total recursive function and this contradicts with the previous theorem. (QED)

Totality Problem is Undecidable

Theorem: For $n > 0$, there is no total recursive function $g: N^{n+1} \to N$ which satisfies the following:

$$\{ g(c, x_1, \dots, x_n): N^{n+1} \to N \mid c \in N \} = \{ f: N^n \to N \mid f \text{ is total and recursive} \}$$

- $\text{comp}_n(z, x_1, \dots, x_n): N^{n+1} \to N$ is the universal function for recursive functions (both partial and total), but there is no universal function for total recursive functions.

Proof:

- In the case for $n = 1$, if $g: N^2 \to N$ exists, $f(x) = g(x, x) + 1$ is a total recursive function.
- Let c be the code of f, $g(c, x) = f(x) = g(x, x) + 1$ and this contradicts when $x = c$.
- In the case for $n > 1$, the proof can be similar. (QED).

Corollary: $\text{total}_n(z) \equiv \forall x_1 \cdots \forall x_n \left(\text{halt}_n(z, x_1, \dots, x_n)\right)$ is not a recursive predicate, i.e. $\text{total}_n(z)$ is undecidable.

Proof: If C_{total_n} is the characteristic function of total_n,

$$g(z, x_1, \dots, x_n) = C_{\text{total}_n}(z) \times \text{comp}_n(z, x_1, \dots, x_n)$$

g is a total recursive function and this contradicts with previous theorem. (QED)

Undecidable Predicates

- $\mathrm{halt}_n(z, x_1, \dots, x_n)$
  - whether a give program z terminates for the input $x_1, \dots, x_n$ or not.
- $\mathrm{total}_n(z)$
  - whether a given program z always terminates or not.
- $\forall x_1 \cdots \forall x_n(\mathrm{comp}_n(z, x_1, \dots, x_n) = 0)$
  - whether a given program z always outputs 0 or not.
- $\exists x_1 \cdots \exists x_n(\mathrm{comp}_n(z, x_1, \dots, x_n) = 0)$
  - whether a given program z outputs 0 for some input or not.
- For z, the domain of $\mathrm{comp}_n(z, x_1, \dots, x_n)$ is finite.
  - whether a program z terminates for finite sets of input or not.
- For z, $\mathrm{comp}_n(z, x_1, \dots, x_n)$ is a constant function.
  - whether a program z outputs always the same number or not.
- For z and z', $\mathrm{comp}_n(z, x_1, \dots, x_n) = \mathrm{comp}_n(z', x_1, \dots, x_n)$
  - whether two programs z and z' are same or not.

s-m-n Theorem

Theorem: For natural numbers m and n, there is a primitive recursive function $S_{m,n}: N^{m+1} \to N$ which satisfies:

$$\mathrm{comp}_{m+n}(z, x_1, \dots, x_n, y_1, \dots, y_m) = \mathrm{comp}_n\left(S_{m,n}(z, y_1, \dots, y_m), x_1, \dots, x_n\right)$$

Proof: $S_{m,n}(z, u_1, \dots, u_m)$ is the function which converts

$$z = \langle \#A_1, \#A_2, \dots, \#A_l \rangle$$

into

$$z' = \left\langle \#\left(\mathrm{input}(x_1, \dots, x_n)\right), \#(y_1 := u_1), \dots, \#(y_m := u_m), \#A_2, \dots, \#A_l \right\rangle$$

which represents:

- $\mathrm{input}(x_1, \dots, x_n)$
- $y_1 := u_1$
- …
- $y_m := u_m$
- A_2
- …
- A_l

The conversion function can be written as a primitive recursive function. (QED)

Recursion Theorem

Theorem: For n and a total recursive function $f: N \to N$, there is a natural number c which makes the following equation true:

$$\mathrm{comp}_n(f(c), x_1, \dots, x_n) = \mathrm{comp}_n(c, x_1, \dots, x_n)$$

Proof:

- Let a be the code for $\mathrm{comp}_{n+1}(y, x_1, \dots, x_n, y)$.
- $\mathrm{comp}_{n+1}(y, x_1, \dots, x_n, y) = \mathrm{comp}_{n+1}(a, x_1, \dots, x_n, y) = \mathrm{comp}_n\left(S_{1,n}(a, y), x_1, \dots, x_n\right)$
- Let b be the code for $\mathrm{comp}_n\left(f\left(S_{1,n}(a, y)\right), x_1, \dots, x_n\right)$
- $\mathrm{comp}_n\left(f\left(S_{1,n}(a, y)\right), x_1, \dots, x_n\right) = \mathrm{comp}_{n+1}(b, x_1, \dots, x_n, y)$
- $\mathrm{comp}_n\left(f\left(S_{1,n}(a, b)\right), x_1, \dots, x_n\right) = \mathrm{comp}_{n+1}(b, x_1, \dots, x_n, b) = \mathrm{comp}_n\left(S_{1,n}(a, b), x_1, \dots, x_n\right)$
- $c = S_{1,n}(a, b)$

(QED)

Rice Theorem

Theorem: Let n be a natural number. If a predicate $p(z)$ satisfies the following two conditions, $p(z)$ is not recursive (i.e. $p(z)$ is undecidable).

(1) $\forall c \forall c' \left(\forall x_1 \cdots \forall x_n \left(\mathrm{comp}_n(c, x_1, \dots, x_n) = \mathrm{comp}_n(c', x_1, \dots, x_n)\right) \Rightarrow p(c) \equiv p(c')\right)$

(2) $\exists c \exists c' \left(p(c) \wedge \neg p(c')\right)$

- (1) means that $p(z)$ truth value is the same for the same program.
- (2) means that $p(z)$ is true for certain number and is false for a different number.

Proof:

- If p is a recursive predicate, let C_p be its characteristic function.
- Let define $f: N \to N$ using c and c' which satisfy (2) as follows:

$$f(z) = C_p(z) \times c' + \left(1 - C_p(z)\right) \times c$$

- From the definition, $p\left(f(z)\right) \not\equiv p(z)$
- Since f is a total recursive function, using recursion theory there exists c'' which makes $\mathrm{comp}_n(f(c''), x_1, \dots, x_n) = \mathrm{comp}_n(c'', x_1, \dots, x_n)$.
- From (1), $p(f(c'')) \equiv p(c'')$, but this contradicts. (QED)

- Using this theorem, we can prove many predicates are undecidable.
  - $p(z) \equiv$ "$\mathrm{comp}_n(z, x_1, \dots, x_n)$ is a constant function."
  - $p(z)$ is same for the same program, and there are a constant program and a not-constant one.

Post Correspondence Problem

Problem: Given a finite set of string pairs,

$$\{(s_1, t_1), (s_2, t_2), \dots, (s_n, t_n)\}$$

using string concatenation, determine whether there is a number sequence $i_1, \dots, i_m$ which makes the following equality hold:

$$s_{i_1} s_{i_2} \dots s_{i_m} = t_{i_1} t_{i_2} \dots t_{i_m}$$

Example:

- $\{(e, abcde), (ababc, ab), (d, cab)\}$

$a\ b\ a\ b\ c$	$a\ b\ a\ b\ c$	d	e
$a\ b$	$a\ b$	$c\ a\ b$	$a\ b\ c\ d\ e$

- This problem (post correspondence problem) is undecidable.
  - There is no program which gives a solution to the problem or none if there is no solution.

Summary

- Decidable Problem
  - A problem for which a program can say yes or no.
- Undecidable Problem
  - A problem which is not decidable.
- Undecidable predicates:
  - Halting problem
  - Totality problem
  - Post correspondence problem