

MATHEMATICS FOR INFORMATION SCIENCE
NO.11 INTRODUCTION TO CATEGORY THEORY

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Slides URL

<https://vu5.sfc.keio.ac.jp/slide/>

Set Theory

- Set Theory
 - Foundation of Modern Mathematics
 - a set \equiv a collection of elements with some property
 - \emptyset
 - $A \cup B$
 - $A \cap B$
 - A^c
 - $\{x \in A \mid \text{logical formula about } x\}$
 - Description of elements is important: i.e. when $x \in A$
- Limit of set theory
 - Russell's Paradox
 - The collection of all sets is not a set.
 - $R = \{x \mid x \notin x\}$
 - $R \in R$ or $R \notin R$?

Category Theory

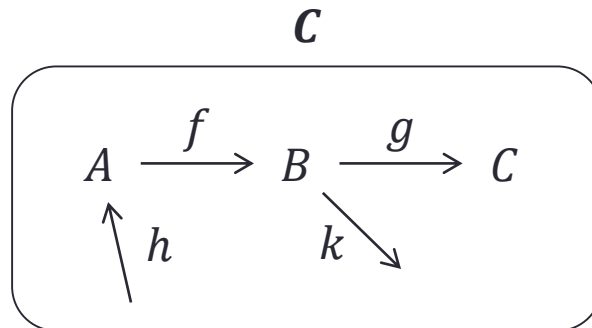
- Alternative foundation of Mathematics
- Some call it Abstract Nonsense.
- Describe things with relationship with others.

Set Theory	Category Theory
$x \in A$	$A \rightarrow B$
description of inside	description from outside
contents	actions

- Unify many concepts in one.
- Can see symmetry easily.

Category

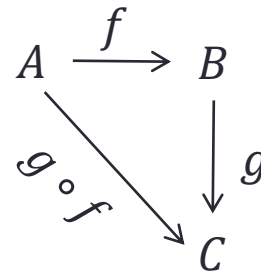
- A **category** \mathcal{C} consists of:
 - A collection of **objects**: $\mathcal{C} = \{A, B, C, \dots\}$
 - For objects A and B , a collection of **arrows** (or **morphisms**): $\text{hom}_{\mathcal{C}}(A, B) = \{f, g, h, \dots\}$
 - If $f \in \text{hom}_{\mathcal{C}}(A, B)$, we may write it as: $f: A \rightarrow B$
 - A is the **domain** of f
 - B is the **codomain** (or **range**) of f



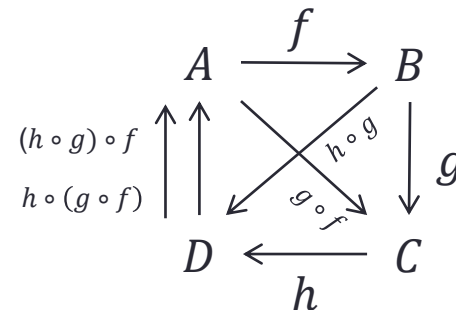
Category (cont)

- A category \mathcal{C} must satisfy the following properties:

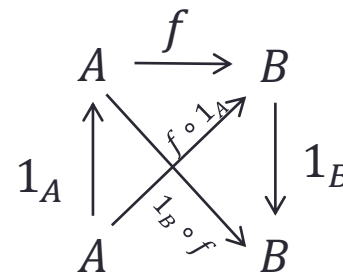
- For $f: A \rightarrow B$ and $g: B \rightarrow C$,
 $g \circ f: A \rightarrow C$



- For $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$,
 $h \circ (g \circ f) = (h \circ g) \circ f$



- For each object A , there exists an identity arrow $1_A: A \rightarrow A$ and for $f: A \rightarrow B$,
 $f \circ 1_A = f$ and $1_B \circ f = f$



Example of Category

- **Set**: the category of sets
 - objects: sets
 - arrows: functions
 - \circ is the function composition
 - 1_A is the identity function of A
- **Grp**: the category of groups
 - objects: groups $(G, \cdot, e, {}^{-1})$
 - arrows: homomorphisms
 - \circ is the function composition
 - 1_A is the identity function of A
- **CPO**: the category of complete partial ordered sets
 - objects: CPO
 - arrows: continuous functions
 - \circ is the function composition.
 - 1_D is the identity function of D

$(G, \cdot, e, {}^{-1})$: group

- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $x \cdot e = e \cdot x = x$
- $x \cdot x^{-1} = x^{-1} \cdot x = e$

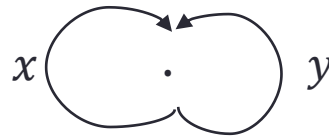
homomorphism: $f: G \rightarrow H$

- $f(x \cdot y) = f(x) \cdot f(y)$

Example of Category (cont.)

- Monoid (M, \cdot, e) as a category

- object: only one object
- arrows: M (i.e. elements in M)
- \circ is \cdot
- 1 is e

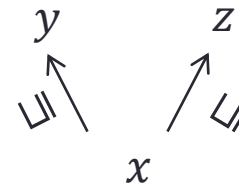


(M, \cdot, e) : monoid

- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $x \cdot e = e \cdot x = x$

- Partially ordered set (D, \sqsubseteq) as a category

- objects: D (i.e. elements in D)
- arrows: \sqsubseteq (i.e. at most one arrow from $x \rightarrow y$)
- \circ is ``if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$ ``
- 1_x is `` $x \sqsubseteq x$ ``

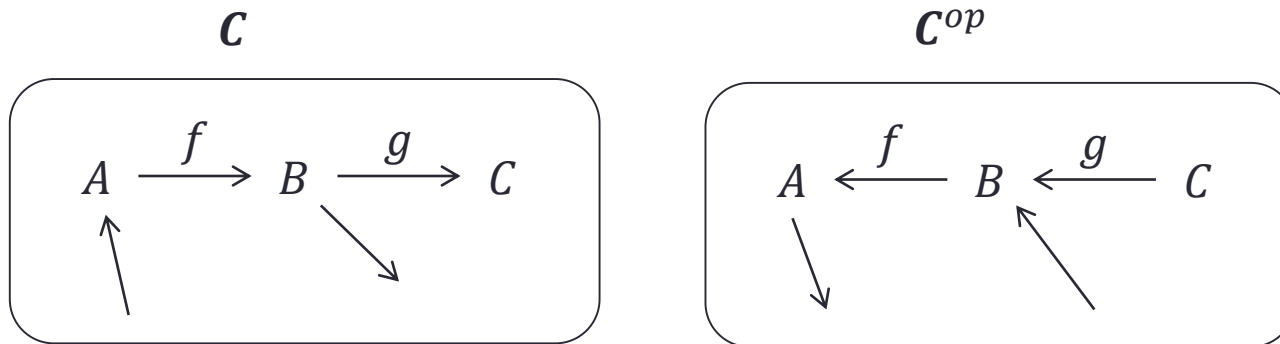


(D, \sqsubseteq) : partially ordered set

- $x \sqsubseteq x$
- if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$
- if $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$

Dual Category

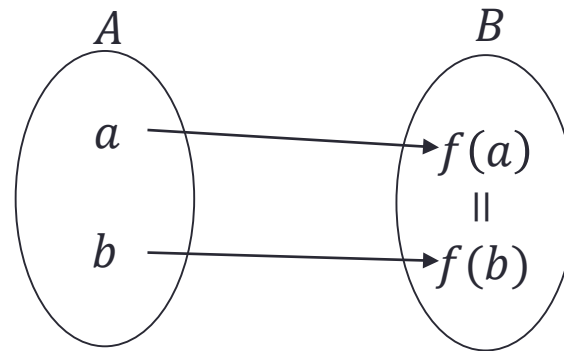
- **Dual Category** \mathcal{C}^{op} of category \mathcal{C}
 - \mathcal{C}^{op} objects = \mathcal{C} objects
 - \mathcal{C}^{op} arrows: $\text{hom}_{\mathcal{C}^{op}}(A, B) = \text{hom}_{\mathcal{C}}(B, A)$
 - Reverse the direction of arrows.



- Any property which is true in category \mathcal{C} is also true in its dual category \mathcal{C}^{op} .
 - $(\mathcal{C}^{op})^{op} = \mathcal{C}$

Mono Morphism

- One-to-one function
 - A function $f: A \rightarrow B$ is one-to-one
 $\Leftrightarrow a = b$ if $f(a) = f(b)$



- **Mono** morphism
 - $f: A \rightarrow B$ is mono-morphism
 \Leftrightarrow for any object D and any arrows $g: D \rightarrow A$ and $h: D \rightarrow A$, if $f \circ g = f \circ h$, then $g = h$.

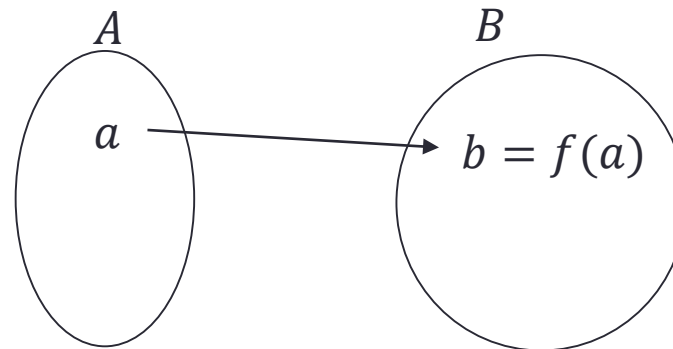
$$D \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{g} \end{array} A \xrightarrow{f} B$$

Epi Morphism

- Onto function

- A function $f: A \rightarrow B$ is onto

\Leftrightarrow for any $b \in B$, there exists $a \in A$ such that $f(a) = b$.



- **Epi** morphism

- $f: A \rightarrow B$ is epi-morphism

\Leftrightarrow for any object D and any arrows $g: B \rightarrow D$ and $h: B \rightarrow D$, if $g \circ f = h \circ f$, then $g = h$.

$$A \xrightarrow{f} B \begin{matrix} \xrightarrow{h} \\ \xrightarrow{g} \end{matrix} D$$

Mono and Epi

- Mono

- $f: A \rightarrow B$ is mono-morphism

\Leftrightarrow for any object D and any arrows $g: D \rightarrow A$ and $h: D \rightarrow A$, if $f \circ g = f \circ h$, then $g = h$.

$$D \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{g} \end{array} A \xrightarrow{f} B$$

Mono in \mathcal{C}^{op} is epi in \mathcal{C} .

- Epi

- $f: A \rightarrow B$ is epi-morphism

\Leftrightarrow for any object D and any arrows $g: B \rightarrow D$ and $h: B \rightarrow D$, if $g \circ f = h \circ f$, then $g = h$.

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{g} \end{array} D$$

Epi in \mathcal{C}^{op} is mono in \mathcal{C} .

Isomorphic

- **Isomorphic** (iso)

- Object A and B is are isomorphic

\Leftrightarrow there are $f: A \rightarrow B$ and $g: B \rightarrow A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

$$A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} B$$

Isomorphic objects play the same role in \mathcal{C} .

Initial and Final Objects

- **Initial** object I

- for any object A , there is a unique arrow from I to A .

$$I \overset{!}{\dashrightarrow} A$$

- **Final** object F

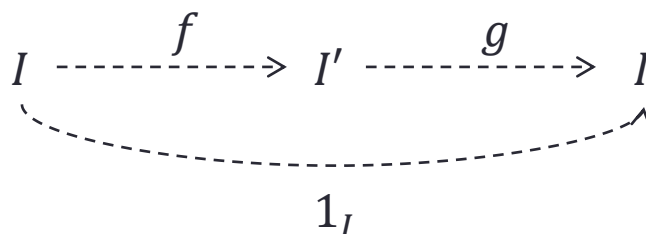
- for any object A , there is a unique arrow from A to F .

$$A \overset{!}{\dashrightarrow} F$$

	Initial Object	Final Object
<i>Set</i>		
<i>Grp</i>		
<i>CPO</i>		
Partially Ordered Set		

Uniqueness of Initial Object

- **Theorem:** The initial object, if it exists, is unique up to isomorphism.
- **Proof:** Assume there are two initial objects I and I' .
 - Since I is an initial object, there is a unique arrow f from I to I' .
 - Since I' is an initial object, there is a unique arrow g from I' to I .
 - $g \circ f$ is an arrow from I to I .
 - Since I is an initial object, it should be the unique arrow 1_I .
 - $g \circ f = 1_I$

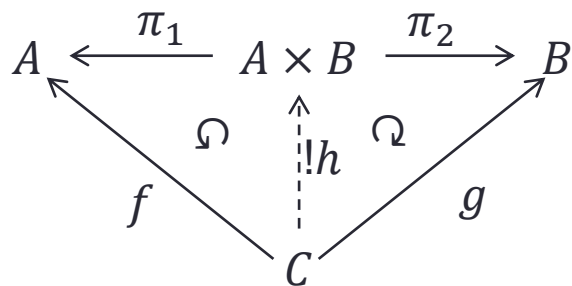


- Similarly, $f \circ g = 1_{I'}$.
- Therefore I and I' are isomorphic. QED
- **Dual Theorem:** The final object, if it exists, is unique up to isomorphism.

Product and Co-Product

- $A \times B$ is the **product** of A and $B \iff$
 - There are two arrows $\pi_1: A \times B \rightarrow A$ and $\pi_2: A \times B \rightarrow B$.
 - For any object C and arrows $f: C \rightarrow A$ and $g: C \rightarrow B$,

there exists a unique arrow $h: C \rightarrow A \times B$ such that the following diagram commutes:

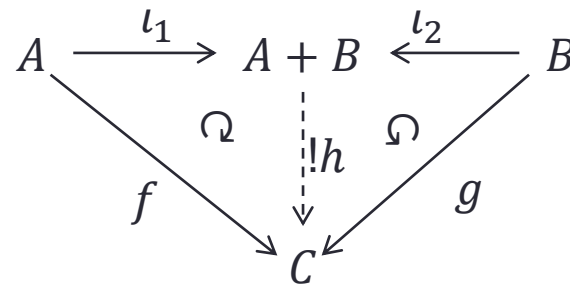


$$\pi_1 \circ h = f$$

$$\pi_2 \circ h = g$$

- $A + B$ is the **co-product** of A and $B \iff$
 - There are two arrows $\iota_1: A \rightarrow A + B$ and $\iota_2: B \rightarrow A + B$.
 - For any object C and arrows $f: A \rightarrow C$ and $g: B \rightarrow C$,

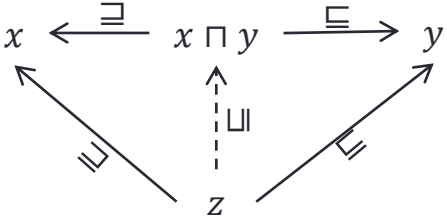
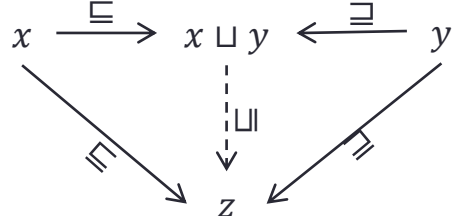
there exists a unique arrow $h: A + B \rightarrow C$ such that the following diagram commutes:



$$h \circ \iota_1 = f$$

$$h \circ \iota_2 = g$$

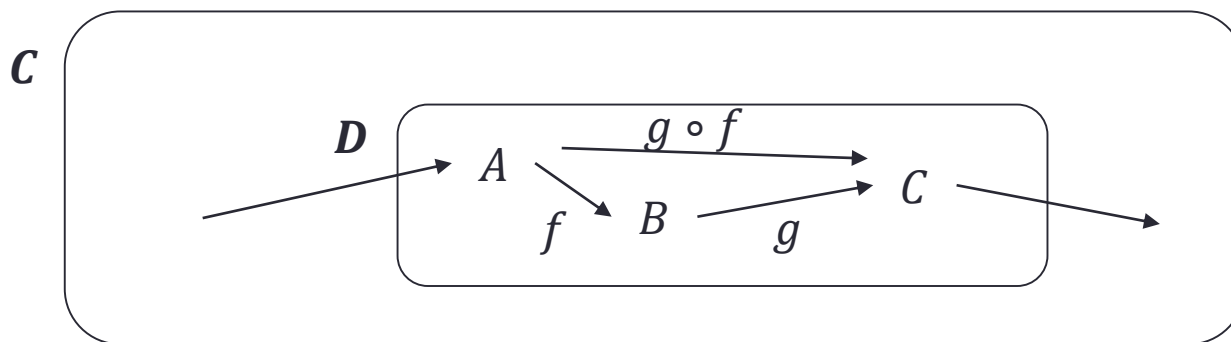
Product and Co-Product

	Product	Co-Product
Set	<ul style="list-style-type: none"> $A \times B = \{(x, y) x \in A \text{ and } y \in B\}$ $\pi_1((x, y)) = x$ $\pi_2((x, y)) = y$ For $f: C \rightarrow A$ and $g: C \rightarrow B$, $h(z) = (f(z), g(z))$ 	<ul style="list-style-type: none"> $A + B = \{(x, 1) x \in A\} \cup \{(y, 2) y \in B\}$ $\iota_1(x) = (x, 1)$ $\iota_2(y) = (y, 2)$ For $f: A \rightarrow C$ and $g: B \rightarrow C$, $h((x, 1)) = f(x)$ and $h((y, 2)) = g(y)$
Partially Ordered Set (D, \sqsubseteq)	<ul style="list-style-type: none"> $x \times y = x \sqcap y$ $\pi_1: x \sqcap y \sqsubseteq x$ $\pi_2: x \sqcap y \sqsubseteq y$ For $f: z \sqsubseteq x$ and $g: z \sqsubseteq y$, $h: z \sqsubseteq x \sqcap y$ 	<ul style="list-style-type: none"> $x + y = x \sqcup y$ $\iota_1: x \sqsubseteq x \sqcup y$ $\iota_2: y \sqsubseteq x \sqcup y$ For $f: x \sqsubseteq z$ and $g: y \sqsubseteq z$, $h: x \sqcup y \sqsubseteq z$ 

- Theorem:** If product $A \times B$ exists, it is unique up to isomorphic.
- Dual Theorem:** If co-product $A + B$ exists, it is unique up to isomorphic.

Subcategory

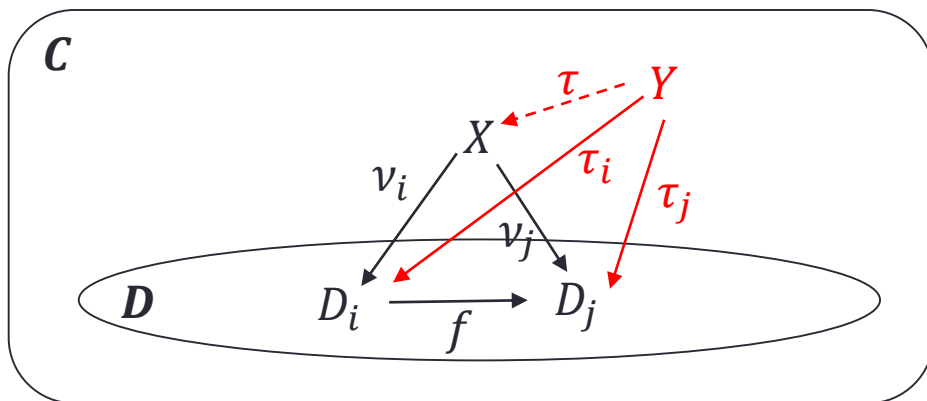
- \mathbf{D} is a **subcategory** of category \mathbf{C} if the following conditions are hold:
 - For any object $A \in \mathbf{D}$, $A \in \mathbf{C}$
 - For any arrow $f: A \rightarrow B \in \mathbf{D}$, $A, B \in \mathbf{D}$
 - For any object $A \in \mathbf{D}$, $1_A \in \mathbf{D}$
 - For any arrows $f, g \in \mathbf{D}$, $g \circ f \in \mathbf{D}$



- A subcategory \mathbf{D} is a category.
- Most of the diagrams we saw are subcategories.

Limit

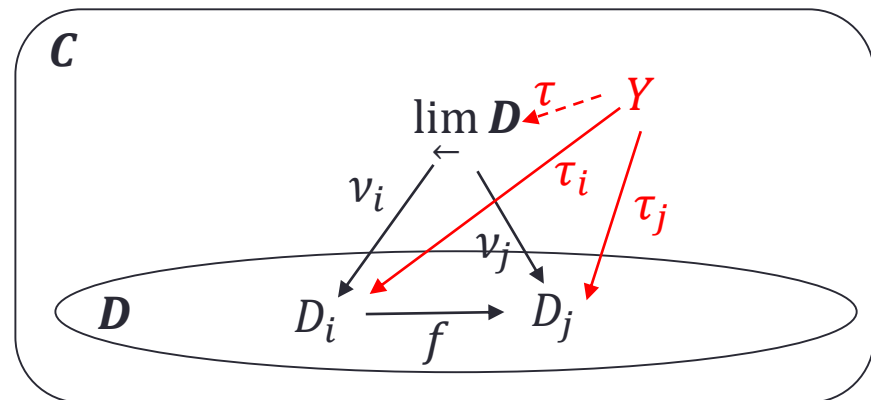
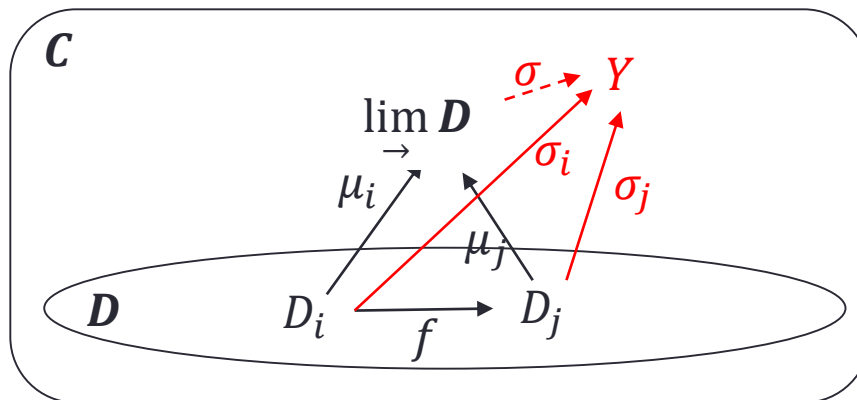
- The **limit** of a subcategory \mathbf{D} of category \mathbf{C} is the object X in \mathbf{C} which satisfies the following conditions:
 - For any object $D_i \in \mathbf{D}$, there exists an arrow $v_i: X \rightarrow D_i$.
 - For any arrow $f: D_i \rightarrow D_j \in \mathbf{D}$, $f \circ v_i = v_j$.
 - For any object Y in \mathbf{C} and arrows $\tau_i: Y \rightarrow D_i$ which satisfies for any arrow $f: D_i \rightarrow D_j \in \mathbf{D}$ of $f \circ \tau_i = \tau_j$, there exists a unique arrow $\tau: Y \rightarrow X$ such that $\tau_i = v_i \circ \tau$.



- The limit X of \mathbf{D} is written as $\varprojlim \mathbf{D}$.

Colimit

- The **colimit** of a subcategory \mathbf{D} of category \mathcal{C} is the object X in \mathcal{C} which satisfies the following conditions:
 - For any object $D_i \in \mathbf{D}$, there exists an arrow $\mu_i: D_i \rightarrow X$.
 - For any arrow $f: D_i \rightarrow D_j \in \mathbf{D}$, $\mu_j \circ f = \mu_i$.
 - For any object Y in \mathcal{C} and arrows $\sigma_i: D_i \rightarrow Y$ which satisfies for any arrow $f: D_i \rightarrow D_j \in \mathbf{D}$ of $\sigma_j \circ f = \tau_i$, , there exists a unique arrow $\sigma: X \rightarrow Y$ such that $\sigma_i = \sigma \circ \mu_i$.



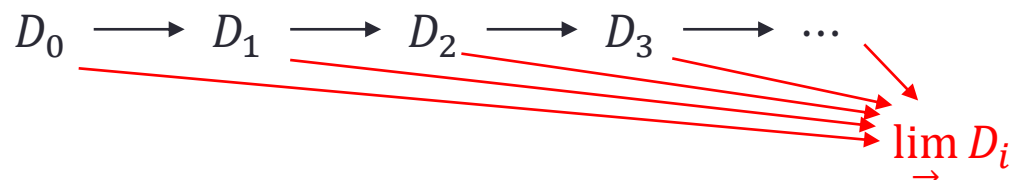
- The colimit X of \mathbf{D} is written as $\lim_{\rightarrow} \mathbf{D}$.

$\lim_{\rightarrow} \mathbf{D}$ and $\lim_{\leftarrow} \mathbf{D}$ are dual

Inductive Limit and Projective Limit

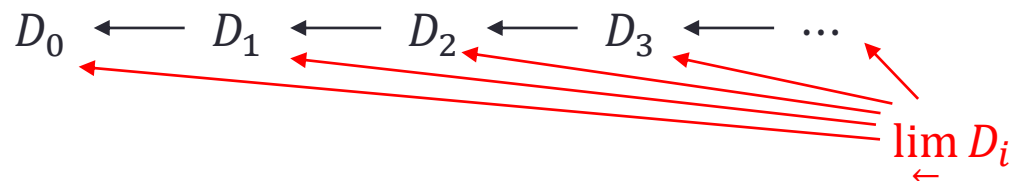
- **Inductive limit**

- The colimit of the following diagram:



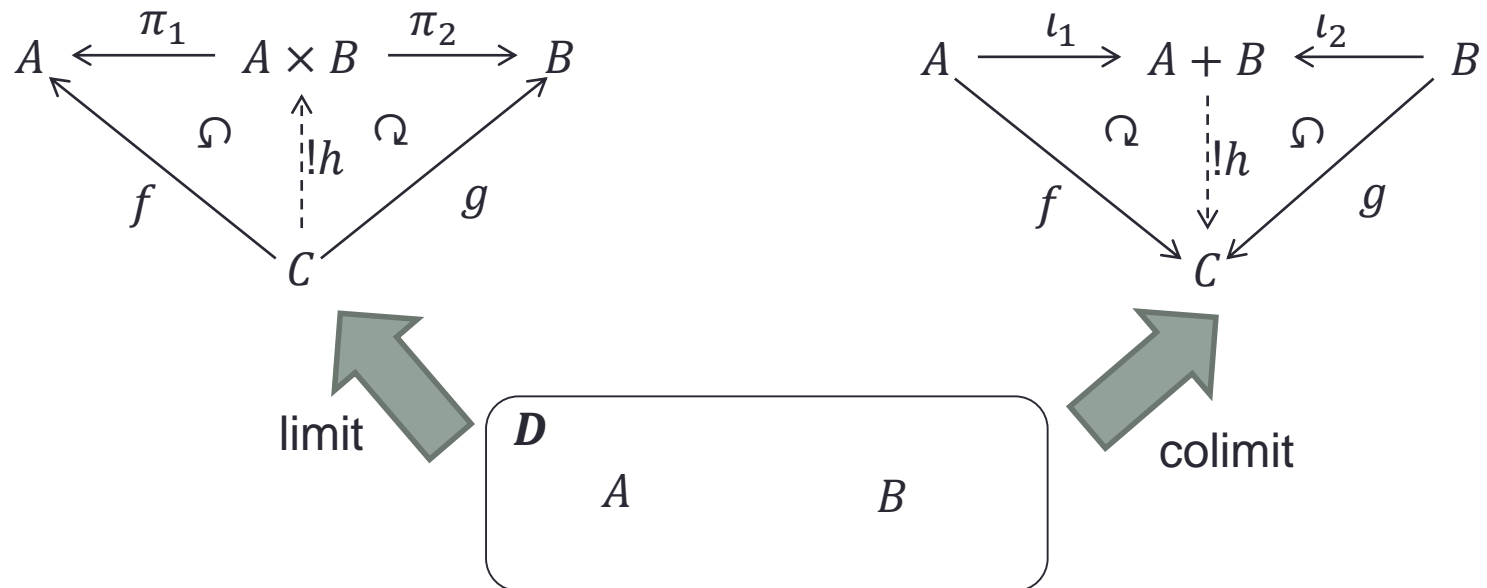
- **Projective limit**

- The limit of the following diagram:



Everything is Limit and Colimit

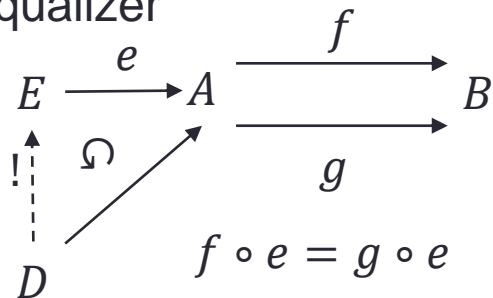
- product and co-product



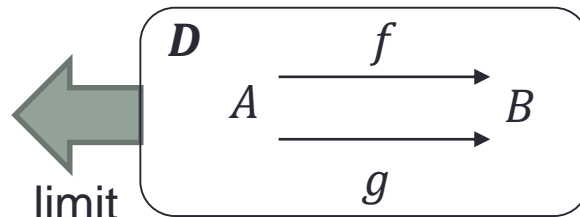
- final object and initial object



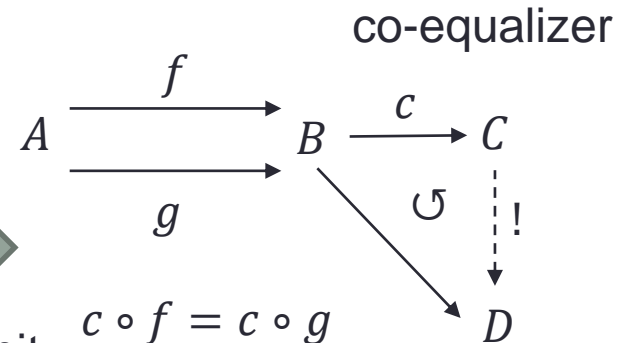
- # equalizer



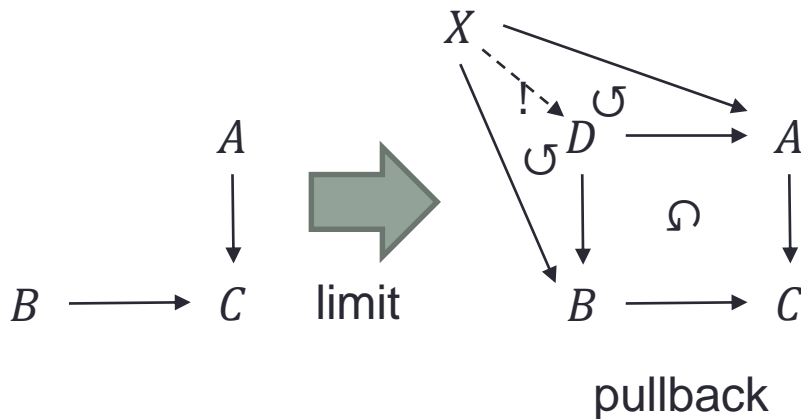
limit



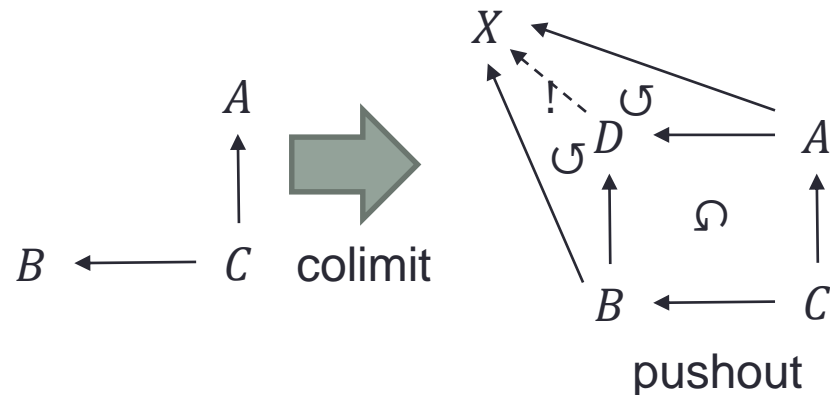
colimit



- pullback and pushout



pullback



pushout

Summary

- Category Theory
 - Alternative foundation of Mathematics
- Category
 - Objects and Arrows
- Special objects
 - Initial and final objects
 - Product and co-product
- Limit and Colimit