MATHEMATICS FOR INFORMATION SCIENCE

NO.9 CPO AND DATA TYPE

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Slides URL

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Complete Partial Ordered Set

- $(D, \sqsubseteq)$ is a partially ordered set when
  - Reflective: $x \sqsubseteq x$
  - Transitive: $x \sqsubseteq y$ and $y \sqsubseteq z$ then $x \sqsubseteq z$
  - Antisymmetric: $x \sqsubseteq y$ and $y \sqsubseteq x$ then $x = y$
- $(D, \sqsubseteq)$ is a complete partial ordered set (cpo) when
  - $(D, \sqsubseteq)$ is a partially ordered set
  - it has the small element $\bot$
    - for any x, $\bot \sqsubseteq x$
  - for $x_1 \sqsubseteq x_2 \sqsubseteq x_3 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$, there is the least upper bound $\sqcup_{i=1}^{\infty} x_i$

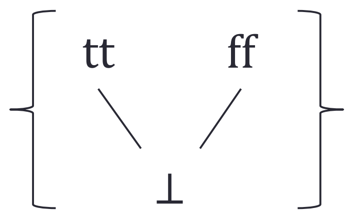
a set of information	$\subseteq$	a complete partial ordered set

Flat Domain

- For a set S, add the smallest element $\perp$
  - $S_\perp = S \cup \{\perp\}$
  - for any $x \in S$, $\perp \sqsubseteq x$
- $S_\perp$ is CPO.

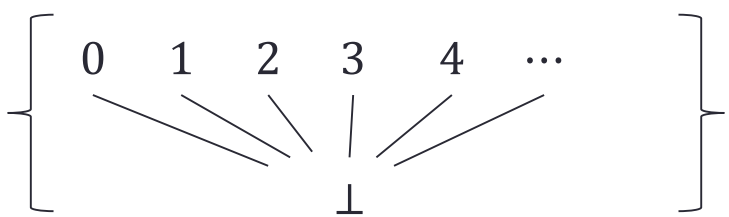
- Example:
  - For Boolean set $B = \{\mathrm{tt}, \mathrm{ff}\}$, its flat domain is:

$B_\perp =$



  - For Natural Number $N = \{0, 1, 2, 3, 4, \cdots\}$, its flat domain is:

$N_\perp =$



Product

- $D_1 \times D_2$ is the product of CPO D_1 and D_2 when:
  - $D_1 \times D_2 = \{\langle x, y\rangle \mid x \in D_1, y \in D_2\}$
  - $\langle x, y\rangle \sqsubseteq \langle x', y'\rangle$ if $x \sqsubseteq x'$ and $y \sqsubseteq y'$

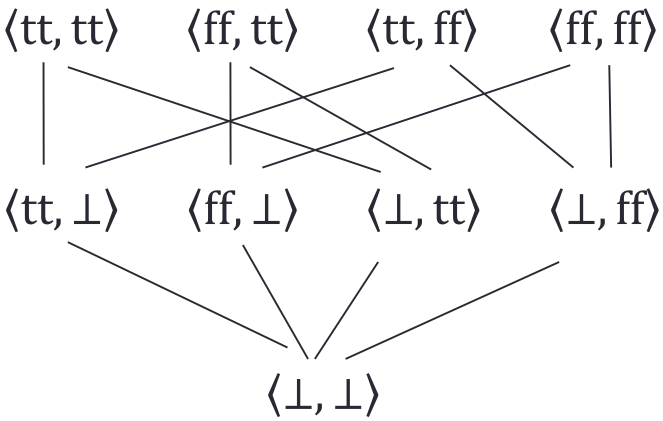
- $D_1 \times D_2$ is also a CPO.
  - The above $\sqsubseteq$ is a partial order.
    - $\langle x, y\rangle \sqsubseteq \langle x, y\rangle$
    - If $\langle x, y\rangle \sqsubseteq \langle x', y'\rangle$ and $\langle x', y'\rangle \sqsubseteq \langle x'', y''\rangle$, then $\langle x, y\rangle \sqsubseteq \langle x'', y''\rangle$
    - If $\langle x, y\rangle \sqsubseteq \langle x', y'\rangle$ and $\langle x', y'\rangle \sqsubseteq \langle x, y\rangle$, then $\langle x, y\rangle = \langle x', y'\rangle$
  - $\bot_{\mathrm{D}_1 \times D_2} = \left\langle \bot_{D_1}, \bot_{\mathrm{D}_2} \right\rangle$
  - For $\langle x_1, y_1\rangle \sqsubseteq \langle x_2, y_2\rangle \sqsubseteq \langle x_3, y_3\rangle \sqsubseteq \cdots \sqsubseteq \langle x_i, y_i\rangle \sqsubseteq \cdots$,

$$\sqcup_{i=1}^{\infty} \langle x_i, y_i\rangle = \langle \sqcup_{i=1}^{\infty} x_i , \sqcup_{i=1}^{\infty} y_i\rangle$$

Example of Product

- $B_\perp \times B_\perp$

Hasse Diagram



- $(B_\perp \times B_\perp) \times B_\perp$

Boolean Functions

- $\text{not}: B \rightarrow B$

	ff	tt
not	tt	ff

- $\text{not}_{\perp}: B_{\perp} \rightarrow B_{\perp}$

	⊥	ff	tt
$\text{not}_{\perp}$	⊥	tt	ff

- $\text{and}: B \times B \rightarrow B$

and	ff	tt
ff	ff	ff
tt	ff	tt

- $\text{or}: B \times B \rightarrow B$

or	ff	tt
ff	ff	tt
tt	tt	tt

Extension of or

- $\text{or}_\perp : B_\perp \times B_\perp \to B_\perp$

$\text{or}_\perp$	⊥	ff	tt
⊥	⊥	⊥	⊥
ff	⊥	ff	tt
tt	⊥	tt	tt

- $\text{or}_R : B_\perp \times B_\perp \to B_\perp$

or_R	⊥	ff	tt
⊥	⊥		
ff		ff	tt
tt		tt	tt

- $\text{or}_L : B_\perp \times B_\perp \to B_\perp$

or_L	⊥	ff	tt
⊥	⊥		
ff		ff	tt
tt		tt	tt

- $\text{or}_P : B_\perp \times B_\perp \to B_\perp$

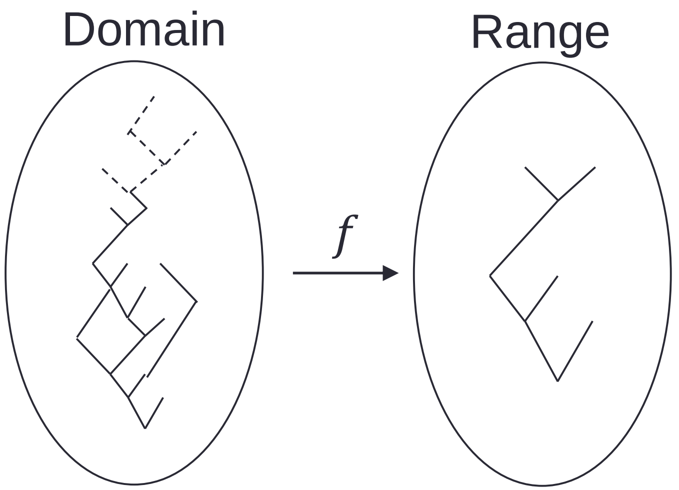
or_P	⊥	ff	tt
⊥	⊥		
ff		ff	tt
tt		tt	tt

Conditional Function

- $\text{cond}: B_{\perp} \times N_{\perp} \times N_{\perp} \to N_{\perp}$
  - $\text{cond}(\perp, x, y) = \perp$
  - $\text{cond}(\text{tt}, x, y) = x$
  - $\text{cond}(\text{ff}, x, y) = y$

- cond is continuous.
  - It is continuous because it is monotonic (using the next property).
  - It is not strict with respect to the second and third arguments.

Property of Continuous Functions

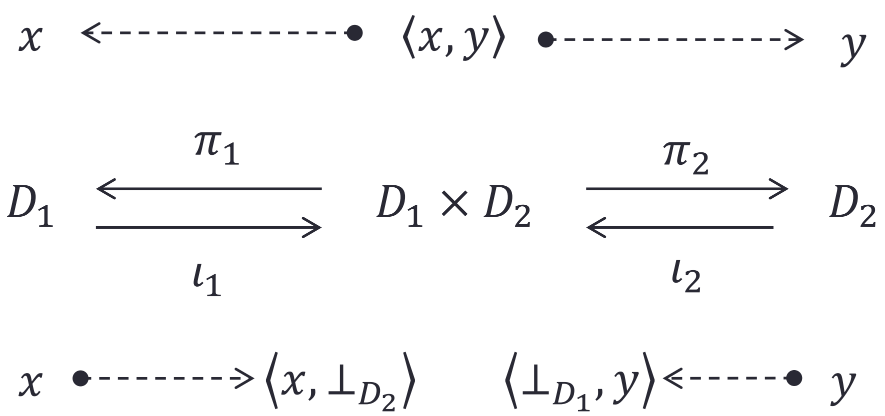


- When the rank of its range is finite,
  - a function is continuous, if it is monotonic.

- $f: D_1 \times D_2 \rightarrow D_3$ is continuous, if and only if
  - for any value $b \in D_2$, $f_b: D_1 \rightarrow D_3$ is continuous
    $$f_b(x) = f(x, b)$$
  - for any value $a \in D_1$, $f^a: D_2 \rightarrow D_3$ is continuous
    $$f^a(y) = f(a, y)$$

Functions Associated with Products

- Projection:
  - $\pi_1 : D_1 \times D_2 \to D_1$
    - $\pi_1(\langle x, y\rangle) = x$
  - $\pi_2 : D_1 \times D_2 \to D_2$
    - $\pi_2(\langle x, y\rangle) = y$

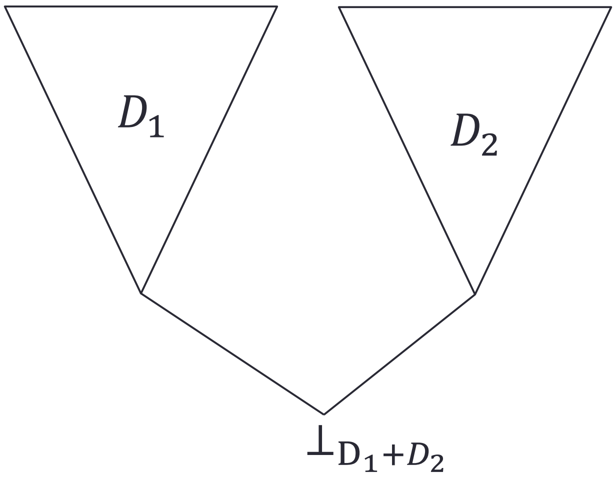


- Injection (embedding):
  - $\iota_1 : D_1 \to D_1 \times D_2$
    - $\iota_1(x) = \langle x, \bot_{D_2}\rangle$
  - $\iota_2 : D_2 \to D_1 \times D_2$
    - $\iota_2(y) = \langle \bot_{D_1}, y\rangle$

- $\iota_1 \circ \pi_1(z) \sqsubseteq z$
- $\iota_2 \circ \pi_2(z) \sqsubseteq z$
- $\pi_1 \circ \iota_1(x) = x$
- $\pi_2 \circ \iota_1(x) = \bot_{D_2}$
- $\pi_1 \circ \iota_2(y) = \bot_{D_1}$
- $\pi_2 \circ \iota_2(y) = y$

Co-Product (Sum, Disjoint Union)

- $D_1 + D_2$ is the co-product of CPO D_1 and D_2 when:
  - $D_1 + D_2 = \{\langle x, 1\rangle \mid x \in D_1\} \cup \{\langle y, 2\rangle \mid y \in D_2\} \cup \{\bot_{D_1+D_2}\}$
    - $\langle x, 1\rangle \sqsubseteq \langle x', 1\rangle \Leftrightarrow x \sqsubseteq x'$
    - $\langle y, 2\rangle \sqsubseteq \langle y', 2\rangle \Leftrightarrow y \sqsubseteq y'$
    - $\bot_{D_1+D_2} \sqsubseteq \langle x, 1\rangle$
    - $\bot_{D_1+D_2} \sqsubseteq \langle y, 2\rangle$



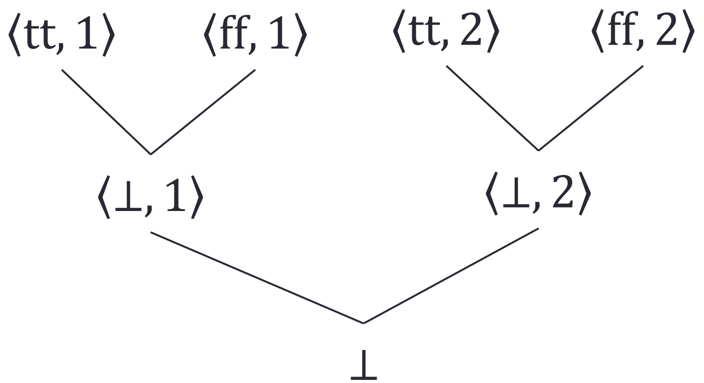
- $D_1 + D_2$ is also a CPO.
  - The above $\sqsubseteq$ is a partial order.
  - $\bot_{D_1+D_2}$ is the bottom element.
  - Any ascending sequence has the least upper bound.

Example of Co-Product

- $B_\perp + B_\perp$

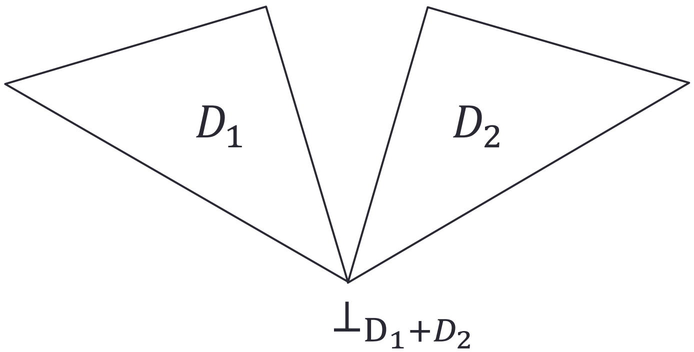
Hasse Diagram

- $(B_\perp + B_\perp) + B_\perp$



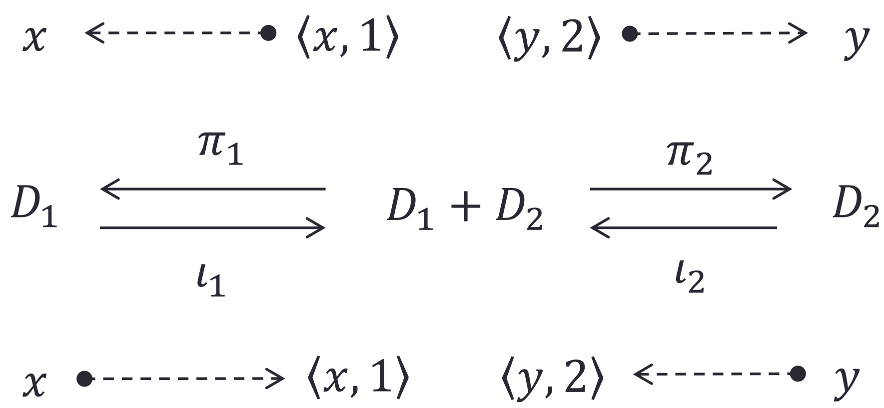
Smashed Co-Product (Smashed Sum)

- $D_1 \oplus D_2$ is the smashed co-product of CPO D_1 and D_2 when:
  - $D_1 \oplus D_2 = \{\langle x,1\rangle \mid x \in D_1,\ x \neq \bot_{D_1}\} \cup \{\langle y,2\rangle \mid y \in D_2,\ y \neq \bot_{D_2}\} \cup \{\bot_{D_1+D_2}\}$
    - $\langle x,1\rangle \sqsubseteq \langle x',1\rangle \Leftrightarrow x \sqsubseteq x'$
    - $\langle y,2\rangle \sqsubseteq \langle y',2\rangle \Leftrightarrow y \sqsubseteq y'$
    - $\bot_{D_1 \oplus D_2} \sqsubseteq \langle x,1\rangle$
    - $\bot_{D_1 \oplus D_2} \sqsubseteq \langle y,2\rangle$



Functions Associated with Co-Products

- Projection:
  - $\pi_1 : D_1 + D_2 \rightarrow D_1$
    - $\pi_1(\langle x, 1\rangle) = x$
    - $\pi_1(\langle y, 2\rangle) = \perp_{D_1}$
    - $\pi_1(\perp_{D_1+D_2}) = \perp_{D_1}$
  - $\pi_2 : D_1 + D_2 \rightarrow D_2$
    - $\pi_2(\langle x, 1\rangle) = \perp_{D_2}$
    - $\pi_2(\langle y, 2\rangle) = y$
    - $\pi_2(\perp_{D_1+D_2}) = \perp_{D_2}$



- Injection (embedding):
  - $\iota_1 : D_1 \rightarrow D_1 + D_2$
    - $\iota_1(x) = \langle x, 1\rangle$
    - $\iota_1(\perp_{D_1}) = \perp_{D_1+D_2}$
  - $\iota_2 : D_2 \rightarrow D_1 + D_2$
    - $\iota_2(y) = \langle y, 2\rangle$
    - $\iota_2(\perp_{D_2}) = \perp_{D_1+D_2}$

- $\iota_1 \circ \pi_1(z) \sqsubseteq z$
- $\iota_2 \circ \pi_2(z) \sqsubseteq z$
- $\pi_1 \circ \iota_1(x) = x$
- $\pi_2 \circ \iota_1(x) = \perp_{D_2}$
- $\pi_1 \circ \iota_2(y) = \perp_{D_1}$
- $\pi_2 \circ \iota_2(y) = y$

Function Space

- $[D_1 \to D_2]$ is the function space from CPO D_1 to D_2 when:
  - $[D_1 \to D_2] = \{f : D_1 \to D_2 \mid f \text{ is continuous}\}$
  - $f \sqsubseteq f'$ if for any $x \in D_1$, $f(x) \sqsubseteq f'(x)$

- $[D_1 \to D_2]$ is also a CPO.
  - The above $\sqsubseteq$ is a partial order.
  - $\bot_{[D_1 \to D_2]}(x) = \bot_{D_2}$
  - For $f_1 \sqsubseteq f_2 \sqsubseteq f_3 \sqsubseteq \cdots \sqsubseteq f_i \sqsubseteq \cdots$,

$$(\sqcup_{i=1}^{\infty} f_i)(x) = \sqcup_{i=1}^{\infty} f_i(x)$$

- Projection and Injection:
  - $\pi : [D_1 \to D_2] \to D_2$
  - $\pi(f) = f(\bot)$
  - $\iota : D_2 \to [D_1 \to D_2]$
  - $\iota(y)(x) = y$

- $\iota \circ \pi(f) \sqsubseteq f$
- $\pi \circ \iota(y) = y$

Projection and Embedding

- $D_1 \lhd D_2$
  - $\pi : D_2 \to D_1$ Projection
  - $\iota : D_1 \to D_2$ Embedding
  - $\iota \circ \pi(y) \sqsubseteq y$ $\iota \circ \pi \sqsubseteq id_{D_2}$
  - $\pi \circ \iota(x) \sqsupseteq x$ $\pi \circ \iota \sqsupseteq id_{D_1}$

- Product: $D_1 \times D_2$
  - $D_1 \lhd D_1 \times D_2$
  - $D_2 \lhd D_1 \times D_2$

- Sum: $D_1 + D_2$
  - $D_1 \lhd D_1 + D_2$
  - $D_2 \lhd D_1 + D_2$

- Function Space: $[D_1 \to D_2]$
  - $D_2 \lhd [D_1 \to D_2]$

Summary

- Domain construction
  - Flat domain
  - Product
  - Co-product
  - Function space

Homework

- Write the Hasse diagram for $(B_\perp + B_\perp) \times B_\perp$
  - There are 7 elements in $B_\perp + B_\perp$ and 3 elements in $B_\perp$, therefore there should be $7 \times 3 = 21$ elements in $(B_\perp + B_\perp) \times B_\perp$.
    - e.g. $<< \mathrm{tt}, 1 >, \mathrm{ff} >$, $<\perp, \mathrm{tt} >$, $\cdots$
  - Write them all and connect $\sqsubseteq$ relation between elements.
    - Place larger elements on top of smaller elements.
    - Write non-trivial relations, omitting reflective and transitive relations.