非可換調和解析における実ハーディ空間の新たな展開
– 表現論・実解析・確率解析の融合

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はしがき

平成16年度から平成19年度への4年間において, 実ランク1な半単純リー群上の K 両側不変関数に関する調和解析とその一般化としてのフーリエ・ヤコビ変換の理論を詳しく研究した. とくに (1) 実ハーディ空間の構成とその特徴付け, (2) 実ハーディ空間の応用としての特異積分作用素の有界性の検証を主たる研究テーマとした. さらに付随した研究成果として, フーリエ・ヤコビ変換に関する (3) ハーディ型の定理の一般化, (4) 不確定性原理の拡張なども得られた.

これらの調和解析は指数増大度を持つウェイト空間（非等質型）における理論である. 等質型空間ではユークリッド空間の諸定理の類型が多く知られているが, 非等質型空間ではあまり研究がなされていなかった. その意味において本研究はあらたな分野を切り開くものであり, ここで用いた手法の継続した研究が期待される.

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研究成果

学会誌等において発表された論文および発表予定の論文をもって研究成果の報告とする. 論文の概略は以下の通りである.

(1) 実ハーディ空間の構成とその特徴付け： [1], [3], [4]

半単純リー群 G 上で最大関数 M_ϕ を考え, $M_\phi(f)$ の可積分性により実ハーディ空間 $H^1(G)$ を定義する. とくに関数を K 両側不変に制限したときはヤコビ解析における実ハーディ空間となる. 今回この空間の特徴付けが [3], [4] で得られた. アーベル変換の逆変換が分数微分で与えられることに [1] で注意し, G 上の K 両側不変関数と $\mathbb{R}$ 上の関数の間の関係式を構成することができた. これにより $H^1(G//K)$ が $\mathbb{R}$ 上の重み付き実ハーディ空間 $H^1_w(\mathbb{R})$ を用いて記述することができた.

(2) 実ハーディ空間の応用としての積分作用素の有界性：[2], [3], [4], [9]

$H^1(G//K)$ の応用として, Poisson 最大関数 $M_\phi(f)$, Littlewood-Paley $g-$ 関数 $g(f)$, Lusin area 関数 $S(f)$ の (H^1, L^1) 有界性を検証した. こらの作用素は $\mathbb{R}$ のときと同様に定義される. M_ϕ と g については (H^1, L^1) 有界性を示すことができた. しかし S に関してはその有界性を得るために定義を若干修正した. 今後この修正の削除が研究課題である.

(3) ハーディ型の定理の一般化： [5], [7]

半単純リー群上の一般の関数に対してはハーディの定理が成立しないことが知られている. K 型を制限しないと熱核と同程度の減少度を持つ関数が, 各 K 型ごとに存在するからである. [5] ではそのような関数族を境界値を用いることにより特徴付けた. K 型を制限すればハーディの定理の拡張として多くの定理が得られているが, [7] では $\mathbb{R}$ 上のボナミ型と宮地型のハーディの定理をヤコビ変換に拡張した.

(4) 不確定性原理の拡張： [6], [8], [10]

$\mathbb{R}$ 上の不確定性原理は f とフーリエ変換 $\hat{f}$ はともに局在できないことを意味する. [6], [10] ではこの定理をヤコビ変換に拡張した. いくつかの拡張が知られているが, 本研究ではヤコビ変換が離散項を持つ場合を考え, 離散項の存在と不確定性原理の関係を詳しく解析した. [8] ではヤコビ変換の拡張として Sturm-Liouville 型 hyper 群を考え, そこでの不確定性原理を示した. とくに離散項の存在条件を不確定性原理の立場で求めた.

Fractional calculus and analytic continuation of the complex Fourier-Jacobi transform

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Abstract

By using the Riemann-Liouville type fractional integral operators we shall reduce the complex Fourier-Jacobi transforms of even functions on **R** to the Euclidean Fourier transforms. As an application of the reduction formula, Parseval's formula and an inversion formula of the complex Jacobi transform are easily obtained. Moreover, we shall introduce a class of even functions, not C^∞ and not compactly supported on **R**, whose transforms have meromorphic extensions on the upper half plane.

1. Introduction.

Let $\alpha, \beta, \lambda \in \mathbf{C}$ and $t \in \mathbf{R}$. For $\alpha \neq -1, -2, -3, \cdots$, $\phi_\lambda^{\alpha,\beta}(t)$ denotes the Jacobi function of the first kind, and for $\lambda \neq -i, -2i, -3i, \cdots$, $\Phi_\lambda^{\alpha,\beta}(t)$ the one of the second kind. Let $C_0^\infty(\mathbf{R})$ denote the space of all even C^∞ functions on $\mathbf{R}$ with compact support. For $f \in C_0^\infty(\mathbf{R})$ and $\Re\alpha > -1$ the Fourier-Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$ and the complex Fourier-Jacobi one $\tilde{f}_{\alpha,\beta}(\lambda)$ are defined by

$$\hat{f}_{\alpha,\beta}(\lambda) = \frac{\sqrt{2}}{\Gamma(\alpha+1)} \int_0^\infty f(t)\phi_\lambda^{\alpha,\beta}(t)\Delta_{\alpha,\beta}(t)dt \tag{1}$$

and

$$\tilde{f}_{\alpha,\beta}(\lambda) = \int_0^\infty f(t)\Phi_\lambda^{\alpha,\beta}(t)\Delta_{\alpha,\beta}(t)dt \tag{2}$$

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respectively, where

$$\Delta_{\alpha,\beta}(t) = (2\mathrm{sh}t)^{2\alpha+1}(2\mathrm{ch}t)^{2\beta+1}. \tag{3}$$

The Fourier-Jacobi transform $f \to \hat{f}_{\alpha,\beta}$ is well-understood. For example, the Paley-Wiener theorem and the inversion formula for $C_0^\infty(\mathbf{R})$ are obtained by Flensted-Jensen [2] and Koornwinder [3]. In particular, Koornwinder reduces the transform $\hat{f}_{\alpha,\beta}$ to the Fourier Cosine transform, which corresponds to the case of $\alpha = \beta = -1/2$:

$$\begin{aligned}\hat{f}_{\alpha,\beta}(\lambda) &= 2^{3(\alpha+1/2)}\frac{2}{\sqrt{2\pi}}\left(W^1_{\alpha-\beta}\circ W^2_{\beta+1/2}(f)\right)^{\wedge}_{-1/2,-1/2}(\lambda) \\ &= 2^{3(\alpha+1/2)}\left(W^1_{\alpha-\beta}\circ W^2_{\beta+1/2}(f)\right)^{\wedge}(\lambda). \end{aligned}\tag{4}$$

Here $W^\sigma_\mu(f)$, $\mu \in \mathbf{C}$, $\sigma > 0$, is the Weyl type fractional integral of f, which is for $\Re\mu > 0$ defined by

$$W^\sigma_\mu(f)(y) = \Gamma(\mu)^{-1}\int_y^\infty f(x)(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu-1}d(\mathrm{ch}\sigma x) \tag{5}$$

and extended to an entire function in μ. Moreover, in the second line of (4) $W^1_{\alpha-\beta}\circ W^2_{\beta+1/2}(f)$ is regarded as an even function on $\mathbf{R}$ and $(\ \cdot\ )^\wedge$ is the Euclidean Fourier transform on $\mathbf{R}$ (see [3, (2.7), (3,7), (3.12)]). One of the aim of this paper is to obtain an analogous formula for the complex Fourier-Jacobi transform $\tilde{f}_{\alpha,\beta}(\lambda)$. Actually, we shall reduce $\tilde{f}_{\alpha,\beta}$ to the Euclidean Fourier transform, which corresponds to the case of $\alpha = \beta = -1/2$ (see [2, (2.7)]). In order to obtain the reduction formula we introduce the Riemann-Liouville type fractional integral $\tilde{W}^\sigma_\mu(f)$: For $f \in C_0^\infty(\mathbf{R})$ and $\Re\mu > 0$, $\tilde{W}^\sigma_\mu(f)$ is defined by

$$\tilde{W}^\sigma_\mu(f)(y) = \sigma\Gamma(\mu)^{-1}\int_0^y f(x)(\mathrm{ch}\sigma y - \mathrm{ch}\sigma x)^{\mu-1}dx \cdot \mathrm{sh}\sigma y \tag{6}$$

and extended to an entire function in μ (see Lemma 3.2). Then the relation between the complex Fourier-Jacobi transform and the Euclidean Fourier one is given by

$$\tilde{f}_{\alpha,\beta}(\lambda) = 2^{-3(\alpha+1/2)}C_{\alpha,\beta}(-\lambda)\left(\tilde{W}^1_{-(\alpha-\beta)}\circ\tilde{W}^2_{-(\beta+1/2)}(f\Delta_{\alpha,\beta})\right)^{\sim}_{-1/2,-1/2}(\lambda),$$

where $C_{\alpha,\beta}$ is the C-function (see (11) and Proposition 4.2). In this formula, if $\Re\alpha > \Re\beta > -1/2$, two operators $\tilde{W}^1_{-(\alpha-\beta)}$ and $\tilde{W}^2_{-(\beta+1/2)}$ correspond to fractional derivatives.

As an application of this formula, Parseval's formula for $C_0^\infty(\mathbf{R})$, which characterizes the inner product $\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta})}$ in terms of $\hat{f}_{\alpha,\beta}$ and $\tilde{g}_{\alpha,\beta}$, easily follows from the one for $L^2(\mathbf{R})$ (see Theorem 5.1). Next, we shall consider analytic continuation of $\tilde{g}_{\alpha,\beta}(\lambda)$ when g is not C^∞ and not compactly supported. We note that, if $\Im\lambda$ is sufficiently large, then $\Phi_\lambda^{\alpha,\beta}$ has exponential decay and thus, $\tilde{g}_{\alpha,\beta}(\lambda)$ is well-defined for a large class of even functions. We shall introduce a class of even functions g on $\mathbf{R}$ for which $\tilde{g}_{\alpha,\beta}(\lambda)$ has a meromorphic extension on $\Im\lambda \geq 0$. Then we can deduce an inversion formula of the complex Fourier-Jacobi transform $g \to \tilde{g}_{\alpha,\beta}$ in a distribution sense (see Theorem 6.5).

Similar result is obtained in [5] by a different and direct approach without using the reduction arguments. Moreover, in [1] the Fourier-Jacobi transform $\tilde{g}_{\alpha,\beta}$ of $g(x) = (\mathrm{ch}x)^\eta$ is explicitly calculated for the group case of $SU(n,1)$ ($\alpha = n-1, \beta = 0$). This function $(\mathrm{ch}x)^\eta$ is a simple example of unbounded functions whose Fourier-Jacobi transform has a meromorphic extension on $\Im\lambda \geq 0$. Compared with these direct approach, if $\Re\alpha > \Re\beta > -1/2$, then the same result follows in our approach, otherwise, some extra conditions on g are required to carry out our reduction method. However, under these extra conditions we see that all poles appeared in our inversion formula are simple and we can distinguish between poles arisen from the C-function and ones from the analytic continuation (see Theorem 6.5 and Remark 6.6).

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2. Notations.

Let $\alpha, \beta, \lambda \in \mathbf{C}$ and $t \in \mathbf{R}$. We shall consider the differential equation

$$(L_{\alpha,\beta} + \lambda^2 + \rho^2) f(t) = 0, \tag{7}$$

where $\rho = \alpha + \beta + 1$ and

$$L_{\alpha,\beta} = \frac{d^2}{dt^2} + ((2\alpha+1)\mathrm{cth}t + (2\beta+1)\mathrm{th}t)\frac{d}{dt}.$$

Then, for $\alpha \neq -\mathbf{N}$, the Jacobi function of the first kind with order (α, β)

$$\phi_\lambda^{\alpha,\beta}(t) = F\left(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -\mathrm{sh}^2 t\right) \tag{8}$$

is a unique solution of (7) satisfying $\phi_\lambda^{\alpha,\beta}(0) = 1$ and $d\phi_\lambda^{\alpha,\beta}/dt(0) = 0$. For $\lambda \neq -i\mathbf{N}$, the Jacobi function of the second kind with order (α, β)

$$\Phi_\lambda^{\alpha,\beta}(t) = (e^t - e^{-t})^{i\lambda-\rho} F\left(\frac{\rho-2\alpha-i\lambda}{2}, \frac{\rho-i\lambda}{2}; 1-i\lambda; -\mathrm{sh}^{-2} t\right) \tag{9}$$

is another solution of (7). Then $\Gamma(\alpha+1)^{-1}\phi_\lambda^{\alpha,\beta}$ is entire of α, β, and for $\lambda \notin i\mathbf{Z}$, we have the identity

$$\sqrt{\pi}\Gamma(\alpha+1)^{-1}\phi_\lambda^{\alpha,\beta}(t) = \frac{1}{2}C_{\alpha,\beta}(\lambda)\Phi_\lambda^{\alpha,\beta}(t) + \frac{1}{2}C_{\alpha,\beta}(-\lambda)\Phi_{-\lambda}^{\alpha,\beta}(t), \tag{10}$$

where $C_{\alpha,\beta}$ is the C-function given by

$$C_{\alpha,\beta}(\lambda) = \frac{2^\rho \Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(\frac{1+i\lambda}{2}\right)}{\Gamma\left(\frac{i\lambda+\rho}{2}\right)\Gamma\left(\frac{i\lambda+\rho-2\beta}{2}\right)}. \tag{11}$$

We recall the following properties of these functions (cf. [3] and [4]).

Lemma 2.1. *Assume that $\alpha, \beta \in \mathbf{C}$ and $\Re\alpha > -1$.*

(1) *For each fixed $t > 0$, as a function of λ, $\phi_\lambda^{\alpha,\beta}(t)$ is an entire function. There exists a constant $K > 0$ such that for all $t \geq 0$ and all $\lambda \in \mathbf{C}$,*

$$|\phi_\lambda^{\alpha,\beta}(t)| \leq K(1+|\lambda|)^\epsilon(1+t)e^{(|\Im\lambda|-\Re\rho)t},$$

where $\epsilon = 0$ if $\Re\alpha > -1/2$ and $\epsilon = 1$ for $-1 < \Re\alpha \leq -1/2$.

(2) *For each fixed $t > 0$, as a function of λ, $\Phi_\lambda^{\alpha,\beta}(t)$ is a holomorphic function in $\mathbf{C}\backslash\{-i\mathbf{N}\}$. For each $c > 0$ there exists a constant $K > 0$ such that for all $t \geq c$ and all $\Im\lambda \geq 0$,*

$$|\Phi_\lambda^{\alpha,\beta}(t)| \leq Ke^{(-\Im\lambda-\Re\rho)t}$$

and for all $0 < t < c$ and all $\Im\lambda \geq 0$,

$$|\Phi_\lambda^{\alpha,\beta}(t)| \leq K\begin{cases} t^{-(2\Re\alpha+1)} & \text{if } \Re\alpha > -1/2, \\ \log|t| & \text{if } \Re\alpha = -1/2, \\ 1 & \text{if } -1 < \Re\alpha < -1/2. \end{cases}$$

(3) *For each $r > 0$, there exists a constant $K > 0$ such that, if $\lambda \in \mathbf{C}$, $\Im\lambda \geq 0$ and λ is at distance larger than r from the poles of $C_{\alpha,\beta}(-\lambda)^{-1}$, then*

$$|C_{\alpha,\beta}(-\lambda)^{-1}| \leq K(1+|\lambda|)^{\Re\alpha+1/2}.$$

Let $C_0^\infty(\mathbf{R})$ denote the set of even C^∞ functions on $\mathbf{R}$ with compact support. For $f \in C_0^\infty(\mathbf{R})$ we define the Fourier-Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$ and the complex Fourier-Jacobi transform $\tilde{f}_{\alpha,\beta}(\lambda)$ by (1) and (2) respectively. From Lemma 2.1 it follows that $\hat{f}_{\alpha,\beta}(\lambda)$ is entire and $\tilde{f}_{\alpha,\beta}(\lambda)$ is holomorphic for $\lambda \neq -i\mathbf{N}$. Especially, (10) implies that for all $\lambda \notin i\mathbf{Z}$,

$$\sqrt{2\pi}\hat{f}_{\alpha,\beta}(\lambda) = C_{\alpha,\beta}(\lambda)\tilde{f}_{\alpha,\beta}(\lambda) + C_{\alpha,\beta}(-\lambda)\tilde{f}_{\alpha,\beta}(-\lambda). \tag{12}$$

In the following we define the Gauss symbol $[z]$ for $z \in \mathbf{C}$ as $[\Re z]$.

3. Fractional integrals.

3.1. Let $C_c^\infty(\mathbf{R}_a)$, $\mathbf{R}_a = [a,\infty)$, $a \in \mathbf{R}$, denote the set of all C^∞ functions F_a on $\mathbf{R}$ with compact support, where F is right differentiable at a. For $F \in C_c^\infty(\mathbf{R}_a)$ and $-n < \Re\mu$, $n = 0, 1, 2, \cdots$, we shall define the Weyl type fractional integral operator $W_\mu^{\mathbf{R}}$ by

$$(W_\mu^{\mathbf{R}}(F))(y) = \frac{(-1)^n}{\Gamma(\mu+n)} \int_y^\infty \frac{d^n F(x)}{dx^n}(x-y)^{\mu+n-1}dx. \tag{13}$$

We extend it as an entire function in μ. Then $W_0^{\mathbf{R}}$ is the identity operator, $W_\mu^{\mathbf{R}} \circ W_\nu^{\mathbf{R}} = W_{\mu+\nu}^{\mathbf{R}}$, and

$$W_\nu^{\mathbf{R}} : C_c^\infty(\mathbf{R}_a) \to C_c^\infty(\mathbf{R}_a)$$

is bijection. We also define the Riemann-Liouville type fractional integral operator $\tilde{W}_\mu^{\mathbf{R}}$ by

$$(\tilde{W}_\mu^{\mathbf{R}}(F))(y) = \frac{1}{\Gamma(\mu+n)} \frac{d^n}{dy^n} \int_a^y F(x)(y-x)^{\mu+n-1}dx \tag{14}$$

and extend it as an entire function in μ. We note that $\tilde{W}_0^{\mathbf{R}}$ is the identity operator and $\tilde{W}_{-\mu}^{\mathbf{R}} \circ \tilde{W}_\mu^{\mathbf{R}}(F) = F$ if $\Re\mu > 0$. For $\Re\mu \leq 0$, $\tilde{W}_{-\mu}^{\mathbf{R}} \circ \tilde{W}_\mu^{\mathbf{R}}(F) = F$ provided $F(a) = F'(a) = \cdots = F^{(n-1)}(a) = 0$. On $C_c^\infty(\mathbf{R}_b)$, $b > a$, $\tilde{W}_\mu^{\mathbf{R}} \circ \tilde{W}_\nu^{\mathbf{R}} = \tilde{W}_{\mu+\nu}^{\mathbf{R}}$. For $\tau, \eta \in \mathbf{C}$ and $m = 0, 1, 2, \cdots$, we define $A_{\tau,\eta}^m(\mathbf{R}_a)$ the class of C^m functions F on $\mathbf{R}_a$ of the form $F = F_0 + F_1$;

$$F_0(x) = (x-a)^\tau G(x), \quad G \in C^m([a, a+2)) \tag{15}$$

and

$$F_1(x) = x^\eta H(x), \quad H \in C^m((a+1,\infty)), \tag{16}$$

where

$$\sup_{0 \leq k \leq m, a+1 \leq x < \infty} \left| x^k \frac{d^k H(x)}{dx^k} \right| \leq c. \tag{17}$$

Moreover, $A^m_{\tau*,\eta}(\mathbf{R}_a)$ denote the class defined by replacing $(x-a)^\tau$ in (15) with $\log(x-a)\cdot(x-a)^\tau$ and $A^m_{\tau,\eta*}(\mathbf{R}_a)$ the one defined by replacing $\sup_{a+1\le x<\infty}|H(x)|\le c$, $k=0$ in (17), with $\sup_{a+1\le x<\infty}|(\log x)H(x)|\le c$.

Lemma 3.1. *For $m=0,1,2,\cdots$ and $\mu,\tau,\eta\in\mathbf{C}$ the fractional operators $W^{\mathbf{R}}_\mu$ and $\tilde{W}^{\mathbf{R}}_\mu$ satisfy the following.*

(1) *If $m+[\mu]-1\ge 0$, $\Re\eta<0$ and $\Re(\eta+\mu)<0$, then*

$$W^{\mathbf{R}}_\mu : A^m_{\tau,\eta}(\mathbf{R}_a)\to A^{m+[\mu]}_{\delta,\eta+\mu}(\mathbf{R}_a),$$

where $\delta=\tau+\mu$ if $\mu=0,-1,-2,\cdots$, and otherwise

$$\delta=\begin{cases} 0 & \text{if } \Re(\tau+\mu)>0,\\ 0* & \text{if } \Re(\tau+\mu)=0,\\ \tau+\mu & \text{if } \Re(\tau+\mu)<0.\end{cases} \tag{18}$$

(2) *If $m+[\mu]\ge 0$ and $\Re\tau>-1$, then*

$$\tilde{W}^{\mathbf{R}}_\mu : A^m_{\tau,\eta}(\mathbf{R}_a)\to A^{m+[\mu]}_{\tau+\mu,\delta}(\mathbf{R}_a),$$

where $\delta=\eta+\mu$ if $\mu=0,-1,-2,\cdots$, and otherwise

$$\delta=\begin{cases} \eta+\mu & \text{if } \Re\eta>-1,\\ (\eta+\mu)* & \text{if } \Re\eta=-1,\\ \mu-1 & \text{if } \Re\eta<-1.\end{cases}$$

Proof. (1) When $\mu=0,-1,-2,\cdots$, $W^{\mathbf{R}}_\mu(F)(y)=c[F^{(-\mu)}]^\infty_y=cF^{(-\mu)}(y)$, because $\Re(\eta+\mu)<0$. Therefore, the assertion for $\mu=0,-1,-2,\cdots$ easily follows. Let $\mu\ne 0,-1,-2,\cdots$. Also we may assume that $\Re\mu>0$. Actually, if $\Re\mu\le 0$, let $W^{\mathbf{R}}_\mu=W^{\mathbf{R}}_{\mu-[\mu]}\circ W^{\mathbf{R}}_{[\mu]}$ and note that $0<\Re(\mu-[\mu])<1$ and $[\mu-[\mu]]+[\mu]=[\mu]$. Hence, the assertion for $\Re\mu\le 0$ follows from the cases of $\Re\mu>0$ and $\mu=0,-1,-2,\cdots$. Let $F\in A^m_{\tau,\eta}(\mathbf{R}_a)$ be of the form $F=F_0+F_1$ in (15) and (16). If $y\ge a+1$, then $W^{\mathbf{R}}_\mu(F)$ is defined as

$$\begin{aligned} W^{\mathbf{R}}_\mu(F)(y) &= c\int_y^\infty (x-a)^\tau G(x)(x-y)^{\mu-1}dx + c\int_y^\infty x^\eta H(x)(x-y)^{\mu-1}dx \\ &= I_1(y)+I_2(y). \end{aligned}$$

Clearly, $I_1(y)=0$ if $y\ge a+2$ and $I_1\in C^{(m+[\mu])}$. Moreover,

$$\begin{aligned} I_2(y) &= cy^{\eta+\mu}\int_1^\infty x^\eta H(yx)(x-1)^{\mu-1}dx \\ &= cy^{\eta+\mu}H_\mu(y). \end{aligned}$$

For $0 \le l \le m$,

$$\begin{aligned} H_\mu^{(l)}(y) &= \int_1^\infty x^{\eta+l} H^{(l)}(yx)(x-1)^{\mu-1} dx \\ &= y^{-(\eta+\mu+l)} \int_y^\infty x^{\eta+l} H^{(l)}(x)(x-y)^{\mu-1} dx \end{aligned}$$

and for $0 \le l' \le [\mu]$,

$$\begin{aligned} H_\mu^{(l+l')}(y) &\sim \sum_{k=0}^{l'} y^{-(\eta+\mu+l+k)} \int_y^\infty x^{\eta+l} H^{(l)}(x)(x-y)^{\mu-1-(l'-k)} dx \\ &\sim \sum_{k=0}^{[\mu]} y^{-l'} \int_1^\infty x^{\eta+l} H^{(l)}(yx)(x-1)^{\mu-1-([\mu]-k)} dx \\ &\sim \sum_{k=0}^{[\mu]} y^{-(l+l')} \int_1^\infty x^{\eta}(xy)^l H^{(l)}(yx)(x-1)^{\mu-1-([\mu]-k)} dx, \end{aligned}$$

where, if μ is positive integer, the term corresponding to $l' = [\mu], k = 0$ equals $y^{-l'} H^{(l)}(y) = y^{-(l+l')} \cdot y^l H^{(l)}(y)$ (see the first line). Hence, (17) implies that $y^{l+l'} H_\mu^{(l+l')}$, $0 \le l + l' \le m + [\mu]$, is bounded on $(a+1, \infty)$. Therefore, $H_\mu(y)$ satisfies (17) replaced m with $m + [\mu]$. If $a < y < a+1$, then $W_\mu^{\mathbf{R}}(F)$ is estimated as

$$\begin{aligned} & \int_y^{a+1} (x-a)^\tau G(x)(x-y)^{\mu-1} dx + \int_{a+1}^\infty x^\eta H(x)(x-y)^{\mu-1} dx \\ \sim\ & (y-a)^{\tau+\mu} \int_1^{1/(y-a)} x^\tau G((y-a)x+a)(x-1)^{\mu-1} dx + y^{\eta+\mu} \\ \sim\ & (y-a)^{\tau+\mu} \left\{ \begin{array}{ll} (y-a)^{-(\tau+\mu)} & \text{if } \Re(\tau+\mu) > 0 \\ \log(y-a) & \text{if } \Re(\tau+\mu) = 0 \\ 1 & \text{if } \Re(\tau+\mu) < 0 \end{array} \right\} + 1 \\ \sim\ & (x-a)^\delta G_\mu(x). \end{aligned}$$

Noting $0 < (y-a) < 1$ and the argument in the previous case, we see that $G_\mu \in C^{m+[\mu]}$. Therefore, $W_\mu^{\mathbf{R}}(F)$ is of the desired form.
(2) When $\mu = 0, -1, -2, \cdots$, $\tilde{W}_\mu^{\mathbf{R}}(F)$ coincides with $cF^{(-\mu)}$ provided $\Re\tau > -1$. Since $\tilde{W}_\mu^{\mathbf{R}} = \tilde{W}_{[\mu]}^{\mathbf{R}} \circ \tilde{W}_{\mu-[\mu]}^{\mathbf{R}}$ if $\Re\mu \le 0$, as in the first case, we may assume that $\Re\mu > 0$. We note that, if $a < y < a+1$, then

$$\begin{aligned} \tilde{W}_\mu^{\mathbf{R}}(F)(y) &= \int_a^y (x-a)^\tau G(x)(y-x)^{\mu-1} dx \\ &= (y-a)^{\tau+\mu} \int_0^1 x^\tau G((y-a)x+a)(1-x)^{\mu-1} dx \\ &= (y-a)^{\tau+\mu} G_\mu(x), \end{aligned}$$

and if $y \geq a+1$, then $\tilde{W}_\mu^{\mathbf{R}}(F)(y)$ is estimated as

$$\begin{aligned}
&\int_a^{a+1} (x-a)^\tau G(x)(y-x)^{\mu-1}dx + \int_{a+1}^y x^\eta H(x)(y-x)^{\mu-1}dx \\
\sim\ & (y-a)^{\tau+\mu}\int_0^{1/(y-a)} x^\tau G((y-a)x+a)(1-x)^{\mu-1}dx \\
&+y^{\eta+\mu}\int_{(a+1)/y}^1 x^\eta H(yx)(x-1)^{\mu-1}dx \\
\sim\ & (y-a)^{\tau+\mu}\int_0^{1/(y-a)} x^\tau dx + y^{\eta+\mu}\int_{(a+1)/y}^1 x^\eta dx \\
\sim\ & (y-a)^{\mu-1} + y^{\eta+\mu}\left\{\begin{array}{ll} 1 & \text{if } \Re\eta > -1 \\ \log(y-a) & \text{if } \Re\eta = -1 \\ (y-a)^{-\eta-1} & \text{if } \Re\eta < -1 \end{array}\right\} \\
\sim\ & y^\delta H_\mu(x).
\end{aligned}$$

Noting $(y-1) \geq 1$ and $-\Re(\tau+\mu-1) < 0$, as in the first case, we see that $G_\mu \in C^{m+[\mu]}$ and H_μ satisfies (17) replaced m with $m+[\mu]$. ■

Remark 3.2. In Lemma 3.1 we note that, if $\Re(\tau+\mu) \geq 0$ and $\mu \neq 1,2,3,\cdots$, then the Weyl type fractional operator $W_\mu^{\mathbf{R}}$ does not keep the zero of F at $x=a$ even if F has sufficiently higher order of zero.

3.2. We shall transfer the operators $W_\mu^{\mathbf{R}}$ and $\tilde{W}_\mu^{\mathbf{R}}$ on $C_c^\infty(\mathbf{R}_1)$, $\mathbf{R}_1 = [1,\infty)$, to ones for $C_0^\infty(\mathbf{R})$. For $f \in C_0^\infty(\mathbf{R})$, $\sigma > 0$ and $-n < \Re\mu$, $n = 0,1,2,\cdots$, we shall define the Weyl type and the Riemann-Liouville type fractional integral operators W_μ^σ and $\tilde{W}_\mu^\sigma$ respectively as follows:

$$W_\mu^\sigma(f)(y) = \frac{(-1)^n}{\Gamma(\mu+n)}\int_y^\infty \frac{d^n f(x)}{d(\mathrm{ch}\sigma x)^n}(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu+n-1}d(\mathrm{ch}\sigma x) \tag{19}$$

and

$$\tilde{W}_\mu^\sigma(f)(y) = \frac{\sigma^{-1}}{\Gamma(\mu+n)}\frac{d^n}{d(\mathrm{ch}\sigma y)^n}\int_0^y f(x)(\mathrm{ch}\sigma y - \mathrm{ch}\sigma x)^{\mu+n-1}dx \cdot \mathrm{sh}\sigma y. \tag{20}$$

Then the change of varibale:

$$f(x) = [f]^\sigma(\mathrm{ch}\sigma x)$$

yields the relation between $W_\mu^{\mathbf{R}}$ and W_μ^σ:

$$W_\mu^\sigma(f)(y) = W_\mu^{\mathbf{R}}([f]^\sigma)(\mathrm{ch}\sigma y) \tag{21}$$

and the one between $\tilde{W}_\mu^{\mathbf{R}}$ and $\tilde{W}_\mu^\sigma$:

$$\tilde{W}_\mu^\sigma(f)(y) = \tilde{W}_\mu^{\mathbf{R}}([f\cdot(\mathrm{sh}\sigma x)^{-1}]^\sigma)(\mathrm{ch}\sigma y)\cdot \mathrm{sh}\sigma y. \tag{22}$$

For $\tau, \eta \in \mathbf{C}$ and $m = 0, 1, 2, \cdots$, let $\mathcal{A}_{\tau,\eta}^{\sigma,m}(\mathbf{R})$ denote the space of all even functions f on $\mathbf{R}$ of the form $f = f_0 + f_1$;

$$f_0(x) = (\mathrm{sh}\sigma x)^{2\tau} g(\mathrm{ch}\sigma x), \quad g \in C^m([1,3)) \tag{23}$$

and

$$f_1(x) = (\mathrm{ch}\sigma x)^{\eta} h(\mathrm{ch}\sigma x), \quad h \in C^m((2,\infty)), \tag{24}$$

where

$$\sup_{0\le k\le m, 2\le x<\infty} \left| x^k \frac{d^k h(x)}{dx^k}\right| \le c. \tag{25}$$

Moreover, $\mathcal{A}_{\tau*,\eta}^{m}(\mathbf{R})$ denote the class defined by replacing $(\mathrm{sh}\sigma x)^{2\tau}$ in (23) with $(\log x)(\mathrm{sh}\sigma x)^{2\tau}$ and $\mathcal{A}_{\tau,\eta*}^{m}(\mathbf{R})$ by replacing $\sup_{2\le x<\infty}|h(x)| \le c$, $k = 0$ in (25), with $\sup_{2\le x<\infty}|(\log x)h(x)| \le c$. Then, using the relations (21) and (22), we can rewrite Lemma 3.1 for $W_\mu^{\mathbf{R}}$ and $\tilde{W}_\mu^{\mathbf{R}}$ to the one for W_μ^σ and $\tilde{W}_\mu^\sigma$:

Lemma 3.3. *Let $\mu, \tau, \eta \in \mathbf{C}$ and $m = 0, 1, 2, \cdots$.*
(1) *If $m + [\mu] - 1 \ge 0$, $\Re\eta < 0$ and $\Re(\eta + \mu) < 0$, then*

$$W_\mu^\sigma : \mathcal{A}_{\tau,\eta}^{\sigma,m}(\mathbf{R}) \to \mathcal{A}_{\delta,\eta+\mu}^{\sigma,m+[\mu]}(\mathbf{R}),$$

where $\delta = \tau + \mu$ if $\mu = 0, -1, -2, \cdots$, and otherwise δ is the same as (18).
(2) *If $m + [\mu] \ge 0$ and $\Re\tau > -1/2$, then*

$$\tilde{W}_\mu^\sigma : \mathcal{A}_{\tau,\eta}^{\sigma,m}(\mathbf{R}) \to \mathcal{A}_{\tau+\mu,\delta}^{\sigma,m+[\mu]}(\mathbf{R}),$$

where $\delta = \eta + \mu$ if $\mu = 0, -1, -2, \cdots$, and otherwise

$$\delta = \begin{cases} \eta + \mu & \text{if } \Re\eta > 0 \\ (\eta+\mu)* & \text{if } \Re\eta = 0 \\ \mu & \text{if } \Re\eta < 0. \end{cases} \tag{26}$$

3.3. As an application of Lemma 3.3, we shall consider the inner product of $f \in A_{\tau_1,\eta_1}^{\sigma,n_1}$ and $g \in A_{\tau_2,\eta_2}^{\sigma,n_2}$, and obtain an adjoint relation between W_μ^σ and $\tilde{W}_\mu^\sigma$.

Proposition 3.4. *Let $\sigma > 0$, $\mu \in \mathbf{C}$ and $\tau_i, \eta_i \in \mathbf{C}, n_i \in \mathbf{N}$ for $i = 1, 2$. Suppose that $\Re\rho \geq 0$, $n_1 + [\mu] - 1 \geq 0$, $n_2 + [\mu] \geq 0$ and*

$$\begin{array}{ll} (a) & \Re(\eta_1 + \mu) < 0, \Re\eta_1 < 0, \\ (b) & \Re(\eta_1 + \eta_2 + \mu + 2\rho/\sigma) < 0 \\ (c) & \Re(\tau_1 + \tau_2 + \alpha + \mu) > -1 \\ (d) & \Re(\tau_2 + \alpha) > -1 \\ (e) & \Re(\tau_2 + \alpha + \mu) > -1. \end{array}$$

Then for $f \in A^{\sigma,n_1}_{\tau_1,\eta_1}$ and $g \in A^{\sigma,n_2}_{\tau_2,\eta_2}$,

$$\langle W^\sigma_\mu(f), g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} = \langle f, \tilde{W}^\sigma_{\bar{\mu}}(g\Delta_{\bar{\alpha},\bar{\beta}})\rangle_{L^2(\mathbf{R}_+,dx)}. \tag{27}$$

Proof. First we check the both sides of (27) are finite. Lemma 3.3 (1) with (a) implies that $W^\sigma_\mu(f) \in \mathcal{A}^{\sigma,n_1+[\mu]}_{\delta,\eta_1+\mu}(\mathbf{R})$ with δ in (18). Since $g\Delta_{\alpha,\beta} \in A^{\sigma,n_2}_{\tau_2+\alpha+1/2,\eta_2+2\rho/\sigma}(\mathbf{R})$, the left hand side of (27) is finite from $(b), (c), (d)$. As for the right hand side, Lemma 3.3 (2) with (d) implies that

$$\tilde{W}^\sigma_\mu(g\Delta_{\alpha,\beta}) \in \mathcal{A}^{\sigma,n_2+[\mu]}_{\tau_2+\alpha+1/2+\mu,\delta}(\mathbf{R})$$

with δ in (26). Then the right hand side of (27) is also finite from (a) and (b). We shall prove the equality. When $\Re\mu > 0$, (27) is clear by changing the order of integration. Let us suppose that $-n < \Re\mu \leq -n+1$, $n = 1, 2, 3, \cdots$. Then, it follows from (19) that

$$\begin{aligned} &\langle W^\sigma_\mu(f), g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} \\ = &\int_0^\infty \frac{(-1)^n}{\Gamma(\mu+n)} \int_y^\infty \frac{d^n f(x)}{d(\mathrm{ch}\sigma x)^n}(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu+n-1} d(\mathrm{ch}\sigma x) \cdot \overline{g(y)}\Delta_{\alpha,\beta}(y)dy \\ = &\int_0^\infty \frac{d^n f(x)}{d(\mathrm{ch}\sigma x)^n} \frac{(-1)^n}{\Gamma(\mu+n)} \int_0^x \overline{g(y)}\Delta_{\alpha,\beta}(y)(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu+n-1} dy \cdot d(\mathrm{ch}\sigma x). \end{aligned}$$

Since $g(y)\Delta_{\alpha,\beta}(y) = O(x^{2\tau_2+2\alpha+1})$ if $0 < x < 1$, the last integral with respect to dy is $O(x^{2(\tau_2+\alpha+\mu+n)})$ if $0 < x < 1$. Thereby, since (e) implies that $2\Re(\tau_2 + \alpha + \mu + n) > -2 + 2n \geq 0$, we can repeat n-times integration by parts with respect to $d(\mathrm{ch}\sigma x)$. This process shifts the differential operator $d/d(\mathrm{ch}\sigma x)$ acting on f to the one acting on the inner integral with respect to dy. Therefore, the desired equality follows from (20). ■

4. Reduction formula.

In order to obtain a reduction formula of $\tilde{f}_{\alpha,\beta}$, we recall some reduction formulas of $\Phi_\lambda^{\alpha,\beta}$ obtained by Koornwinder [3]. Let $\Re\mu > 0$ and $\Im\lambda > -\Re\rho$. Then for $x > 0$,

$$C_{\alpha,\beta}(-\lambda)^{-1}\Phi_\lambda^{\alpha,\beta} = 2^{3\mu+1}C_{\alpha+\mu,\beta+\mu}(-\lambda)^{-1}W_\mu^2(\Phi_\lambda^{\alpha+\mu,\beta+\mu})$$

(see [3, (2.15)]). Hence, applying Proposition 3.4 with Lemma 2.1 (2), we see that for $f \in C_0^\infty(\mathbf{R})$,

$$\begin{aligned}
& C_{\alpha,\beta}(-\lambda)^{-1}\tilde{f}_{\alpha,\beta}(\lambda) \\
= \; & C_{\alpha,\beta}(-\lambda)^{-1}\langle f, \overline{\Phi_\lambda^{\alpha,\beta}}\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} \\
= \; & 2^{3\mu+1}C_{\alpha+\mu,\beta+\mu}(-\lambda)^{-1}\langle f, W_{\bar{\mu}}^2(\overline{\Phi_\lambda^{\alpha+\mu,\beta+\mu}})\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} \\
= \; & 2^{3\mu+1}C_{\alpha+\mu,\beta+\mu}(-\lambda)^{-1}\langle \tilde{W}_\mu^2(f\Delta_{\alpha,\beta}), \overline{\Phi_\lambda^{\alpha+\mu,\beta+\mu}}\rangle_{L^2(\mathbf{R}_+,dx)} \\
= \; & 2^{3\mu+1}C_{\alpha+\mu,\beta+\mu}(-\lambda)^{-1}\left(\tilde{W}_\mu^2(f\Delta_{\alpha,\beta})\Delta_{\alpha+\mu,\beta+\mu}^{-1}\right)^\sim_{\alpha+\mu,\beta+\mu}(\lambda).
\end{aligned}$$

Clearly, this equation is meromorphically extended to $\alpha, \beta, \lambda, \mu \in \mathbf{C}$.

Proposition 4.1. *Let $\Re\alpha > -1$ and $f \in C_0^\infty(\mathbf{R})$. As a meromorphic function of $\alpha, \beta, \lambda, \mu \in \mathbf{C}$,*

$$\tilde{f}_{\alpha,\beta}(\lambda) = 2^{3\mu+1}\frac{C_{\alpha,\beta}(-\lambda)}{C_{\alpha+\mu,\beta+\mu}(-\lambda)}\left(\frac{\tilde{W}_\mu^2(f\Delta_{\alpha,\beta})}{\Delta_{\alpha+\mu,\beta+\mu}}\right)^\sim_{\alpha+\mu,\beta+\mu}(\lambda).$$

Now we shall reduce the complex Fourier-Jacobi transform $\tilde{f}_{\alpha,\beta}$ to the Euclidean Fourier transform.

One way to obtain the reduction is to use Proposition 4.1 repeatedly and to reduce the parameters (α,β) to $(-1/2,-1/2)$. We here apply another way, but essentially it is the same way. We note the following formula: Let $\Re\alpha > \Re\beta > -1/2$, $s > 0$ and $\Im\lambda > 0$. Then

$$e^{i\lambda s} = C_{\alpha,\beta}(-\lambda)^{-1}\int_s^\infty \Phi_\lambda^{\alpha,\beta}(t)A_{\alpha,\beta}(s,t)dt, \tag{28}$$

where $A_{\alpha,\beta}(s,t)$ is given by

$$\frac{2^{3(\alpha+1/2)+1}\mathrm{sh}2t}{\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}\int_s^t(\mathrm{ch}2t-\mathrm{ch}2w)^{\beta-1/2}(\mathrm{ch}w-\mathrm{ch}s)^{\alpha-\beta-1}\mathrm{sh}w\,dw.$$

(see [2, (2.17)]). In particular, it follows from [2, (3.5), (3.12)] that (28) can be rewritten as

$$e^{i\lambda s} = C_{\alpha,\beta}(-\lambda)^{-1}2^{3(\alpha+1/2)}W_{\alpha-\beta}^1 \circ W_{\beta+1/2}^2(\Phi_\lambda^{\alpha,\beta})(s).$$

Since $(\alpha-\beta)+2(\beta-1/2)=\rho$, Lemma 2.1 (2) and Lemma 3.3 (1) imply that the right hand side is well-defined if $\Im\lambda>0$. Furthermore, it follows from Lemma 3.3 (1) that, if $\Im\lambda$ is sufficiently large, then

$$\Phi_\lambda^{\alpha,\beta}=C_{\alpha,\beta}(-\lambda)2^{-3(\alpha+1/2)}W^2_{-(\beta+1/2)}\circ W^1_{-(\alpha-\beta)}(e^{i\lambda(\cdot)}).$$

Since $\Re\alpha>\Re\beta>-1/2$ means that $\Re(-(\alpha-\beta))<0$ and $\Re(-(\beta+1/2))<0$, $e^{i\lambda x}$ for a sufficiently large $\Im\lambda$ and $f\in C_0^\infty(\mathbf{R})$ satisfy the assumptions on f,g in Proposition 3.4. Thereby, it is easy to see that

$$\begin{aligned}
&C_{\alpha,\beta}(-\lambda)^{-1}\tilde{f}_{\alpha,\beta}(\lambda)\\
=\;&C_{\alpha,\beta}(-\lambda)^{-1}\langle f,\overline{\Phi_\lambda^{\alpha,\beta}}\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)}\\
=\;&2^{-3(\alpha+1/2}\langle f,W^2_{-(\bar\beta+1/2)}\circ W^1_{-(\bar\alpha-\bar\beta)}(e^{-i\lambda(\cdot)})\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)}\\
=\;&2^{-3(\alpha+1/2}\langle\tilde{W}^1_{-(\alpha-\beta)}\circ\tilde{W}^2_{-(\beta+1/2)}(f\Delta_{\alpha,\beta}),e^{-i\lambda(\cdot)}\rangle_{L^2(\mathbf{R}_+,dx)}\\
=\;&2^{-3(\alpha+1/2}\left(\tilde{W}^1_{-(\alpha-\beta)}\circ\tilde{W}^2_{-(\beta+1/2)}(f\Delta_{\alpha,\beta})\right)^\sim_{-1/2,-1/2}(\lambda).
\end{aligned}$$

If, for simplicity, we put

$$W_{\alpha,\beta}=W^1_{\alpha-\beta}\circ W^2_{\beta+1/2}\text{ and }\tilde{W}_{\alpha,\beta}=\tilde{W}^2_{\beta+1/2}\circ\tilde{W}^1_{\alpha-\beta},\tag{29}$$

then we have the following.

Proposition 4.2. *Let $f\in C_0^\infty(\mathbf{R})$ and $\Re\alpha>\Re\beta>-1/2$. Then, as a meromorphic function of $\alpha,\beta,\lambda\in\mathbf{C}$,*

$$\tilde{f}_{\alpha,\beta}(\lambda)=2^{-3(\alpha+1/2)}C_{\alpha,\beta}(-\lambda)\left(\tilde{W}^{-1}_{\alpha,\beta}(f\Delta_{\alpha,\beta})\right)^\sim_{-1/2,-1/2}(\lambda).\tag{30}$$

We shall extend this formula for $f\in\mathcal{A}^{\sigma,m}_{\tau,\eta}(\mathbf{R})$. We recall that $\tilde{W}^{-1}_{\alpha,\beta}$ is a composition of two fractional operators $\tilde{W}^1_{-(\alpha-\beta)}$ and $\tilde{W}^2_{-(\beta+1/2)}$ (see (29)) and these operators change smoothness according to Lemma 3.3 (2). We take $m=N_{\alpha,\beta}$ defined by

$$\begin{cases}
-[-(\beta+1/2)]-[-(\alpha-\beta)] & \text{if }\Re(\beta+1/2)\geq0,\Re(\alpha-\beta)\geq0\\
-[-(\beta+1/2)] & \text{if }\Re(\beta+1/2)\geq0,\Re(\alpha-\beta)<0\\
[-(\beta+1/2)]-[-(\alpha-\beta)] & \text{if }(\Re\beta+1/2)<0,\Re(\alpha-\beta)\geq0\\
0 & \text{if }(\Re\beta+1/2)<0,\Re(\alpha-\beta)<0.
\end{cases}$$

Corollary 4.3. *Let $\alpha,\beta,\tau,\eta\in\mathbf{C}$, $\Re\alpha>-1$, $\Re\tau\geq0$, $\Re\tau>-\Re(\alpha-\beta)-1/2$ and $\Re(\eta+\rho)>\max\{-\Re\rho,-\Re(\alpha-\beta)\}$. Then for $f\in\mathcal{A}^{1,N_{\alpha,\beta}}_{\tau,\eta}(\mathbf{R})$, $\tilde{f}_{\alpha,\beta}(\lambda)$ is holomorphic on $\Im\lambda>\Re(\eta+\rho)$ and satisfies* (30).

Proof. It follows from Lemma 2.1 (2) that $\tilde{f}_{\alpha,\beta}(\lambda)$ is well-defined if $\Re\alpha > -1$, $\Re\tau \geq 0$ and $\Im\lambda > \Re(\eta+\rho)$. On the other hand, we note that

$$f(x)\Delta_{\alpha,\beta}(x) \sim (\mathrm{ch}x)^{\eta+2\rho}(\mathrm{th}x)^{2\tau+2\alpha+1}.$$

Since $\Re(\tau+\alpha+1/2) > \Re(\tau-1/2) \geq -1/2$ and $\Re(\eta+2\rho) > 0$, Lemma 3.3 (2) implies that

$$\tilde{W}^2_{-(\beta+1/2)}(f\Delta_{\alpha,\beta})(x) \sim (\mathrm{ch}x)^{\eta+2\rho-2(\beta+1/2)}(\mathrm{th}x)^{2\tau+2\alpha+1-2(\beta+1/2)}.$$

Since $\Re(\tau+\alpha+1/2-(\beta+1/2)) = \Re(\tau+(\alpha-\beta)) > -1/2$ and $\Re(\eta+2\rho-2(\beta+1/2)) = \Re(\eta+\rho+(\alpha-\beta)) > 0$, Lemma 3.3 (2) again implies that

$$\tilde{W}^1_{-(\alpha-\beta)} \circ \tilde{W}^2_{-(\beta+1/2)}(f\Delta_{\alpha,\beta})(x) \sim (\mathrm{ch}x)^{\eta+\rho}(\mathrm{th}x)^{2\tau}.$$

Therefore, the Euclidean Fourier transform of $\tilde{W}^{-1}_{\alpha,\beta}(f\Delta_{\alpha,\beta})$ is well-defined if $\Im\lambda > \Re(\eta+\rho)$. ■

Remark 4.4. If $-(\beta+1/2)$ and $-(\alpha-\beta)$ are $0,-1,-2,\cdots$, then the condition $\Re(\eta+\rho) > \max\{-\Re\rho, -\Re(\alpha-\beta)\}$ is not necessary.

5. Inversion formula.

Let $\alpha,\beta \in \mathbf{C}$ and $\Re\alpha > -1$. The inversion formula of the Fourier-Jacobi transform $f \to \hat{f}_{\alpha,\beta}$, $f \in C_0^\infty(\mathbf{R})$, is obtained by Flensted-Jensen [3] and Koornwinder [4]. We recall their inversion formula and give a simple proof.

Let $D_{\alpha,\beta}$ denote the set of poles of $C_{\alpha,\beta}(-\lambda)^{-1}$ located in $\Im\lambda \geq 0$:

$$D_{\alpha,\beta} = \{\gamma_m = i(\varepsilon\beta-\alpha-1-2m)\ ;\ m=0,1,2,\cdots,\Im\gamma_m \geq 0\}, \qquad (31)$$

where $\varepsilon = 1$ if $\Re\beta > 0$ and $\varepsilon = -1$ if $\Re\beta \leq 0$. Let $R_{\alpha,\beta}(\gamma_m)$ denote the residue of $C_{\alpha,\beta}(-\lambda)^{-1}$ at γ_m, explicitly given by

$$R_{\alpha,\beta}(\gamma_m) = \frac{(-1)^m 2^{-\rho+(\varepsilon\beta-\alpha-1-2m)}i}{m!\sqrt{\pi}}\frac{\Gamma(\varepsilon\beta-m)}{\Gamma(\varepsilon\beta-\alpha-1-2m)}.$$

Then it follows from [4, Theorems 2.2, 2.3, 2.4] that

Theorem 5.1 *Let $\alpha,\beta \in \mathbf{C}$ and $\nu \in \mathbf{R}$. Suppose that $\Re\alpha > -1$, $\nu \geq 0$, and $\nu > -\Re(\alpha\pm\beta+1)$.*

(1) *For each $f \in C_0^\infty(\mathbf{R})$ and $t > 0$,*

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}_{\alpha,\beta}(\lambda+i\nu)\Phi^{\alpha,\beta}_{\lambda+i\nu}(t)C_{\alpha,\beta}(-\lambda-i\nu)^{-1}d\lambda$$

$$= \frac{\sqrt{2}}{\Gamma(\alpha+1)}\int_0^\infty \hat{f}_{\alpha,\beta}(\lambda)\phi_\lambda^{\alpha,\beta}(t)(C_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda))^{-1}d\lambda$$
$$-\frac{2\sqrt{2}\pi i}{\Gamma(\alpha+1)}\sum_{\gamma\in D_{\alpha,\beta}}\frac{\hat{f}_{\alpha,\beta}(\gamma)}{C_{\alpha,\beta}(\gamma)}\phi_\gamma^{\alpha,\beta}(t)R_{\alpha,\beta}(\gamma).$$

(2) *For each* $f, g \in C_0^\infty(\mathbf{R})$,

$$\begin{aligned}
&\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} \\
= &\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}_{\alpha,\beta}(\lambda+i\nu)\tilde{\overline{g}}_{\alpha,\beta}(\lambda+i\nu)C_{\alpha,\beta}(-\lambda-i\nu)^{-1}d\lambda \\
= &\frac{1}{\sqrt{2\pi}}\int_0^\infty \hat{f}_{\alpha,\beta}(\lambda)\tilde{\overline{g}}_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda)^{-1}d\lambda \\
&-\sqrt{2\pi}i\sum_{\gamma\in D_{\alpha,\beta}}\hat{f}_{\alpha,\beta}(\gamma)\tilde{\overline{g}}_{\alpha,\beta}(\gamma)R_{\alpha,\beta}(\gamma).
\end{aligned}$$

(3) *If* $\alpha, \beta \in \mathbf{R}$ *and* $\alpha \geq \beta > -1/2$*, then* $D_{\alpha,\beta} = \emptyset$ *and for* $f, g \in C_0^\infty(\mathbf{R})$,

$$\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} = \langle \hat{f}_{\alpha,\beta}, \hat{g}_{\alpha,\beta})\rangle_{L^2(\mathbf{R}_+,|C_{\alpha,\beta}(\lambda)|^{-2}d\lambda)}.$$

Proof. We shall give a simple proof based on Proposition 3.4 and the reduction formula in Corollary 4.3. Obviously, it is enough to prove the first equation in (2). We note that $|\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)}| \leq \|f\|_2\|g\|_2$, where $\|\cdot\|_2$ is the $L^2(\mathbf{R}_+, \Delta_{\alpha,\beta}dx)$-norm, and $|\hat{f}_{\alpha,\beta}(\lambda+i\nu)| \leq \|f\|_{L^1(\mathbf{R}_+)}e^{(\nu+2\Re\rho)R}$ if $\mathrm{supp}(f) \subset [-R, R]$ and $|\hat{g}_{\alpha,\beta}(\lambda+i\nu)| \leq \|g\|_{L^1(\mathbf{R}_+)}$ (see Lemma 2.1). Therefore, by using approximation argument, we may suppose that f, g belong to $\mathcal{A}^{1,N}_{\tau,\eta}(\mathbf{R})$ for sufficiently large positive numbers τ and N. We take $\nu > \Re(\eta+\rho) > \max\{-\Re\rho, -\Re(\alpha-\beta)\}$. Hence, Proposition 3.4, (4), (30) and the Plancherel formula for $L^2(\mathbf{R})$ yield that

$$\begin{aligned}
&\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} \\
= &\langle W_{\alpha,\beta}(f), \tilde{W}_{\overline{\alpha},\overline{\beta}}(g\Delta_{\overline{\alpha},\overline{\beta}})\rangle_{L^2(\mathbf{R}_+,dx)} \\
= &\frac{1}{2}\langle W_{\alpha,\beta}(f)e^{\nu x}, \tilde{W}_{\overline{\alpha},\overline{\beta}}(g\Delta_{\overline{\alpha},\overline{\beta}})e^{-\nu x}\rangle_{L^2(\mathbf{R},dx)} \\
= &\frac{1}{\sqrt{2\pi}}\langle \hat{f}_{\alpha,\beta}(\lambda+i\nu), \tilde{g}_{\overline{\alpha},\overline{\beta}}(\lambda-i\nu)C_{\overline{\alpha},\overline{\beta}}(-\lambda+i\nu)^{-1}\rangle_{L^2(\mathbf{R},dx)},
\end{aligned}$$

where $W_{\alpha,\beta}(f)$ and $\tilde{W}_{\overline{\alpha},\overline{\beta}}(g\Delta_{\overline{\alpha},\overline{\beta}})$ in the third line are regarded as even functions on $\mathbf{R}$. ∎

Similarly, we can deduce the following.

Corollary 5.2. *Let $\alpha, \beta, \eta \in \mathbf{C}$ and $\Re\alpha > -1$. Let $\nu \geq 0$ and $\nu > \Re(\eta + \rho) > \max\{-\Re\rho, -\Re(\alpha - \beta)\}$. Then, for all $f \in C_0^\infty(\mathbf{R})$ and $g \in \mathcal{A}_{0,\eta}^{1,0}(\mathbf{R})$,*

$$\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta})} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_{\alpha,\beta}(\lambda + i\nu)\tilde{\bar{g}}_{\alpha,\beta}(\lambda + i\nu)C_{\alpha,\beta}(-\lambda - i\nu)^{-1} d\lambda.$$

6. Analytic continuation.

6.1. We shall consider analytic continuation of the formula in Corollary 5.2. For $\theta \in \mathbf{C}$ let $W_{-\theta}^{\mathbf{R}}$ be the Weyl type fractional operator on $\mathbf{R}_0 = [0, \infty)$ (see (13)) and let $C_{[0,1]}^\theta$ (resp. $C_{[0,1)}^\theta$) denote the space of all functions H on $\mathbf{R}$ such that $\mathrm{supp}(H) \subset [0, 1]$ (resp. $[0, 1)$), $W_{-\theta}^{\mathbf{R}}(H)$ is well-defined, and

$$\sup_{0\leq w\leq 1} |W_{-\theta}^{\mathbf{R}}(H)(w)| \leq c.$$

For $\sigma > 0, \theta, \eta \in \mathbf{C}$ let $\mathcal{B}_{0,\eta}^{\sigma,\theta}(\mathbf{R})$ denote the space consisting of all even functions f on $\mathbf{R}$ of the form

$$f(x) = (\mathrm{ch}\sigma x)^\eta H((\mathrm{ch}\sigma x)^{-1}), \quad H \in C_{[0,1]}^\theta. \tag{32}$$

We note that, if $\mathrm{supp}(H) \subset [0, 1]$, then $\mathrm{supp}(W_{-\theta}^{\mathbf{R}}(H)) \subset [0, 1]$ and thus, if $H \in C_{[0,1]}^\theta$, then $H \in C_{[0,1]}^{\theta'}$ for all θ' such that $\Re\theta' \leq \Re\theta$. When $\Re\theta \geq 0$, if we put $h(t) = H(1/t)$ as in the form of (24), we see that h satisfies (25) with $m = [\theta]$. Hence, if $\Re\theta \geq 0$, then

$$\mathcal{B}_{0,\eta}^{\sigma,\theta}(\mathbf{R}) \subset \mathcal{A}_{0,\eta}^{\sigma,[\theta]}(\mathbf{R}).$$

Let $\mathcal{B}_{\flat,\eta}^{\sigma,\theta}(\mathbf{R})$ denote the set of $f \in \mathcal{B}_{0,\eta}^{\sigma,\theta}(\mathbf{R})$ such that $0 \notin \mathrm{supp}(f)$, that is f is identically zero around 0. We may suppose that $f \in \mathcal{B}_{\flat,\eta}^{\sigma,\theta}(\mathbf{R})$ is of the form

$$f(x) = (\mathrm{ch}\sigma x)^{\eta-1}(\mathrm{sh}\sigma x)H_f((\mathrm{ch}\sigma x)^{-1}), \quad H_f \in C_{[0,1)}^\theta. \tag{33}$$

Obviously, we may suppose that $f \in \mathcal{B}_{0,\eta}^{\sigma,\theta}(\mathbf{R})$ is of the form

$$f = f_0 + f_1, \quad f_0 \in C_0^\infty(\mathbf{R}), \quad f_1 \in \mathcal{B}_{\flat,\eta}^{\sigma,\theta}(\mathbf{R}). \tag{34}$$

6.2. For $\mu = 0, -1, -2, \cdots$, it follows from the definition that

$$\tilde{W}_\mu^\sigma : \mathcal{B}_{\flat,\eta}^{\sigma,\theta}(\mathbf{R}) \to \mathcal{B}_{\flat,\eta+\mu}^{\sigma,\theta+\mu}(\mathbf{R}) \tag{35}$$

and $\mathcal{B}^{\sigma,\theta}_{0,\eta}(\mathbf{R}) \to \mathcal{B}^{\sigma,\theta+\mu}_{\mu,\eta+\mu}(\mathbf{R})$ (see Lemma 3.3 (2)). For general $\mu \in \mathbf{C}$ we have the following.

Lemma 6.1. *Let $\mu \in \mathbf{C}$ and $f \in \mathcal{B}^{\sigma,\theta}_{\flat,\eta}(\mathbf{R})$.*
(1) *If $\Re\mu > 0$ and H_f satisfies*

$$W^{\mathbf{R}}_{\mu}\left(s^{-(\eta+\mu-\theta)}W^{\mathbf{R}}_{-\theta}(H_f)\right)(w) = O(w^{-(\eta-\theta)}) \quad \text{if } \Re(\eta-\theta) \leq 0,$$

then $\tilde{W}^{\sigma}_{\mu}(f) \in \mathcal{B}^{\sigma,\theta}_{\flat,\eta+\mu}(\mathbf{R})$.
(2) *If $\Re\mu \leq 0$ and H_f satisfies the above conditions replaced μ, θ and η with $\mu-[\mu]$, $\theta+[\mu]$ and $\eta+[\mu]$ respectively, then $\tilde{W}^{\sigma}_{\mu}(f) \in \mathcal{B}^{\sigma,\theta+[\mu]}_{\flat,\eta+\mu}(\mathbf{R})$.*

Proof. Let $\Re\mu > 0$ and f be of the form (33). Clearly, if f is identically zero around 0, then $\tilde{W}^{\sigma}_{\mu}(f)$ is also identically zero around 0. Letting $\mathrm{ch}\sigma x = t$ in (20), we see that

$$\begin{aligned}
\tilde{W}^{\sigma}_{\mu}(f)(x) &= c\int_1^t s^{\eta-1}H_f(s^{-1})(t-s)^{\mu-1}ds \cdot \sqrt{t^2-1} \\
&= ct^{\eta+\mu-1}\sqrt{t^2-1}\int_{1/t}^1 s^{\eta-1}H_f(1/ts)(1-s)^{\mu-1}ds \\
&= ct^{\eta+\mu-1}\sqrt{t^2-1}H_1(t^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
H_1(w) &= \int_w^1 s^{\eta-1}H_f(w/s)(1-s)^{\mu-1}ds \\
&= \int_1^\infty s^{-(\eta+\mu)}H_f(ws)(s-1)^{\mu-1}ds \\
&= w^\eta \int_w^\infty s^{-(\eta+\mu)}H_f(s)(s-w)^{\mu-1}ds \\
&= w^\eta W^{\mathbf{R}}_{\mu}(s^{-(\eta+\mu)}H_f)(w).
\end{aligned}$$

Since $H_f \in C^{\theta}_{[0,1)}$ and $W^{\mathbf{R}}_{-\gamma}(H_f(ws)) = s^\gamma W^{\mathbf{R}}_{-\gamma}(H_f)(ws)$, $\Re\gamma \leq \Re\theta$, as a function of w, it follows that

$$\begin{aligned}
W^{\mathbf{R}}_{-\theta}(H_1)(w) &= \int_1^\infty s^{-(\eta+\mu-\theta)}W^{\mathbf{R}}_{-\theta}(H_f)(ws)(s-1)^{\mu-1}ds \\
&= w^{\eta-\theta}W^{\mathbf{R}}_{\mu}\left(s^{-(\eta+\mu-\theta)}W^{\mathbf{R}}_{-\theta}(H_f)\right)(w). \qquad (36)
\end{aligned}$$

Therefore, if $\Re(\eta-\theta) > 0$ and $W^{\mathbf{R}}_{-\theta}(H_f)$ is bounded, then Lemma 3.1 (1) yields that $W^{\mathbf{R}}_{-\theta}(H_1)$ is bounded. On the other hand, if $\Re(\eta-\theta) \leq 0$, then the assumption on H_f also yields that $W^{\mathbf{R}}_{-\theta}(H_1)$ is bounded. Hence, $H_1 \in C^{\theta}_{[0,1)}$

and the desired result follows. Let $\Re\mu \le 0$. When $\mu = 0, -1, -2, \cdots$, the assertion is nothing but (35). Otherwise, since $\tilde{W}^\sigma_\mu = \tilde{W}^\sigma_{\mu-[\mu]} \circ \tilde{W}^\sigma_{[\mu]}$, the desired result follows from (35) and the first case. ■

Corollary 6.2. *Let* $\mu \in \mathbf{C}$ *and* $f \in \mathcal{B}^{\sigma,\eta}_{\flat,\eta}(\mathbf{R})$.

(1) *If* $\Re\mu < 0$, *then* $\tilde{W}^\sigma_\mu(f) \in \mathcal{B}^{\sigma,\eta+\mu}_{\flat,\eta+\mu}(\mathbf{R})$.

(2) *If* $\Re\mu = 0$, *then* $\tilde{W}^\sigma_\mu(f) \in \mathcal{B}^{\sigma,\eta+\mu-\delta}_{\flat,\eta+\mu}(\mathbf{R})$ *for* $\delta > 0$.

(3) *If* $\Re\mu > 0$ *and* $W^{\mathbf{R}}_{-\eta}(H_f)(w) = O(w^{\mu-\delta_1})$, *then* $\tilde{W}^\sigma_\mu(f) \in \mathcal{B}^{\sigma,\eta+\mu-\delta_2}_{\flat,\eta+\mu}(\mathbf{R})$, *where* $\delta_2 > \delta_1 \ge 0$ *or* $\delta_1 = \delta_2 = 0$.

Proof. (3) Let $\Re\mu > 0$ and suppose that $W^{\mathbf{R}}_{-\eta}(H_f)(w) = O(w^\mu)$. Then, letteing $\theta = \eta$ in (36), it follows that

$$W^{\mathbf{R}}_{-\eta}(H_1) = W^{\mathbf{R}}_\mu(s^{-\mu} W^{\mathbf{R}}_{-\eta}(H_f))$$

and thus,

$$W^{\mathbf{R}}_{-(\eta+\mu)}(H_1)(w) = w^{-\mu} W^{\mathbf{R}}_{-\eta}(H_f)(w). \tag{37}$$

Hence $H_1 \in C^{\eta+\mu}_{[0,1)}$ follows. When $W^{\mathbf{R}}_{-\eta}(H_f)(w) = O(w^{\mu-\delta_1})$, (37) is replaced with $W^{\mathbf{R}}_{-(\eta+\mu-\delta_2)}(H_1) = W^{\mathbf{R}}_{\delta_2}(s^{-\mu} W^{\mathbf{R}}_{-\eta}(H_f))$. Therefore, $H_1 \in C^{\eta+\mu-\delta_2}_{[0,1)}$ provided $\delta_2 > \delta_1$. (1) Let $\Re\mu < 0$. When $\mu = -1, -2, \cdots$, the assertion is obvious from (35). We may suppose that $\mu \ne -1, -2, \cdots$. Because of (35) and $\tilde{W}^\sigma_\mu = \tilde{W}^\sigma_{\mu-[\mu]-1} \circ \tilde{W}^\sigma_{[\mu]+1}$, we may suppose that $-1 < \Re\mu < 0$. Then it is easy to see that

$$\begin{aligned} \tilde{W}^\sigma_\mu(f)(x) &= c\frac{d}{dx}\left(t^{\eta+\mu} G_{\eta,\mu}(t^{-1})\right) \cdot \sqrt{t^2-1} \\ &= ct^{\eta+\mu-1}\sqrt{t^2-1}((\eta+\mu)G_{\eta,\mu}(t^{-1}) - t^{-1}G'_{\eta,\mu}(t^{-1})), \end{aligned}$$

where $\mathrm{ch}\sigma x = t$ and

$$G_{\eta,\mu}(w) = w^\eta W^{\mathbf{R}}_{\mu+1}(s^{-(\eta+\mu+1)} H_f))(w).$$

Therefore,

$$H_1(w) = (\eta+\mu)G_{\eta,\mu}(w) - wG'_{\eta,\mu}(w) = \mu G_{\eta,\mu}(w) - G_{\eta+1,\mu-1}(w). \tag{38}$$

Let $G = G_{\eta,\mu}$. As before, $W^{\mathbf{R}}_{-\eta}(G) = W^{\mathbf{R}}_{\mu+1}(s^{-(\mu+1)} W^{\mathbf{R}}_{-\eta}(H_f))$ and thus, $W^{\mathbf{R}}_{-(\eta+\mu)}(G) = W^{\mathbf{R}}_1(s^{-(\mu+1)} W^{\mathbf{R}}_{-\eta}(H_f))$. Since $\Re(\mu+1) > 0$, $\mathrm{supp}(H_f) \subset [0,1)$ and $W^{\mathbf{R}}_{-\eta}(H_f)$ is bounded, Lemma 3.1 means that $W^{\mathbf{R}}_{-(\eta+\mu)}(G)$ is bounded. Let $G = G_{\eta+1,\mu-1}$. Then the same process yields that $W^{\mathbf{R}}_{-(\eta+\mu)}(G)$

$= W_1^{\mathbf{R}}(s^{-\mu} W_{-1}^{\mathbf{R}} \circ W_{-\eta}^{\mathbf{R}}(H_f))$. This function is again bounded. Hence, $H_1 \in C_{[0,1)}^{\eta+\mu}$. (2) The case of $\Re\mu = 0$ follows from the same process in (1) replaced $\eta + \mu$ with $\eta + \mu - \delta$ and $W_1^{\mathbf{R}}$ with $W_{1+\delta}^{\mathbf{R}}$ respectively. ■

6.3. Now we shall consider analytic continuation of $\tilde{g}_{\alpha,\beta}(\lambda)$ in Corollary 5.2 provided $g \in \mathcal{B}_{0,\eta/2}^{2,\theta}(\mathbf{R})$ where θ will be suitably determined. We recall (34). When $g \in C_0^\infty(\mathbf{R})$, Lemma 2.1 (2) and the fact that

$$\Delta_{\alpha,\beta}(x) = (\mathrm{ch} x)^{2\rho}(\mathrm{th} x)^{2\alpha+1}$$

has zero of order $2\alpha + 1$ at $x = 0$ mean that, if $\Re\alpha > -1$, $\tilde{g}_{\alpha,\beta}(\lambda)$ is a holomorphic function on $\Im\lambda \geq 0$ of exponential type (see §5). Therefore, it is enough to consider the analytic continuation of $\tilde{g}_{\alpha,\beta}(\lambda)$ for $g \in \mathcal{B}_{\flat,\eta/2}^{2,\theta}(\mathbf{R})$. Since g is identically zero around 0, Corollary 4.3 yields that, if $\Re\alpha > -1$, then $\tilde{g}_{\alpha,\beta}(\lambda)$ is holomorphic on $\Im\lambda > \Re(\eta+\rho)$ and $2^{3(\alpha+1/2)}\tilde{g}_{\alpha,\beta}(\lambda)\, C_{\alpha,\beta}(-\lambda)^{-1}$ is the Euclidean Fourier transform of

$$\tilde{W}_{-(\alpha-\beta)}^1(\tilde{W}_{-(\beta+1/2)}^2(g\Delta_{\alpha,\beta})).$$

In the following, let $\Re\alpha > -1, \epsilon > 0$ and

$$f = g\Delta_{\alpha,\beta}.$$

Obviously, $f \in \mathcal{B}_{\flat,\eta/2+\rho}^{2,\theta}(\mathbf{R})$ and is of the form

$$f(x) = (\mathrm{ch}2x)^{\eta/2+\rho-1}(\mathrm{sh}2x)H_f((\mathrm{ch}2x)^{-1}), \quad H_f \in C_{[0,1)}^\theta. \tag{39}$$

We here take θ as

$$\theta_{\alpha,\beta}^\eta = \frac{\eta}{2} + \rho \tag{40}$$

and assume that,

$$\text{if } -\Re(\beta + 1/2) > 0, \text{ then } W_{-(\eta/2+\rho)}^{\mathbf{R}}(H_f)(w) = O(w^{-(\beta+1/2)}). \tag{41}$$

Then, by taking η and μ in Corollary 6.2 as $\theta_{\alpha,\beta}^\eta = \eta/2 + \rho$ and $-(\beta + 1/2)$ respectively, it follows that for a sufficiently small $\epsilon_1 > 0$

$$\tilde{W}_{-(\beta+1/2)}^2(f) \in \mathcal{B}_{\flat,\eta_1}^{2,\eta_1-\epsilon_1}(\mathbf{R}),$$

where $\eta_1 = (\eta + \rho + (\alpha - \beta))/2$. This means that $\tilde{W}_{-(\beta+1/2)}^2(f)$ is of the form

$$\tilde{W}_{-(\beta+1/2)}^2(f)(x) = (\mathrm{ch}2x)^{\eta_1-1}(\mathrm{sh}2x)H_f^1(\mathrm{ch}^{-1}2x), \quad H_f^1 \in C_{[0,1)}^{\eta_1-\epsilon_1}.$$

We here rewrite this function as

$$\tilde{W}^2_{-(\beta+1/2)}(f)(x) = (\mathrm{ch}x)^{2\eta_1-1}(\mathrm{sh}x)H^2_f(\mathrm{ch}^{-2}x), \quad H^2_f \in C^{\eta_1-\epsilon_1}_{[0,1)}, \tag{42}$$

where

$$H^2_f(w) = 2(2-w)^{\eta_1}H^1_f(w/(2-w)). \tag{43}$$

Before applying $\tilde{W}^1_{-(\alpha-\beta)}$ to $\tilde{W}^2_{-(\beta+1/2)}(f)$, we prepare the following lemma.

Lemma 6.3. *Let $\mu \in \mathbf{C}, \epsilon > 0$ and f be of the form*

$$f(x) = (\mathrm{ch}x)^{2\eta-1}(\mathrm{sh}x)H(\mathrm{ch}^{-2}x), \quad H \in C^{\eta-\epsilon}_{[0,1)}. \tag{44}$$

(1) *If $\Re\mu < 0$, then $\tilde{W}^1_\mu(f)$ is of the form*

$$\tilde{W}^1_\mu(f)(x) = (\mathrm{ch}x)^{2\eta+\mu-1}(\mathrm{sh}x)H_1(\mathrm{ch}^{-2}x), \quad H_1 \in C^{\eta+\mu/2-\epsilon}_{[0,1)}. \tag{45}$$

(2) *If $\Re\mu = 0$, then (45) holds with $H_1 \in C^{\eta+\mu/2-\delta}_{[0,1)}$ for $\delta > \epsilon$.*

(3) *If $\Re\mu > 0$ and $W^{\mathbf{R}}_\mu(W^{\mathbf{R}}_{-(\mu/2+\eta)}(H)(s^2))(w) = O(w^{\mu-\delta_1})$, then (45) holds with $H_1 \in C^{\eta+\mu/2-\delta_2}_{[0,1)}$, where $\delta_2 > \delta_1 \geq 0$ or $\delta_1 = \delta_2 = 0$.*

Proof. We repeat the similar arguments in the proof of Corollary 6.2. (3) Let $\Re\mu > 0$ and suppose that $W^{\mathbf{R}}_\mu(W^{\mathbf{R}}_{-(\eta+\mu/2)}(H)(s^2))(w) = O(w^\mu)$. From the proof of Lemma 6.1, letting $\mathrm{ch}^{-2}x = w$, it follows that

$$\begin{aligned} H_1(w) &= c\int_{\sqrt{w}}^1 s^{2\eta-1}H(w/s^2)(1-s)^{\mu-1}ds \\ &= c\int_0^1 s^{2\eta-1}H(w/s^2)(1-s)^{\mu-1}ds \\ &= cw^\eta W^{\mathbf{R}}_\mu(s^{-(2\eta+\mu)}\tilde{H}))(\sqrt{w}), \end{aligned}$$

where $\tilde{H}(w) = H(w^2)$. Since $W^{\mathbf{R}}_{-\gamma}(H(w/s^2))(w) = cs^{-2\gamma}W^{\mathbf{R}}_{-\gamma}(H)(w/s^2)$, $\Re\gamma \leq \Re\eta$, as a function of w, we see that

$$\begin{aligned} W^{\mathbf{R}}_{-(\eta+\mu/2)}(H_1)(w) &= c\int_0^1 s^{-\mu}W^{\mathbf{R}}_{-(\eta+\mu/2)}(H)(w/s^2)(1-s)^{\mu-1}ds \\ &= cw^{-\mu/2}W^{\mathbf{R}}_\mu((W^{\mathbf{R}}_{-(\eta+\mu/2)}(H))^\sim)(\sqrt{w}). \end{aligned} \tag{46}$$

Hence $H_1 \in C^{\eta+\mu/2}_{[0,1)}$ follows. The case of $\delta_2 > \delta_1 \geq 0$ also follows as in (3) of Corollary 6.2. (1) Let $\Re\mu < 0$. As before, we may assume that $-1 < \Re\mu < 0$.

Then, using (45) replaced μ with $\mu+1$, we can repeat the proof of Corollary 6.2. Actually, $G_{\eta,\mu}$ is replaced by

$$w^{\eta} W^{\mathbf{R}}_{\mu+1}\left(s^{-(2\eta+\mu+1)}\tilde{H}\right)(\sqrt{w})$$

and $G_{\eta+1,\mu-1}$ by $G_{\eta+1/2,\mu-1}$. Hence, applying $W^{\mathbf{R}}_{-(\eta+\mu/2-\epsilon)} = W^{\mathbf{R}}_{1/2}\circ W^{\mathbf{R}}_{-(\mu+1)/2}\circ W^{\mathbf{R}}_{-(\eta-\epsilon)}$ to these functions, the desired result similarly follows as in Corollary 6.2. (2) The case of $\Re\mu = 0$ also follows from the above argument. ■

We apply $W^{\mathbf{R}}_{-(\alpha-\beta)}$ to $\tilde{W}^2_{-(\beta+1/2)}(f)$ (see (42)) under the assumption that,

$$\text{if } -\Re(\alpha-\beta) > 0, \text{ then } W^{\mathbf{R}}_{-(\alpha-\beta)}(W^{\mathbf{R}}_{-(\eta+\rho)/2}(H^2_f)(s^2))(w) = O(w^{-(\alpha-\beta)}). \quad (47)$$

Then, by taking η, μ and H in Lemma 6.3 as $(\eta+\rho+(\alpha-\beta))/2$, $-(\alpha-\beta)$ and H^2_f respectively, it follows that for a sufficiently small $\epsilon_2 > 0$,

$$\tilde{W}^1_{-(\alpha-\beta)}(\tilde{W}^2_{-(\beta+1/2)}(f))(x) = (\mathrm{ch}x)^{\eta_2-1}(\mathrm{sh}x)H^3_f(w^2), \quad H^3_f \in C^{\eta_2/2-\epsilon_2}_{[0,1)}, \quad (48)$$

where $w = \mathrm{ch}^{-1}x$ and $\eta_2 = \eta+\rho$.

Remark 6.4. (1) If $-\Re(\beta+1/2) \le 0$ and $-\Re(\alpha-\beta) \le 0$, then no extra conditions on zero of H_f and H^2_f (see (41) and (47)) are required.

(2) If $-\Re(\beta+1/2) > 0$ and $-\Re(\alpha-\beta) > 0$, then the both extra conditions on zero of H_f and H^2_f are required. However, the extra condition on zero of H^2_f means the one of H_f. First we note that (37) implies that

$$W^{\mathbf{R}}_{-(\eta/2+\rho)}(H_f)(w) = w^{-(\beta+1/2)} W^{\mathbf{R}}_{\theta}(H^1_f)(w),$$

where $\theta = -(\eta+\rho+(\alpha-\beta))/2$. Thereby, if $W^{\mathbf{R}}_{\theta}(H^1_f)$ is bounded, then $W^{\mathbf{R}}_{-(\eta/2+\rho)}(H_f)$ has zero of order $-(\beta+1/2)$ at $w=0$ and thus, the extra condition on zero of H_f follows. Now, let us suppose the extra condition on zero of H^2_f: $W^{\mathbf{R}}_{-(\alpha-\beta)}(W^{\mathbf{R}}_{-(\eta+\rho)/2}(H^2_f)(s^2))(w) = O(w^{-(\alpha-\beta)})$. We denote this function by $h(w)$. Then $H^2_f = W^{\mathbf{R}}_{(\eta+\rho)/2}(W^{\mathbf{R}}_{(\alpha-\beta)}(h)(\sqrt{s}))$ and thus

$$W^{\mathbf{R}}_{\theta}(H^2_f) = W^{\mathbf{R}}_{-(\alpha-\beta)/2}(W^{\mathbf{R}}_{(\alpha-\beta)}(h)(\sqrt{s})).$$

Hence, from Lemma 3.1 (1) it is easy to see that the right hand side is bounded. Similarly, $W^{\mathbf{R}}_{\theta-\delta}(H^2_f)$ is bounded for $\delta > 0$ such that $-\Re(\alpha-\beta)/2-\delta > 0$. Let $\theta = -n+\gamma$, $n = 0,1,2,\cdots$ and $0 < \Re\gamma \le 1$. Since $H^1_f(w) = G(w)H^2_f(w/(2-w))$ for $G \in C^{\infty}_c$ (see (43)) and $W^{\mathbf{R}}_{\theta} = W^{\mathbf{R}}_{\gamma}\circ W^{\mathbf{R}}_{-n}$, it follows that

$$\begin{aligned} W_{\theta}(H^1_f) &\sim \sum_{k=0}^{n} W^{\mathbf{R}}_{\gamma}(G_k W^{\mathbf{R}}_{-k}(H^2_f)) \\ &= \sum_{k=0}^{n} W^{\mathbf{R}}_{\gamma}(G_k W^{\mathbf{R}}_{-\gamma+\delta+(n-k)}\circ W^{\mathbf{R}}_{\theta-\delta}(H^2_f)), \end{aligned}$$

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where $G_k \in C_c^\infty$. Therefore, since $W^{\mathbf{R}}_{\theta-\delta}(H_f^2)$ is bounded, Lemma 3.1 (1) implies that $W_\theta(H_f^1)$ is bounded as desired.

(3) From (39) and (42) H_f and H_f^2 can be written as

$$\begin{aligned} H_f(w) &= f((\mathrm{arccosh}\ w^{-1})/2)\frac{w^{\eta/2+\rho}}{\sqrt{1-w^2}}, \\ H_f^2(w) &= \tilde{W}^2_{-(\beta+1/2)}(f)(\mathrm{arccosh}\ w^{-1/2})\frac{w^{(\eta+\rho+(\alpha-\beta))/2}}{\sqrt{1-w}}. \end{aligned}$$

Since $2^{3(\alpha+1/2)}\tilde{g}_{\alpha,\beta}(\lambda)\ C_{\alpha,\beta}(-\lambda)^{-1}$ is the Euclidean Fourier transform of (48), in order to carry out the analytic continuation of $\tilde{g}_{\alpha,\beta}(\lambda)\ C_{\alpha,\beta}(-\lambda)^{-1}$, it is enough to consider $\hat{F}(\lambda)$ of

$$F(x) = (\mathrm{ch}x)^{\gamma-1}(\mathrm{sh}x)H(w^2), \quad H \in C^{\gamma/2-\epsilon}_{[0,1)},\ \Re\gamma > 0.$$

For simplicity, put $\theta = \gamma/2-\epsilon$. Since $e^{i\lambda x} = (\mathrm{ch}x)^{i\lambda}(1+\mathrm{th}x)^{i\lambda}$, by changing the variable as $w = \mathrm{ch}^{-1}x$, it follows that for $\Im\lambda > \Re\gamma$,

$$\begin{aligned} \hat{F}(\lambda) &= \int_0^1 H(w^2)\left(1+\sqrt{1-w^2}\right)^{i\lambda} w^{-i(\lambda-i\gamma)-1}dw, \\ &= \int_0^1 I(w)w^{-i(\lambda-i\gamma)/2-1}dw. \end{aligned}$$

Here, $I \in C^\theta_{[0,1)}$. Then, applying Proposition 3.4 with $\alpha=\beta=-1/2$ and (14), we see that

$$\begin{aligned} \hat{F}(\lambda) &= \int_0^1 W^{\mathbf{R}}_{-\theta}(I)(w)\tilde{W}^{\mathbf{R}}_\theta(w^{-i(\lambda-i\gamma)/2-1})dw \\ &= \frac{\Gamma(-i(\lambda-i\gamma)/2)}{\Gamma(-i(\lambda-i\gamma)/2)+\theta+1)}\int_0^1 W^{\mathbf{R}}_{-\theta}(I)(w)w^{-i(\lambda-i\gamma)/2+\theta}dw \end{aligned}$$

Since $I \in C^\theta_{[0,1)}$ and $-i(\lambda-i\gamma)/2+\theta = -i\lambda-\epsilon$, this integral is bounded if $\Im\lambda > -1+\epsilon$. Therefore, $\hat{F}(\lambda)$ has a meromorphic extension in $\Im\lambda \geq 0$ with simple poles lie in

$$F_\gamma = \{\xi_m = i(\gamma-2m)\ ;\ m=0,1,2,\cdots,\Im\xi_m \geq 0\}$$

and

$$\mathrm{Res}_{\lambda=\xi_m}(\hat{F}(\lambda)) = \frac{(-1)^m}{m!\Gamma(-m+\theta+1)}\int_0^1 W^{\mathbf{R}}_{-\theta}(I)(w)w^{-m+\theta}dw.$$

Finally, noting (41), (47) and Remark 6.4 (2), we have the following.

Theorem 6.5. *Let* $\alpha, \beta, \eta \in \mathbf{C}$, $\Re\alpha > -1$ *and* $g \in \mathcal{B}^{2,\eta/2+\rho}_{0,\eta/2}(\mathbf{R})$. *We suppose that there exists a decomposition* $g = g_0 + g_1$, $g_0 \in C_0^\infty(\mathbf{R})$ *and* $g_1 \in \mathcal{B}^{2,\eta/2+\rho}_{\flat,\eta/2}(\mathbf{R})$, *such that* $f = g_1\Delta_{\alpha,\beta}$ *satisfies that, if* $-\Re(\alpha-\beta) > 0$, *then* $W^{\mathbf{R}}_{-(\alpha-\beta)}(W^{\mathbf{R}}_{-(\eta+\rho)/2}(H_f^2)(s^2))(w) = O(w^{-(\alpha-\beta)})$ *and, if* $-\Re(\alpha-\beta) \le 0$ *and* $-\Re(\beta+1/2) > 0$, *then* $W^{\mathbf{R}}_{-(\eta/2+\rho)}(H_f)\ (w) = O(w^{-(\beta+1/2)})$. *Then, for all* $\phi \in C_0^\infty(\mathbf{R})$,

$$\begin{aligned}\langle\phi, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\phi}_{\alpha,\beta}(\lambda)\tilde{\overline{g}}_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-(\lambda))^{-1}d\lambda \\ &\quad -\sqrt{2\pi}i\sum_{D_{\alpha,\beta}\cup F_{\eta+\rho}}\hat{\phi}_{\alpha,\beta}(\gamma)\mathrm{Res}_{\lambda=\gamma}\left(\tilde{\overline{g}}_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda)^{-1}\right),\end{aligned}$$

where we supposed that $(D_{\alpha,\beta}\cup F_{\eta+\rho})\cap\mathbf{R} = \emptyset$. *All poles appered in the second sum are simple. If* $g_0 = 0$ *(resp.* $g_1 = 0$*), then the second sum corresponding to* $D_{\alpha,\beta}$ *(resp.* $F_{\eta+\rho}$*) vanishes.*

Remark 6.6. (1) If $-\Re(\alpha-\beta) \le 0$ and $-\Re(\beta+1/2) \le 0$, then there are no assumptions on f and $D_{\alpha,\beta} = \emptyset$. This case perfectly coincides with (3) of Theorem 5.1.

(2) In [5] the analytic continuation of $\tilde{g}_{\alpha,\beta}(\lambda)$ is also calculated directly; the poles of $\tilde{g}_{\alpha,\beta}(\lambda)$ lie in $F_{\eta+\rho}$ and if $D_{\alpha,\beta}\cap F_{\eta+\rho} \ne \emptyset$, then $\tilde{g}_{\alpha,\beta}(\lambda)C(-\lambda)^{-1}$ has double poles. However, in Theorem 6.5, no double poles appear, because we use the reduction formula in Corollary 4.3 and we assume the extra conditions on zero of H_f and H_f^2.

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On the Lusin area function and the Littlewood-Paley g function on real rank 1 semisimple Lie groups

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Abstract

Let G be a real rank one connected semisimple Lie group with finite center. Using the spherical Fourier transform and the classical one, we shall consider a pull back on G of $H^1(\mathbf{R})$ and introduce a real Hardy space $W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ on G as a subspace of $L^1(G//K)$. We also define the Lusin area function $S_+(f)$ and the Littlewood-Paley g function $g(f)$ on G as analogues of the classical theory. We show that S_+ and g are bounded from $W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ to $L^1(G//K)$.

1. Notation. Let G be a real rank one connected semisimple Lie group with finite center and $G = KAN = K\overline{A^+}K$ respectively an Iwasawa and the Cartan decompositions of G. Let $\mathfrak{a}$ be the Lie algebra of A and $\mathcal{F} = \mathfrak{a}^*$ the dual space of $\mathfrak{a}$. Let γ be the positive simple root of (G, A) determined by N and H the unique element in $\mathfrak{a}$ satisfying $\gamma(H) = 1$. Let m_1 and m_2 denote the multiplicities of γ and 2γ respectively. We put

$$\alpha = \frac{m_1 + m_2 - 1}{2}, \quad \beta = \frac{m_2 - 1}{2}, \quad \rho = \alpha + \beta + 1.$$

We parameterize each element in A, $\mathfrak{a}$, and $\mathcal{F}$ as $a_x = \exp(xH)$, xH, and $x\gamma$ ($x \in \mathbf{R}$) respectively, and identify A, $\mathfrak{a}$, and $\mathcal{F}$ with $\mathbf{R}$. In this paper we shall treat only K-bi-invariant functions on G. Since $A^+ = \{a_x; x > 0\}$, all K-bi-invariant functions can be identified with even functions on $\mathbf{R}$.

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Let $dg = e^{2\rho x} dk dx dn = \Delta(x) dk dx dk'$ denote the decompositions of a Haar measure dg on G respectively corresponding to the Iwasawa and Cartan decompositions of G, where dk, dx, dn denote Haar measures on K, A, N respectively, and $\Delta(x)$, $x \geq 0$, is explicitly given as

$$\Delta(x) = 2^{2\rho}(\mathrm{sh}x)^{2\alpha+1}(\mathrm{ch}x)^{2\beta+1}.$$

We extend this function on $\mathbf{R}_+$ as an even function on $\mathbf{R}$. Let $L^p(G//K)$ denote the space of K-bi-invariant functions on G with finite L^p-norm: $\|f\|_p = \left(\int_0^\infty |f(x)|^p \Delta(x)dx\right)^{1/p}$ and $L^1_{\mathrm{loc}}(G//K)$ the space of locally integrable, K-bi-invariant functions on G. Let $C_c^\infty(G//K)$ be the space of compactly supported C^∞, K-bi-invariant functions on G. We denote by $\hat{f}$ the spherical Fourier transform of f and by $f * h$ the convolution of f, h in $L^1(G//K)$ (cf. [**2**], [**9**, Chap.9]). Similarly, we denote by $\tilde{F}$ and $F \circledast H$ the Euclidean Fourier transform of F and the convolution of F, H in $L^1(\mathbf{R})$ respectively.

2. Real Hardy spaces. We shall introduce a real Hardy space on G by using a radial maximal function on G. Let ϕ be a positive compactly supported C^∞, K-bi-invariant function on G with $\|\phi\|_1 = 1$. We define the dilation ϕ_t, $t > 0$, of ϕ as

$$\phi_t(x) = \frac{1}{t}\frac{1}{\Delta(x)}\Delta\left(\frac{x}{t}\right)\phi\left(\frac{x}{t}\right). \tag{1}$$

Since this dilation has the same properties as in the Euclidean case; $\|\phi_t\|_1 = \|\phi\|_1$ and $\{\phi_t; t > 0\}$ approximates the identity in $L^p(G//K)$, $0 < p \leq \infty$, it is quite natural to introduce a radial maximal function $M_\phi f$ on G as

$$(M_\phi f)(g) = \sup_{0<t<\infty} |(f * \phi_t)(g)|, \quad g \in G.$$

As shown in [**3**, Theorem 3.4], this maximal operator M_ϕ satisfies the so-called maximal theorem: M_ϕ is bounded on $L^p(G//K)$ $(1 < p \leq \infty)$ and satisfies the weak type L^1 estimate. Analogously as the definition of the real Hardy space $H^1(\mathbf{R})$ on $\mathbf{R}$, we define the real Hardy space on G by

$$H^1(G//K) = \{f \in L^1_{\mathrm{loc}}(G//K)\ ;\ M_\phi f \in L^1(G//K)\}$$

and the norm by $\|f\|_{H^1(G)} = \|M_\phi f\|_1$. Then $H^1(G//K) \subset L^1(G//K)$.

For $f \in C_c^\infty(G//K)$, we define the Abel transform F_f^s, $s \in \mathbf{R}$, of f as

$$F_f^s(x) = e^{\rho(1+s)x}\int_N f(a_x n)dn.$$

For simplicity, we put $W_+(f) = F_f^1$ and we denote by W_- the inverse operator of W_+. Explicitly forms of $W_\pm$ are given by a composition of the Weyl type fractional integral transforms (see [**6**, §6]). We recall (cf. [**6**, (3.7)]) that both $\hat{f}$ and $(F_f^s)^\sim$ are holomorphic functions on $\mathbf{C}$ of exponential type and

$$\hat{f}(\lambda + is\rho) = (F_f^s)^\sim(\lambda), \quad \lambda \in \mathbf{C}. \tag{2}$$

Let $C(\lambda)$ denote Harish-Chandra's C-function (cf. [**6**, (2.6)]) and $\mathcal{M}_{C_\rho}$ the Euclidean Fourier multiplier corresponding to $C_\rho(\lambda) = C(-(\lambda + i\rho))$, that is, $\mathcal{M}_{C_\rho}(F)^\sim(\lambda) = C(-(\lambda + i\rho))\tilde{F}(\lambda)$. We define

$$W_-(H^1(\mathbf{R})) = \{f \in L^1_{\text{loc}}(G//K) \; ; \; W_+(f) \in H^1(\mathbf{R})\}$$

and the norm by $\|f\|_{W_-} = \|W_+(f)\|_{H^1(\mathbf{R})}$. We also define $W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ by replacing the condition that $W_+(f) \in H^1(\mathbf{R})$ in the above definition with $\mathcal{M}_{C_\rho}^{-1} \circ W_+(f) \in H^1(\mathbf{R})$ and the norm by $\|f\|_{W_-^{C_\rho}} = \|\mathcal{M}_{C_\rho}^{-1} \circ W_+(f)\|_{H^1(\mathbf{R})}$.

We note that $f * \phi_t = W_-(W_+(f * \phi_t)) = W_-(F \circledast W_+(\phi_t))$, where $F = W_+(f)$. Hence the H^1-norm $\|f\|_{H^1(G)}$ of f on G is related to an L^1-norm of $F = W_+(f)$ on $\mathbf{R}$. Actually, let $\alpha - \beta = [\alpha - \beta] + \delta$ and $\beta + 1/2 = [\beta + 1/2] + \delta'$, where $[\ ]$ is the Gauss symbol, and set $\underline{n} = [\alpha - \beta] + [\beta - 1/2]$ and $\underline{D} = \{0, \delta, \delta', \delta + \delta'\}$. Then it follows from [**5**, Theorem 4.6] that

$$\|f\|_{H^1(G)} \sim \sum_{m=0}^{\underline{n}} \sum_{\xi \in \underline{D}} \|M_\phi^{\mathbf{R}} \circ W_{-(m+\xi)}^{\mathbf{R}}(F)(x)(\text{th}x)^{m+\xi}\|_{L^1(\mathbf{R})}, \tag{3}$$

where $W_{-\gamma}^{\mathbf{R}}$ is the Weyl type fractional integral transform on $\mathbf{R}$ and $M_\phi^{\mathbf{R}}$ is the maximal operator on $\mathbf{R}$ defined by

$$(M_\phi^{\mathbf{R}} F)(x) = \sup_{0<t<\infty} |(F \circledast W_+(\phi_t)(x)|, \quad x \in \mathbf{R}.$$

From the equivalence (3) it follows that

$$W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R}))) \subset H^1(G//K) \subset W_-(H^1(\mathbf{R}))$$

(see [**5**, Remark 4.7(1) and Corollary 4.3]).

3. Estimate of W_-. We retain the previous notations. In the process to deduce (3) we use a relation between the Weyl type fractional integral transforms W_- on G and $W_{-\gamma}^{\mathbf{R}}$ on $\mathbf{R}_+$. As shown in [**5**, Proposition 4.5, Lemma 4.4], if F is smooth, then $W_-(F)$ is estimated as, for $x > 0$

$$\begin{aligned} |W_-(F)(x)| \;\; &\leq \;\; c\Delta(x)^{-1} \sum_{m=0}^{\underline{n}} \sum_{\xi \in \underline{D}} \Big(|W_{-(m+\xi)}^{\mathbf{R}}(F)(x)| \\ &\quad + \int_x^\infty |W_{-(m+\xi)}^{\mathbf{R}}(F)(s)| A_{m,\xi}(x,s) ds \Big), \end{aligned} \tag{4}$$

where $A_{m,0}(x,s) \equiv 0$, $A_{m,\xi}(x,s) \geq 0$ and there exists a constant c such that $\int_0^s A_{m,\xi}(x,s)dx \leq c$ for all $s \geq 0$. More precisely, $A_{m,\xi}(x,s)$ is dominated by $\chi_{[0,\infty)}(s-x)\chi_{[0,1]}(s)$ or $B_{m,\xi}(s-x)$, where $B_{m,\xi}(x)$ is integrable on $\mathbf{R}_+$. Let $f \in W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ and put $F = W_+(f)$. By the definition, $\mathcal{M}_{C_\rho}^{-1}(F)$ belongs to $H^1(\mathbf{R})$. We note that $C(-(\lambda + i\rho)) \sim (1+|\lambda|)^{-(\alpha+1/2)}$ (cf. [**3**, Theorem 2]) and $(i\lambda)^\gamma/(\lambda+i\rho)^{\alpha+1/2}$, $0 \leq \gamma \leq \alpha+1/2$, satisfies the Hörmander condition (cf. [**8**, p.318]). Therefore, $W_{-\gamma}^{\mathbf{R}}(F)$ belongs to $H^1(\mathbf{R})$ (cf. [**8**, p.363]). Since $m+\xi \leq \underline{n}+\delta+\delta' = \alpha+1/2$, each $W_{-(m+\xi)}^{\mathbf{R}}(F)$ in (3) and (4) belongs to $H^1(\mathbf{R})$, that is,

$$\|W_{-(m+\xi)}^{\mathbf{R}}(F)\|_{H^1(\mathbf{R})} \leq c\|f\|_{W_-^{C_\rho}} \tag{5}$$

for all $0 \leq m \leq \underline{n}$ and $\xi \in \underline{D}$.

4. Area and g functions. Let p_t denote the Poisson kernel on G, which is a K-bi-invariant function on G given by

$$\hat{p}_t(\lambda) = e^{-t\sqrt{\lambda^2+\rho^2}}.$$

We define the Littlewood-Paley g function $g(f)$ on G, $f \in C_c^\infty(G//K)$, as

$$g(f)(x) = \left(\int_0^\infty \left|t\frac{\partial}{\partial t} f * p_t(x)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

As shown in [**1**], [**7**] and [**8**], g satisfies the maximal theorem. We define the Lusin area function on G as an analogue of the classical theory (cf. [**9**, p.314]). Let $B(t)$ denote the ball on G with radius t centered at the origin and $|B(t)|$ the volume of the ball. Let $\chi_{B(t)}$ denote the characteristic function of $B(t)$ and put

$$\chi_t(x) = \frac{1}{|B(t)|}\chi_{B(t)}(x).$$

We define the Lusin area function $S(f)$ on G as

$$S(f)(x) = \left(\int_0^\infty \chi_t * \left|t\frac{\partial}{\partial t} p_t * f(x)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

As shown in [**7**], S is bounded on $L^p(G//K)$, $0 < p < \infty$. We also define the modified one as

$$S_+(f)(x) = \left(\int\int_{\{\sigma(y)\geq\sigma(x)\}} \chi_t(xy^{-1}) \left|t\frac{\partial}{\partial t} f * p_t(y)\right|^2 dy\frac{dt}{t}\right)^{1/2},$$

where σ is the distance function on G (cf. [**10**, 8.1.2]). Our main theorem is the following.

Theorem. g and S_+ are bounded from $W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ to $L^1(G//K)$.

5. Sketch of the proof. We suppose that $f \in W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ and put $F = W_+(f)$. For simplicity we denote $K_t = t(\partial/\partial t)p_t$. Since $t(\partial/\partial t)f * p_t = f * K_t = W_-(W_+(f * K_t)) = W_-(F * W_+(K_t))$, it follows from (4) that

$$\|g(f)\|_1 \leq c \sum_{m=0}^{n} \sum_{\xi \in \underline{D}} \int_0^\infty \left(\int_0^\infty |W^{\mathbf{R}}_{-(m+\xi)}(F) \circledast W_+(K_t)(x)|^2 \frac{dt}{t} \right)^{1/2} dx,$$

because

$$\begin{aligned}
& \int_0^\infty \left(\int_0^\infty \left| \int_x^\infty H(s,t) A(x,s) ds \right|^2 \frac{dt}{t} \right)^{1/2} dx \\
\leq \; & \int_0^\infty \int_s^\infty \left(\int_0^\infty |H(s,t)|^2 \frac{dt}{t} \right)^{1/2} A(x,s) ds dx \\
= \; & \int_0^\infty \left(\int_0^\infty |H(s,t)|^2 \frac{dt}{t} \right)^{1/2} \int_0^s A(x,s) dx ds \\
\leq \; & c \int_0^\infty \left(\int_0^\infty |H(s,t)|^2 \frac{dt}{t} \right)^{1/2} ds.
\end{aligned}$$

We here put

$$g_{\mathbf{R}}(H)(x) = \left(\int_0^\infty |H \circledast W_+(K_t)(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad H \in C_c^\infty(\mathbf{R})$$

Since each $W^{\mathbf{R}}_{-(m+\xi)}(F)$ belongs to $H^1(\mathbf{R})$ (see (5)), according to the $(1, \infty, 1)$-atomic decomposition of $H^1(\mathbf{R})$, it is enough to show that there exists a constant C such that for all $(1, \infty, 1)$-atoms A on $\mathbf{R}$,

$$\int_0^\infty g_{\mathbf{R}}(A)(x) dx \leq C. \tag{6}$$

Obviously, we may suppose that A is centered, that is, A is supported on $[-r, r]$, $\|A\|_\infty \leq (2r)^{-1}$ and $\int_{-\infty}^\infty A(x) x^k dx = 0$, $k = 0, 1$. First we shall prove that $g_{\mathbf{R}}$ is bounded on $L^2(\mathbf{R})$: For $H \in L^2(\mathbf{R})$

$$\begin{aligned}
\|g_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} &= \int_0^\infty \|H \circledast W_+(K_t)\|^2_{L^2(\mathbf{R})} \frac{dt}{t} \\
&= \int_0^\infty \|\tilde{H} \cdot W_+(K_t)^\sim\|^2_{L^2(\mathbf{R})} \frac{dt}{t} \\
&= \int_0^\infty \|\tilde{H}(\lambda) \cdot t\sqrt{\lambda(\lambda+2i\rho)} e^{-t\sqrt{\lambda(\lambda+2i\rho)}}\|^2_{L^2(\mathbf{R})} \frac{dt}{t} \\
&= \int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 \left(\int_0^\infty t|\lambda(\lambda+2i\rho)| e^{-2t\Re\sqrt{\lambda(\lambda+2i\rho)}} dt \right) d\lambda \\
&= \int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 \left(\int_0^\infty tre^{-2t\sqrt{r}\cos(\theta/2)} dt \right) d\lambda \\
&\le c \int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 d\lambda = c\|H\|^2_{L^2(\mathbf{R})},
\end{aligned}$$

where we set $\lambda(\lambda+2i\rho) = re^{i\theta}$ and we used the fact that $\cos\theta \ge 0$ and $\cos(\theta/2) = \sqrt{(\cos\theta+1)/2} \ge 1/\sqrt{2}$. Hence, by Schwarz' inequality, we have

$$\int_0^{2r} g_{\mathbf{R}}(A)(x)dx \le c\|A\|_{L^2(\mathbf{R})}(2r)^{1/2} \le C. \tag{7}$$

Next we suppose that $x \ge 2r$. We recall $W_+(K_t)(x) = te^{\rho x}(\partial/\partial t)F^0_{p_t}(x)$ and $F^0_{p_t}(x) = Ct(t^2+x^2)^{-1/2}\mathcal{K}_1(\rho(t^2+x^2)^{1/2})$, where $\mathcal{K}_\nu$ is the modified Bessel function (see [**1**], p. 289). Since $\mathcal{K}_\nu(x) = O(x^{-1/2}e^{-x})$ if $x \to \infty$, and $O(x^{-\nu})$ if $x \to 0$, and $x-y \ge x-r > r$ if $|y| \le r$, it follows that

$$\begin{aligned}
|A \circledast W_+(K_t)(x)| &= \Big| \int_{-\infty}^\infty A(y)W_+(K_t)(x-y)dy \Big| \\
&\le c\int_{-\infty}^\infty |A(y)|t^3(t^2+(x-y)^2)^{-3/4-1/2-\epsilon} e^{-\rho(t^2+(x-y)^2)^{1/2}} e^{\rho(x-y)} dy \\
&\le ct(t^2+(x-r)^2)^{-3/4} \le ct(x-r)^{-3/2},
\end{aligned} \tag{8}$$

where $\epsilon = 0$ if $t^2+(x-y)^2 \ge 1$, and $\epsilon = 1/4$ if $t^2+(x-y)^2 < 1$. Actually, when $\epsilon = 0$, we used the fact that $t^{2\ell}e^{-\rho(t^2+(x-y)^2)^{1/2}}$, $\ell \in \mathbf{R}$, has the maximum $O((x-y)^\ell e^{-\rho(x-y)})$ at $t \sim (x-y)^{1/2}$. Thereby, letting $\ell = 1$, we see that $t^2(t^2+(x-y)^2)^{-1/2}e^{-\rho(t^2+(x-y)^2)^{1/2}} \le t^2(x-y)^{-1}e^{-\rho(t^2+(x-y)^2)^{1/2}} \le ce^{-\rho(x-y)}$. When $\epsilon = 1/4$, we used the fact that $t^2(t^2+(x-y)^2)^{-3/4} \le t^2(t^2+(x-y)^2)^{-1} \le 1$. Next we note the moment condition of A, which implies that $B(x) = \int_{-\infty}^x \int_{-\infty}^u A(v)dvdu$ is supported on $[-r, r]$ and $\|B\|_\infty \le 2r$.

Since $(d/dx)^k(K_\nu(x)e^x) = O(x^{-1/2-k})$ if $x \to \infty$, and $O(x^{-\nu-k})$ if $x \to 0$, integration by parts yields that

$$\begin{aligned} & |A \circledast W_+(K_t)(x)| \\ \leq\ & c\int_{-\infty}^{\infty} |B(y)| t^3(t^2 + t(x-y) + (x-y)^2) \\ & \qquad \times (t^2 + (x-y)^2)^{-3/4-1/2-2-\epsilon} e^{-\rho(t^2+(x-y)^2)^{1/2}} e^{\rho(x-y)} dy \\ \leq\ & cr^2 t^{-2}(t^2 + (x-r)^2)^{-1} \leq cr^2 t^{-2}(x-r)^{-2}, \end{aligned} \tag{9}$$

where $\epsilon = 0$ if $t^2 + (x-y)^2 \geq 1$, and $\epsilon = 5/4$ if $t^2 + (x-y)^2 < 1$. Actually, when $\epsilon = 0$, letting $\ell = 5/2$, we see that $t^5(t^2 + (x-y)^2)^{-5/4} e^{-\rho(t^2+(x-y)^2)^{1/2}} \leq e^{-\rho(x-y)}$ and when $\epsilon = 5/4$, we have $t^5(t^2 + (x-y)^2)^{-5/2} \leq 1$. Hence, from (8) and (9) we see that

$$\begin{aligned} & \int_0^\infty |A \circledast W_+(K_t)(x)|^2 \frac{dt}{t} \\ \leq\ & c(x-r)^{-3} \int_0^{\sqrt{r}} t\,dt + cr^4(x-r)^{-4} \int_{\sqrt{r}}^\infty t^{-5} dt \\ \leq\ & cr(x-r)^{-3} + cr^2(x-r)^{-4} \end{aligned}$$

and thus

$$\int_{2r}^\infty g_{\mathbf{R}}(A)(x)dx \leq c\int_{2r}^\infty (r^{1/2}(x-r)^{-3/2} + r(x-r)^{-2})dx \leq C. \tag{10}$$

Then (7) and (10) imply the desired estimate (6).

As for the area function $S_+(f)$, it follows from (4) that it is enough to show that for all centered $(1,\infty,1)$-atoms A on $\mathbf{R}$,

$$\int_0^\infty S^1_{\mathbf{R}}(A)(x)dx \leq C \quad \text{and} \quad \int_0^\infty S^2_{\mathbf{R}}(A)(x)dx \leq C, \tag{11}$$

where

$$\begin{aligned} S^1_{\mathbf{R}}(H)(x) &= \Big(\int_0^\infty \int_{\{y \geq |x|\}} \int_K \chi_t(a_x k a_y^{-1}) dk \\ & \qquad \times \Delta(y)^{-1} |H \circledast W_+(K_t)(y)|^2 dy \frac{dt}{t}\Big)^{1/2} \Delta(x). \end{aligned}$$

and

$$\begin{aligned} S^2_{\mathbf{R}}(H)(x) &= \Big(\int_0^\infty \int_{\{y \geq |x|\}} \int_K \chi_t(a_x k a_y^{-1}) dk \\ & \qquad \times \Delta(y)^{-1} \left|\int_y^\infty H \circledast W_+(K_t)(s) A(y,s) ds\right|^2 dy \frac{dt}{t}\Big)^{1/2} \Delta(x). \end{aligned}$$

Here $A(y,s) \geq 0$ and $\int_0^s A(y,s)dy \leq c$ for all $s \geq 0$. More precisely, $A(y,s)$ is dominated by $\chi_{[0,\infty)}(s-y)\chi_{[0,1]}(s)$ or $B(s-y)$, where $B(y)$ is integrable on $\mathbf{R}_+$. First we shall estimate $S^1_{\mathbf{R}}(A)$. Since $\Delta(y)^{-1} \leq \Delta(x)^{-1}$ if $y \geq |x|$ and $\int_0^\infty \int_K \chi_t(a_x k a_y^{-1}) dk \Delta(x) dx = \int_G \chi_t(g a_y) dg = \|\chi_t\|_1 = 1$, it follows that for $H \in C_c^\infty(\mathbf{R})$,

$$\begin{aligned} & \|S^1_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} \\ \leq\ & \int_0^\infty \int_0^\infty \Big(\int_0^\infty \int_K \chi_t(a_x k a_y^{-1}) dk \Delta(x) dx \Big) |H \circledast W_+(K_t)(y)|^2 dy \frac{dt}{t} \\ =\ & \|g_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} \end{aligned}$$

and thus, $S^1_{\mathbf{R}}$ is bounded on $L^2(\mathbf{R})$. Then $\int_0^{2r} S^1_{\mathbf{R}}(A)(x)dx \leq C$ as before. We suppose that $x \geq 2r$. We recall that, if $y \geq |x|$, then $\Delta(y)^{-1}\Delta(x)^2 \leq \Delta(y)$ and $|A \circledast W_+(K_t)(y)|$ is dominated by $t(x-r)^{-3/2}$ and $r^2 t^{-2}(x-r)^{-2}$. Since $\|\chi_t\|_1 = 1$, as in the case of $g_{\mathbf{R}}$, $S^1_{\mathbf{R}}(A)(x)$ is estimated as $r^{1/2}(x-r)^{-3/2} + r(x-r)^{-2}$ and then $\int_{2r}^\infty S^1_{\mathbf{R}}(A)(x)dx \leq C$. Therefore, we can deduce (11) for $S^1_{\mathbf{R}}$. Next we shall estimate $S^2_{\mathbf{R}}$. As before, we see that

$$\|S^2_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} \leq \| \int_y^\infty g_{\mathbf{R}}(H)(s) A(y,s) ds \|^2_{L^2(\mathbf{R})}.$$

When $A(y,s)$ is dominated by $\chi_{[0,\infty)}(s-y)\chi_{[0,1]}(s)$, we see that $0 \leq |x| \leq y \leq s \leq 1$ and thus,

$$\|S^2_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} \leq \Big(\int_0^1 g_{\mathbf{R}}(H)(s) \|A(\cdot,s)\|_{L^2(\mathbf{R})} ds \Big)^2 \leq c \|g_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})}$$

by Schwarz's inequality. When $A(y,s)$ is dominated by $B(s-y)$, we change the variable s to $s+y$ and thus,

$$\begin{aligned} \|S^2_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} & \leq \| \int_0^\infty g_{\mathbf{R}}(H)(s+y) B(s) ds \|^2_{L^2(\mathbf{R})} \\ & \leq c \|g_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} \Big(\int_0^\infty B(s) ds \Big)^2 \leq c \|g_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})}. \end{aligned}$$

Hence $S^2_{\mathbf{R}}$ is bounded on $L^2(\mathbf{R})$ and $\int_0^{2r} S^2_{\mathbf{R}}(A)(x)dx \leq C$ as before. We suppose that $x \geq 2r$. When $A(y,s)$ is dominated by $\chi_{[0,\infty)}(s-y)\chi_{[0,1]}(s)$, we see that $2r \leq x \leq y \leq s \leq 1$ and thus, $A \circledast W_+(K_t)(s)$ is estimated as $t(x-r)^{-3/2}$ and $r^2 t^{-2}(x-r)^{-2}$. Moreover, $\Delta(y)^{-1}\Delta(x)^2 \leq \Delta(y)$, $\|\chi_t\|_1 = 1$, and $\int_y^\infty A(y,s)ds \leq 1$. Therefore, $S^2_{\mathbf{R}}(A)(x)$ is dominated by $r^{1/2}(x-r)^{-3/2} + r(x-r)^{-2}$ and thereby, $\int_{2r}^\infty S^2_{\mathbf{R}}(A)(x)dx \leq C$ as in the case of $g_{\mathbf{R}}(A)$. When $A(y,s)$ is dominated by $B(s-y)$, we change the variable s to $s+(y-x)$.

We recall that, since $s+(y-x) \geq s \geq x > 2r$, $|A \circledast W_+(K_t)(s+(y-x))|$ is dominated by $H(t,s,r) = \min\{t(s-r)^{-3/2}, r^2t^{-2}(s-r)^{-2}\}$, which is independent of x, y, and $\Delta(y)^{-1}\Delta(x)^2 \leq \Delta(y)$. Therefore, noting $\|\chi_t\|_1 = 1$ and $A(y, s+(y-x)) \leq B(s-x)$, we can deduce that

$$S^2_{\mathbf{R}}(A)(x) \leq \Big(\int_0^\infty \Big| \int_x^\infty H(t,s,r)B(s-x)ds \Big|^2 \frac{dt}{t} \Big)^{1/2}.$$

Since $\int_0^\infty |H(t,s,r)|^2 dt/t \leq r(s-r)^3 + r^2(s-r)^{-4}$ as before, it follows that

$$\begin{aligned} & \int_{2r}^\infty S^2_{\mathbf{R}}(A)(x)dx \\ \leq \; & \int_{2r}^\infty \int_x^\infty (r^{1/2}(s-r)^{3/2} + r(s-r)^{-2})B(s-x)dsdx \\ = \; & \int_{2r}^\infty (r^{1/2}(s-r)^{3/2} + r(s-r)^{-2}) \left(\int_{2r}^s B(s-x)dx \right) ds \leq C. \end{aligned}$$

Therefore, we have (11) for $S^2_{\mathbf{R}}$. This completes the proof of the theorem.

Remark. We put $D_x = W_- \circ (d/dx) \circ W_+$. Then the operators g' and S'_+ defined by replaced $t(d/dt)p_t$ in the definitions of g and S_+ with tD_xp_t are also bounded from $W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ to $L^1(G//K)$. Moreover, in the definition of S_+ we may replace χ_t with ϕ_t in (1).

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Real Hardy spaces on real rank 1 semisimple Lie groups

Takeshi KAWAZOE *

Abstract

Let G be a real rank one connected semisimple Lie group with finite center. We introduce a real Hardy space $H^1(G//K)$ on G as the space consisting of all K-bi-invariant functions f on G whose radial maximal functions $M_\phi f$ are integrable on G. We shall obtain a relation between $H^1(G//K)$ and $H^1(\mathbf{R})$, the real Hardy space on the real line $\mathbf{R}$, via the Abel transform on G and give a characterization of $H^1(G//K)$.

1. Introduction. In the study of the classical Hardy spaces on the unit disk and the upper half plane, real variable characterizations of their boundary values are called the real variable method. In the 1970's these boundary values were completely characterized by various maximal functions without using the complex variable method and their atomic characterizations were also given at the same time. This was a significant breakthrough in harmonic analysis. Nowadays, this fruitful theory of real Hardy spaces, which are defined by maximal functions and atoms, has been extended to the spaces of homogeneous type: A topological space X with measure μ and distance d is of homogeneous type if there exists a constant $c > 0$ such that for all $x \in X$ and $r > 0$

$$\mu(B(x, 2r)) \leq c\mu(B(x, r)),$$

where $B(x, r)$ is the ball defined by $\{y \in X \mid d(x, y) < r\}$ and $\mu(B(x, r))$ the volume of the ball (cf. [**2**, §1]).

When the space X is not of homogeneous type, little work on real Hardy spaces on X has been done. In this paper, looking at the example of a

*Supported by Grant-in-Aid for Scientific Research (C), No. 13640190, Japan Society for the Promotion of Science. 2000 Mathematics Subject Classification: Primary 22E30; Secondary 43A30, 43A80 semisimple Lie group G as a space of non-homogeneous type, we shall consider real Hardy spaces on G. Actually, when $X = G$, $\mu(B(x,r))$ has an exponential growth order as r goes to infinity and hence G is not of homogeneous type. We shall introduce a real Hardy space on G by using a radial maximal function on G. Our goal is to give an atomic characterization of the space and to obtain a relation with the real Hardy space $H^1(\mathbf{R})$ on $\mathbf{R}$.

We shall overview some results obtained in the previous papers [**7**], [**8**], [**9**] and state our main theorem. Let G be a real rank one connected semisimple Lie group with finite center and $G = KAN = KCL(A^+)K$ respectively an Iwasawa and the Cartan decompositions of G. Let $dg = dkdadn = \Delta(a)dkdadk'$ denote the corresponding decompositions of a Haar measure dg on G. In this paper we shall treat only K-bi-invariant functions on G. Since A is identified with $\mathbf{R}$ as $A = \{a_x; x \in \mathbf{R}\}$, all K-bi-invariant functions can be identified with even functions on $\mathbf{R}$. We also denote $\Delta(a_x)$ by $\Delta(x)$ for $x \geq 0$ and extend it as an even function on $\mathbf{R}$. Then the one dimensional space $\mathbf{R}$ with normal distance and weighted measure $\Delta(x)dx$ is not of homogeneous type, because $\Delta(x) \sim e^{2\rho|x|}$ as $|x| \to \infty$, where ρ is a positive constant determined by a group structure (see (**??**), (**??**), (**??**)). Let $L^p(G//K)$ denote the space of all K-bi-invariant functions on G with finite L^p-norm:

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p \Delta(x) dx \right)^{1/p} \tag{1}$$

and $L^1_{\text{loc}}(G//K)$ the space of all locally integrable, K-bi-invariant functions on G.

We shall introduce a real Hardy space on G by using a radial maximal function. As in the Euclidean case, to define a radial maximal function we need to define a dilation ϕ_t, $t > 0$, of a function ϕ on G. Let ϕ be a positive compactly supported C^∞, K-bi-invariant function on G such that

$$\int_G \phi(g)dg = \int_0^\infty \phi(x)\Delta(x)dx = 1.$$

We define the dilation ϕ_t of ϕ as

$$\phi_t(x) = \frac{1}{t}\frac{1}{\Delta(x)}\Delta\left(\frac{x}{t}\right)\phi\left(\frac{x}{t}\right).$$

Clearly, ϕ_t has the same L^1-norm as ϕ: $\|\phi_t\|_1 = \|\phi\|_1$ and, for $1 \leq p \leq \infty$, it gives an approximate identity in $L^p(G//K)$ (see [**3**, Lemma 16]). Since this dilation has the same properties as in the Euclidean case, it is quite natural to introduce a radial maximal function $M_\phi f$ on G as

$$(M_\phi f)(g) = \sup_{0<t<\infty} |(f * \phi_t)(g)| \quad (g \in G).$$

As shown in [**7**, Theorem 3.4], this maximal operator M_ϕ satisfies the so-called maximal theorem: M_ϕ is bounded on $L^p(G//K)$ $(1 < p \leq \infty)$ and satisfies the weak type L^1 estimate. Now we define the *real* Hardy space $H^1(G//K)$ on G analogously as in the real Hardy space $H^1(\mathbf{R})$ on $\mathbf{R}$:

$$H^1(G//K) = \{f \in L^1_{\text{loc}}(G//K)\ ;\ M_\phi f \in L^1(G//K)\}$$

and the norm is given by

$$\|f\|_{H^1(G)} = \|M_\phi f\|_1.$$

The aim of this paper is to characterize $H^1(G//K)$.

For a compactly supported C^∞, K-bi-invariant function f on G, we shall define the Abel transform F_f^s, $s \in \mathbf{R}$, of f as

$$F_f^s(x) = e^{\rho(1+s)x} \int_N f(a_x n) dn.$$

This integral over N is explicitly given by a composition of generalized Weyl type fractional integrals (see [**11**, Corollary 3.3] and (**??**)). Especially, we put

$$W_+(f) = F_f^1$$

and denote by W_- the inverse operator of W_+, which is given by a composition of Weyl type fractional derivatives (see (**??**), (**??**)). Let $\hat{f}$ denote the spherical Fourier transform of f on G and $F^\sim$ the Euclidean Fourier transform of F on $\mathbf{R}$. Both $\hat{f}$ and $(F_f^s)^\sim$ are regarded as functions on the dual space F of the Lie algebra of A, which is identified with $\mathbf{R}$. Then they are extended to holomorphic functions on F_c, the complexification of F, of exponential type and

$$\hat{f}(\lambda + is\rho) = (F_f^s)^\sim(\lambda), \quad \lambda \in \mathsf{F}_c$$

(cf. [**11**, (3.17)]). Let $C(\lambda)$ denote Harish-Chandra's C-function (see (**??**)) and M_{C_ρ} the Euclidean Fourier multiplier corresponding to $C_\rho(\lambda) = C(\lambda + i\rho)$:

$$\mathsf{M}_{C_\rho}(F)^\sim(\lambda) = C(\lambda + i\rho)F^\sim(\lambda).$$

We here define

$$W_-(H^1(\mathbf{R})) = \{f \in L^1_{\text{loc}}(G//K)\ ; W_+(f) \in H^1(\mathbf{R})\}$$

and also $W_-(\mathsf{M}_{C_\rho}(H^1(\mathbf{R})))$ by replacing the condition that $W_+(f) \in H^1(\mathbf{R})$ in the above definition with $\mathsf{M}_{C_\rho}^{-1} \circ W_+(f) \in H^1(\mathbf{R})$.

Theorem A. *Let notations be as above.*

$$W_-(\mathsf{M}_{C_\rho}(H^1(\mathbf{R}))) \subset H^1(G//K) \subset W_-(H^1(\mathbf{R})).$$

This is one of the main results in [**8**]. However, the proof was a little bit complicated, because to obtain the first inclusion we used the Harish-Chandra expansion of the zonal spherical function and also the Gangolli expansion (see (**??**), (**??**)). Thereby, to sum up the estimates of each expanded terms we required a sharp estimate and a deep theory of H^1 Fourier multipliers on $\mathbf{R}$. In this paper we shall give a simple proof based on fractional calculus on G. Actually, Theorem A follows from the next Theorem B: We define a maximal operator $M_\phi^{\mathbf{R}}$ and a fractional operator $W_\gamma^{\mathbf{R}}$ on $\mathbf{R}$ by Definition 3.2 and (**??**) respectively, and $\underline{n} \in \mathbf{N}$ and the set $\underline{D}$ by (**??**) and (**??**) respectively.

Theorem B. *There exist* $0 < c_1 \le c_2$ *such that for all* $f \in H^1(G//K)$

$$c_1 \|f\|_{H^1(G)} \le \sum_{m=0}^{\underline{n}} \sum_{\xi \in \underline{D}} \|M_\phi^{\mathbf{R}} \circ W_{-(m+\xi)}^{\mathbf{R}}(F)(x)(\mathrm{th}x)^{m+\xi}\|_{L^1(\mathbf{R})} \le c_2 \|f\|_{H^1(G)},$$

where $F = W_+(f)$.

We next introduce *atomic* Hardy spaces on G. In the Euclidean space the atomic Hardy space $H^1_{\infty,0}(\mathbf{R})$ coincides with $H^1(\mathbf{R})$ (cf. [**6**, Theorem 3.30], [**15**, §2 in Chap.3]). However, it may not be true in our setting, because the Lebesgue measure dx is replaced by the weighted measure $\Delta(x)dx$. We denote the interval $[x_0 - r, x_0 + r]$ by $R(x_0, r)$ and set the volume by

$$|R(x_0, r)| = \int_{x_0 - r}^{x_0 + r} \Delta(x) dx.$$

We say that a K-bi-invariant function a on G is a $(1, \infty, 0)$-atom on G provided that there exist $x_0 \ge 0$ and $r > 0$ such that

$$\begin{aligned} &(i) \quad \mathrm{supp}(a) \subset R(x_0, r), \\ &(ii) \quad \|a\|_\infty \le |R(x_0, r)|^{-1}, \\ &(iii) \quad \int_0^\infty a(x)\Delta(x) dx = 0. \end{aligned}$$

Here a is identified with a function on $\mathbf{R}_+$. Then $H^1_{\infty,0}(G//K)$ is defined by

$$H^1_{\infty,0}(G//K) = \{f = \sum_i \lambda_i a_i \ ; \ a_i \text{ is } (1, \infty, 0)\text{-atom on } G \text{ and } \textstyle\sum_i |\lambda_i| < \infty\}$$

and the norm is given by

$$\|f\|_{H^1_{\infty,0}(G)} = \inf \sum_i |\lambda_i|,$$

where the infimum is taken over all such representations $f = \sum_i \lambda_i a_i$. Moreover, we define $H^{1,\epsilon}_{\infty,0}(G//K)$ $(\epsilon \geq 0)$ and $H^{1,+}_{\infty,0}(G//K)$ by replacing (ii) and (iii) of the above definition of $(1,\infty,0)$-atom a on G, respectively, with

$$(ii)_\epsilon \quad \|a\|_\infty \leq |R(x_0, r)|^{-1}(1+r)^{-\epsilon}$$

and

$$(iii)_+ \int_0^\infty a(x)\Delta(x)dx = 0 \;\text{ if } r \leq 1.$$

Clearly, for $\epsilon \geq 0$,

$$H^{1,\epsilon}_{\infty,0}(G//K) \subset H^1_{\infty,0}(G//K) \subset H^{1,+}_{\infty,0}(G//K).$$

We define a truncated maximal operator M_ϕ^{loc} on G as

$$(M_\phi^{\mathrm{loc}} f)(g) = \sup_{0<t<1} |(f * \phi_t)(g)| \quad (g \in G).$$

In [**9**] we essentially proved that M_ϕ^{loc} is bounded from $H^{1,+}_{\infty,0}(G//K)$ to $L^1(G//K)$. As for M_ϕ, we shall prove that M_ϕ is bounded from $H^{1,+}_{\infty,0}(G//K)$ $\cap\, W_-(H^1(\mathbf{R}))$ to $L^1(G//K)$. This means that $H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ $\subset H^1(G//K)$. Finally, as a refinement, we have the following main theorem.

Theorem C. *Let notations be as above. Then*

$$H^1(G//K) = H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R})).$$

Furthermore, $H^{1,\epsilon}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$, $\epsilon \geq 0$, *is dense in* $W_-(H^1(\mathbf{R}))$ *and especially,* $H^1(G//K)$ *is dense in* $W_-(H^1(\mathbf{R}))$.

This paper is organized as follows. We recall some basic notations in §2. Then we shall define a radial maximal function $M_\phi f$ on G in §3 and obtain a key formula which reduces $M_\phi f$ to a Euclidean maximal function $M_\phi^{\mathbf{R}}(W_+(f))$ on $\mathbf{R}$ (see Proposition 3.3). Theorem B follows from this formula and fractional calculus on G and $\mathbf{R}$ in §4 (see Theorem 4.6). As an easy consequence of Theorem B, we can obtain a simple proof of Theorem A (see Remark 4.7), and moreover, a norm-equivalence between $\|f\|_{H^1(G)}$ and $\|M_\phi^{\mathrm{loc}} f\|_1$ $+ \|W_+(f)\|_{H^1(\mathbf{R})}$ (see Theorem 4.9). In §5 we introduce atomic Hardy spaces on G. Then the inclusion $H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R})) \subset H^1(G//K)$ follows from the estimates obtained in §4 and [**9**] (see Proposition 5.9). Next we shall consider a $(1,\infty,+)$-atomic decomposition of functions f in $H^1(G//K)$ (see Proposition 6.2). Then, combining with Theorem A, we can deduce our main Theorem C (see Theorem 6.5 and Theorem 6.6). In §7 we shall consider (H^1, L^1)-boundedness of other operators on G; singular integrals, modified heat and the Poisson maximal operators (see Definition 7.2) and the Riesz transform on G (see Definition 7.8). We summarize some calculations and make a comment on the dual space of $W_-(H^1(\mathbf{R}))$ in §8. In §8.A, we shall obtain a sharp estimate of $\Gamma_m(\lambda)$, meromorphic functions appearing in the Gangolli expansion of a spherical function (see (**??**)), and in §8.B, we shall obtain a norm-equivalence among some Euclidean maximal operators. In §8.C, we shall introduce a dual space of $W_-(H^1(\mathbf{R}))$ as a pull-back of $BMO(\mathbf{R})$ via the complex Fourier-Jacobi transform on G (see (**??**) and Definition 8.7). In this paper we introduce many real Hardy spaces on G as subspaces of $L^1(G//K)$. We refer the reader to the figure in §8.D for the understanding of main relationship among them.

2. Notations. Let G be a real rank one connected semisimple Lie group with finite center and $G = KAN$ an Iwasawa decomposition of G. Let $\mathfrak{a}$ be the Lie algebra of A and $\mathsf{F} = \mathfrak{a}^*$ the dual space of $\mathfrak{a}$. Let γ be the positive simple root of (G, A) determined by N and H the unique element in $\mathfrak{a}$ satisfying $\gamma(H) = 1$. Let m_1 and m_2 denote the multiplicities of γ and 2γ respectively. We put

$$\alpha = \frac{m_1 + m_2 - 1}{2}, \quad \beta = \frac{m_2 - 1}{2}, \quad \rho = \alpha + \beta + 1. \tag{2}$$

We parameterize each element in A, $\mathfrak{a}$, and F as $a_x = \exp(xH)$, xH, and $x\gamma$ ($x \in \mathbf{R}$) respectively. In what follows we often identify these spaces $A, \mathfrak{a}, \mathsf{F}$ with $\mathbf{R}$ and their complexifications with $\mathbf{C}$. We put $A_+ = \{a_x; x \in \mathbf{R}_+\}$. Then, according to the Cartan decomposition $G = KCL(A_+)K$ of G, every K-bi-invariant functions f on G are determined by their restrictions to $CL(A_+)$ and hence, they are identified with even functions on $\mathbf{R}$. We denote them by the same letter, that is, if $g \in Ka_{\sigma(g)}K$ with $\sigma(g) \in \mathbf{R}_+$, then

$$f(g) = f(a_{\sigma(g)}) = f(\sigma(g)) = f(-\sigma(g)). \tag{3}$$

Let dg denote the Haar measure on G. We denote by $L^1_{\text{loc}}(G//K)$ and $L^p(G//K)$, respectively, the spaces of locally integrable and p-th power integrable K-bi-invariant functions on G. Let $C_c^\infty(G//K)$ denote the space of compactly supported C^∞, K-bi-invariant functions on G. Then for $f \in L^1(G//K)$ the Cartan decomposition of G yields that

$$\int_G f(g)dg = \int_0^\infty f(x)\Delta(x)dx,$$

where

$$\Delta(x) = c(\mathrm{sh}x)^{2\alpha+1}(\mathrm{sh}2x)^{2\beta+1},\ x \geq 0. \tag{4}$$

We note that the order of $\Delta(x)$ is given by

$$\Delta(x) \sim \begin{cases} x^{2\alpha+1} & \text{if } 0 \leq x \leq 1 \\ e^{2\rho x} & \text{if } 1 < x < \infty, \end{cases} \tag{5}$$

where the symbol " $\sim$ " means that the ratio of the left side and the right side is bounded uniformly below and above by positive constants. We extend $\Delta(x)$ as an even function on $\mathbf{R}$. By using this weight function $\Delta(x)$ the L^p-norm $\|f\|_p$ of f on G can be rewritten as in (**??**). We denote the Euclidean L^p-norm on $\mathbf{R}$ by $\|\cdot\|_{L^p(\mathbf{R})}$.

Let Ω denote the Laplace-Beltrami operator on G and φ_λ ($\lambda \in \mathsf{F}$) the normalized zonal spherical function on G, that is, the K-bi-invariant eigenfunction of Ω satisfying

$$\Omega\varphi_\lambda = -(\lambda^2+\rho^2)\varphi_\lambda \quad \text{and} \quad \varphi_\lambda(0) = 1.$$

By restricting φ_λ to A, $\varphi_\lambda(x)$, $x \in \mathbf{R}$, is a solution of

$$\frac{d^2}{dx^2} + ((2\alpha+1)\mathrm{cth}x + (2\beta+1)\mathrm{th}x)\,\frac{d}{dx}, \tag{6}$$

which satisfies $\varphi_\lambda(0) = 1$ and $\varphi'_\lambda(0) = 0$. Hence, if $\alpha \notin -\mathbf{N}$, it is explicitly given by the Jacobi function of the first kind with order (α, β):

$$\varphi_\lambda(x) = F\left(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -\mathrm{sh}^2 x\right).$$

Clearly, $\varphi_\lambda(x)$ is even with respect to λ, x and it is uniformly bounded on $x \in \mathbf{R}$ if λ is in the tube domain

$$\mathsf{F}(\rho) = \{\lambda \in \mathsf{F}_c; |\Im\lambda| \leq \rho\}$$

(cf. [**5**, Lemma 11]). For $\lambda \notin -i\mathbf{N}$, let $\Phi(\lambda, x)$ denote another solution, which is given by the Jacobi function of the second kind with order (α, β):

$$\Phi(\lambda, x) = (e^x - e^{-x})^{i\lambda-\rho} F\left(\frac{\rho-2\alpha-i\lambda}{2}, \frac{\rho-i\lambda}{2}; 1-i\lambda; -\mathrm{sh}^{-2}x\right). \tag{7}$$

Then for $\lambda \notin \mathbf{Z}$, $\varphi_\lambda(x)$ has the so-called Harish-Chandra expansion:

$$\varphi_\lambda(x) = e^{-\rho x}\left(\Phi(\lambda, x)C(\lambda)e^{i\lambda x} + \Phi(-\lambda, x)C(-\lambda)e^{-i\lambda x}\right), \tag{8}$$

where

$$C(\lambda) = \frac{\Gamma(\alpha+1)}{2\sqrt{\pi}} \frac{2^\rho \Gamma\left(\dfrac{i\lambda}{2}\right)\Gamma\left(\dfrac{1+i\lambda}{2}\right)}{\Gamma\left(\dfrac{i\lambda+\rho}{2}\right)\Gamma\left(\dfrac{i\lambda+\rho-2\beta}{2}\right)}. \tag{9}$$

We denote the Gangolli expansion of $\Phi(\lambda, x)$, $x > 0$, as

$$\Phi(\lambda, x) = \sum_{m=0}^{\infty} \Gamma_m(\lambda) e^{-2mx}. \tag{10}$$

For some basic properties of $\varphi_\lambda(x)$, $\Phi(\lambda, x)$, and the recursive definition of $\Gamma_m(\lambda)$ we refer to [**3**, §2, §3] and [**17**, 9.1.4, 9.1.5].

For $f \in L^1(G//K)$ the spherical Fourier transform $\hat{f}(\lambda)$, $\lambda \in \mathsf{F}$, of f is defined by

$$\hat{f}(\lambda) = \int_G f(g)\varphi_\lambda(g)dg. \tag{11}$$

Then $\hat{f}(\lambda)$ is even and continuously extended on $\mathsf{F}(\rho)$, which is holomorphic in the interior and

$$|\hat{f}(\lambda)| \le \|f\|_1, \quad \lambda \in \mathsf{F}(\rho).$$

For $f \in C_c^\infty(G//K)$ the Paley-Wiener theorem (cf. [**3**, Theorem 4]) implies that $\hat{f}(\lambda)$ is holomorphic on F_c of exponential type. Furthermore, it satisfies the inversion formula

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda)\varphi_\lambda(x)|C(\lambda)|^{-2}d\lambda$$

and the Plancherel formula

$$\int_0^\infty |f(x)|^2\Delta(x)dx = \int_0^\infty |\hat{f}(\lambda)|^2|C(\lambda)|^{-2}d\lambda.$$

Therefore, the spherical Fourier transform $f \mapsto \hat{f}$ of $C_c^\infty(G//K)$ is uniquely extended to an isometry between $L^2(G//K) = L^2(\mathbf{R}_+, \Delta(x)dx)$ and $L^2(\mathbf{R}_+, |C(\lambda)|^{-2}d\lambda)$ (cf. [**3**, Proposition 3], [**17**, Theorem 9.2.2.13]).

For $f \in C_c^\infty(G//K)$ we define the Abel transform F_f^s, $s \in \mathbf{R}$, of f as

$$F_f^s(x) = e^{\rho(1+s)x} \int_N f(a_x n)dn. \tag{12}$$

Then the Euclidean Fourier transform $(F_f^s)^\sim(\lambda)$ is holomorphic on F_c of exponential type, because $F_f^s(f) \in C_c^\infty(\mathbf{R})$, and coincides with the spherical Fourier transform:

$$\hat{f}(\lambda + is\rho) = (F_f^s)^\sim(\lambda), \quad \lambda \in \mathsf{F}_c. \tag{13}$$

(cf. [**11**, §3]). Especially, F_f^0 is even on $\mathbf{R}$. The integral over N in (**??**) can be explicitly rewritten by using a generalized Weyl type fractional integral operator W_μ^σ: For $\sigma > 0$ and $\mu \in \mathbf{C}$, we define $W_\mu^\sigma(f)(y)$, $y > 0$, as

$$W_\mu^\sigma(f)(y) = \frac{(-1)^n}{\Gamma(\mu+n)} \int_y^\infty \frac{d^n f(x)}{d(\mathrm{ch}\sigma x)^n} (\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu+n-1} d(\mathrm{ch}\sigma x), \tag{14}$$

where $n = 0$ if $\Re\mu > 0$ and $-n < \Re\mu \leq -n+1$, $n = 0, 1, 2, \cdots$, if $\Re\mu \leq 0$ (see [**11**, (3.11)]). Koornwinder [**11**, Corollary 3.3] obtains that for $x > 0$,

$$\begin{aligned} F_f^0(x) &= W_{\alpha-\beta}^1 \circ W_{\beta+1/2}^2(f)(x) \\ &= c\int_x^\infty f(s)A(x,s)ds, \end{aligned} \tag{15}$$

where $A(x,s)$ is given by

$$\begin{aligned} & (\mathrm{sh}2s)(\mathrm{ch}s)^{\beta-1/2}(\mathrm{ch}s - \mathrm{ch}x)^{\alpha-1/2} F\left(\frac{1}{2}+\beta, \frac{1}{2}-\beta; \alpha+\frac{1}{2}; \frac{\mathrm{ch}s - \mathrm{ch}x}{2\mathrm{ch}s}\right) \\ = \; & (\mathrm{sh}2s)\int_x^s (\mathrm{ch}2s - \mathrm{ch}2w)^{\beta-1/2}(\mathrm{ch}w - \mathrm{ch}x)^{\alpha-\beta-1}(\mathrm{sh}w)dw \end{aligned} \tag{16}$$

(see [**11**, (2.18), (2.19), (3.5)]). We note that for $0 \leq \gamma \leq s_\alpha$ and $0 < x \leq s$,

$$|A(x,s)| \leq ce^{(\rho-2)s}(\mathrm{sh}2s)(\mathrm{th}s)^{2\alpha-1} \leq ce^{\rho s}(\mathrm{th}s)^{2\alpha}. \tag{17}$$

In the following, for simplicity, we denote $W_+(f)(x) = F_f^1(|x|)$, that is,

$$W_+(f)(x) = e^{\rho x} W_{\alpha-\beta}^1 \circ W_{\beta+1/2}^2(f)(|x|), \; x \in \mathbf{R} \tag{18}$$

and for a function F on $\mathbf{R}_+$,

$$W_-(F)(x) = W_{-(\beta+1/2)}^2 \circ W_{-(\alpha-\beta)}^1(e^{-\rho x}F), \; x \in \mathbf{R}_+. \tag{19}$$

Then $W_- \circ W_+(f) = f$ and $W_+ \circ W_-(F) = F$. For $f \in L^1(G//K)$, $W_+(f)$ belongs to $L^1(\mathbf{R})$, because (**??**) and the integral formula for the Iwasawa decomposition of G yield that

$$\|W_+(f)\|_{L^1(\mathbf{R})} \leq \|f\|_1 \tag{20}$$

(cf. [**11**, (3.5), (2.20)]). Hence $W_+(f)^\sim(\lambda)$, $\lambda \in \mathsf{F}$, is well-defined and

$$\hat{f}(\lambda + i\rho) = W_+(f)^\sim(\lambda), \quad \lambda \in \mathsf{F}. \tag{21}$$

For $f, g \in L^1(G//K)$, since $f * g \in L^1(G//K)$ and $(f * g)^\wedge(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$ (cf. [**3**, Theorem 5]), it follows that

$$W_+(f * g) = W_+(f) * W_+(g), \tag{22}$$

where the symbol "$*$" in the left and right hand sides denotes the convolution on G and on $\mathbf{R}$ respectively. We denote them by the same symbol. We say that a function F on $\mathbf{R}$ is W_+-smooth if $W_-(F)$ is well-defined and continuous. Then, for W_+-smooth functions F, G on $\mathbf{R}$ with compact support such that $e^{-\rho x}F$ and $e^{-\rho x}G$ are even, it follows that

$$W_-(F * G) = W_-(F) * W_-(G).$$

Let $(X_i, \|\cdot\|_i)$, $i = 1, 2$, be normed spaces and $T : X_1 \to X_2$ a sublinear operator. We use the symbol $\|x_1\|_1 \approx \|T(x_1)\|_2$ if there exist constants $0 < c_1 \le c_2$ such that $c_1\|x_1\|_1 \le \|T(x_1)\|_2 \le c_2\|x_1\|_1$ for all $x_1 \in X_1$. We denote by $[\,\cdot\,]$ the Gauss symbol and by $f = O(g)$ the Landau symbol. We use letters c, C, $c_1, c_2, \cdots$ to denote many different constants.

3. Maximal functions and a reduction formula. We shall introduce a radial maximal function $M_\phi f$ on G and give a reduction formula, which relates $M_\phi f$ with a maximal function of $W_+(f)$ on $\mathbf{R}$. We suppose that ϕ is a positive C^∞, K-bi-invariant function on G such that, after identifying it with an even function on $\mathbf{R}$, it is supported on $[-1, 1]$,

$$\int_0^\infty \phi(x)\Delta(x)dx = 1, \quad \int_0^1 \phi(x)x\Delta(x)dx \le \frac{1}{4}, \tag{23}$$

and furthermore, there exists $M \in \mathbf{N}$ such that

$$\phi(x) = O(x^{2M}). \tag{24}$$

For $t > 0$ we define a dilation ϕ_t of ϕ as

$$\phi_t(x) = \frac{1}{t}\frac{1}{\Delta(x)}\Delta\left(\frac{x}{t}\right)\phi\left(\frac{x}{t}\right).$$

As mentioned in §1 this dilation keeps the L^1-norm of ϕ and gives an approximate identity in $L^p(G//K)$, $1 \le p \le \infty$. We here introduce the radial maximal function $M_\phi f$ on G as follows.

Definition 3.1. *For* $f \in L^1_{\text{loc}}(G//K)$,

$$(M_\phi f)(g) = \sup_{0<t<\infty} |(f * \phi_t)(g)|, \quad g \in G.$$

As shown in [**7**, Theorem 3.4 and Theorem 3.5], M_ϕ satisfies the maximal theorem and $\|f\|_p \leq \|M_\phi f\|_p$, if the both sides exist, for $1 \leq p \leq \infty$. Next, by using $W_+(\phi_t)$, we define a maximal function on $\mathbf{R}$ as follows.

Definition 3.2. *For* $F \in L^1_{\text{loc}}(\mathbf{R})$,

$$(M^{\mathbf{R}}_\phi F)(x) = \sup_{0<t<\infty} |(F * W_+(\phi_t))(x)|, \quad x \in \mathbf{R}.$$

Since $W_+(f * \phi_t) = W_+(f) * W_+(\phi_t)$ (see (**??**)) and W_+ is an integral operator with a positive kernel (see (**??**), (**??**)), it follows that

$$\begin{aligned} \sup_{0<t<\infty} |W_+(f) * W_+(\phi_t)(x)| &= \sup_{0<t<\infty} |W_+(f * \phi_t)(x)| \\ &\leq W_+\left(\sup_{0<t<\infty} |f * \phi_t|\right)(x). \end{aligned}$$

This fact and (**??**) yield a relation between M_ϕ and $M^{\mathbf{R}}_\phi$:

Proposition 3.3. *For* $f \in L^1_{\text{loc}}(G//K)$,

$$(M^{\mathbf{R}}_\phi W_+(f))(x) \leq W_+(M_\phi f)(x), \quad x \in \mathbf{R}.$$

In particular,

$$\|M^{\mathbf{R}}_\phi W_+(f)\|_{L^1(\mathbf{R})} \leq c\|M_\phi f\|_1$$

if the both sides exist.

In what follows we shall prove that the maximal operator $M^{\mathbf{R}}_\phi$ characterizes $H^1(\mathbf{R})$, that is, $F \in H^1(\mathbf{R})$ if and only if $M^{\mathbf{R}}_\phi(F) \in L^1(\mathbf{R})$ (see Theorem 3.7 below). We first obtain some properties of $W_+(\phi_t)^\sim(\lambda) = \hat{\phi}_t(\lambda + i\rho)$ (see (**??**)), which guarantee that $W_+(\phi_t)$ behaves like a dilation on $\mathbf{R}$. Let M be the same in (**??**) and $N \in \mathbf{N}$.

Lemma 3.4. *There exists a constant* c *such that for all* $t > 0$, $\lambda \in \mathbf{R}$, $0 \leq n \leq N$ *and* $0 \leq k \leq M$,

$$\left|\left(\frac{d}{d\lambda}\right)^n \hat{\phi}_t(\lambda + i\rho)\right| \leq ct^n(1+t)^k(1+|t\lambda|)^{-2k}.$$

Proof. For simplicity, we denote $\psi = \Delta\phi$ and $\psi_{[t]} = \phi_t$, that is

$$\psi_{[t]}(x) = \frac{1}{t\Delta(x)}\psi\left(\frac{x}{t}\right)$$

and moreover, we put for $u, j \in \mathbf{N}$

$$\psi_{(j)}^{(u)}(x) = \left(\frac{d}{dx}\right)^u \psi(x) \cdot x^j.$$

We recall that, by identifying a K-bi-invariant function f on G with an even function on $\mathbf{R}$, the Laplace-Beltrami operator Ω acts on f as $\Omega(f) = \Delta^{-1}(\Delta f')' = \mathbf{D} \cdot f' + f''$, where

$$\mathbf{D}(x) = \Delta'(x)\Delta(x)^{-1} = m_1(\mathrm{cth}x) + 2m_2(\mathrm{cth}2x)$$

(see (**??**), (**??**)). Then it is easy to see that for each k,

$$(\Omega^k \phi_t)(x) = \frac{1}{t^{2k}} \left(\sum_{u=0}^{2k} (\phi_{(u-2k)}^{(u)})_{[t]}(x) Q_u^k(x) \right),$$

where $Q_u^k(x)$ is a polynomial with degree $\leq k$ of $(d/dx)^{\ell-1}\mathbf{D}(x) \cdot x^\ell$ $(1 \leq \ell \leq k)$. We note that

$$\left| \left(\frac{d}{dx}\right)^{\ell-1} \mathbf{D}(x) \cdot x^\ell \right| \leq c \begin{cases} 1 + |x| & \text{if } \ell = 1 \\ 1 & \text{if } \ell > 1. \end{cases}$$

Since ϕ is supported on $[-1, 1]$, we may suppose that $|x| \leq t$. Thereby, we can deduce that $|Q_u^k(x)| \leq c(1+t)^k$. Then the argument used in the proof of [**8**, Lemma 6.5] yields the desired estimate. ■

Lemma 3.5. *There exists a constant c such that for all $t > 1$, $\lambda \in \mathbf{R}$, and $0 \leq n \leq N$,*

$$\left| \left(\frac{d}{d\lambda}\right)^n \hat{\phi}_t(\lambda + i\rho) \right| \leq ct^n(1 + |t\lambda|)^{-(2M+\alpha+1/2)}.$$

Proof. Substituting the Harish-Chandra and Gangolli expansions of φ_λ (see (**??**) and (**??**) respectively) with (**??**), we can expand $\hat{\phi}_t(\lambda)$ as

$$\begin{aligned} \hat{\phi}_t(\lambda) = \frac{1}{t}\int_0^\infty \Delta\left(\frac{x}{t}\right)\phi\left(\frac{x}{t}\right)\varphi_\lambda(x)dx = \int_0^1 \Delta(x)\phi(x)\varphi_\lambda(tx)dx \\ = \int_0^1 \Delta(x)\phi(x)e^{-\rho tx}\left(\Phi(\lambda, tx)C(\lambda)e^{i\lambda tx} + \Phi(-\lambda, tx)C(-\lambda)e^{-i\lambda tx}\right)dx \\ = I(\lambda) + I(-\lambda), \end{aligned}$$

where

$$\begin{aligned} I(\lambda) &= \int_0^1 \Delta(x)\phi(x)e^{-\rho tx}\Phi(\lambda,tx)C(\lambda)e^{i\lambda tx}dx \\ &= \sum_{m=0}^{\infty}\Gamma_m(\lambda)C(\lambda)\int_0^1 \Delta(x)\phi(x)e^{-\rho tx}e^{-2mtx}e^{i\lambda tx}dx \\ &= \sum_{m=0}^{\infty}\Gamma_m(\lambda)C(\lambda)I_m(\lambda). \end{aligned} \tag{25}$$

In order to obtain the desired estimate of $\hat{\phi}_t(\lambda+i\rho)$, we first estimate $I_m(\xi)$, $\xi = \pm(\lambda+i\rho)$, in (**??**) and combine it with estimates for $\Gamma_m(\xi)$ and $C(\xi)$. Then we sum up these estimates of $\Gamma_m(\xi)C(\xi)I_m(\xi)$ with respect to m.

As for $I_m(\lambda+i\rho)$, since $(\phi\Delta)(x) = O(x^{2M+2\alpha+1})$ (see (**??**), (**??**)), integration by parts yields that for $0 \le l \le 2M$,

$$\begin{aligned} &(it\lambda)^l\left(\frac{d}{d\lambda}\right)^n I_m(\lambda+i\rho) \\ =\ &(it\lambda)^l\int_0^1(\phi\Delta)(x)e^{-2(m+\rho)tx}(itx)^n e^{i\lambda tx}dx \\ =\ &(it)^n\int_0^1(\phi\Delta)(x)x^n e^{-2(m+\rho)tx}\cdot\left(\frac{d}{dx}\right)^l e^{i\lambda tx}dx \\ =\ &(it)^n(-1)^l\int_0^1\left(\frac{d}{dx}\right)^l\left((\phi\Delta)(x)\cdot x^n e^{-2(m+\rho)tx}\right)e^{i\lambda tx}dx \end{aligned}$$

and it is dominated by

$$ct^n\sum_{p=0}^{l}((m+\rho)t)^{l-p}\int_0^1\left|\left(\frac{d}{dx}\right)^p((\phi\Delta)(x)x^n)\right|e^{-2(m+\rho)tx}dx. \tag{26}$$

Let $0<\delta<1$ and set $\beta_\delta(p) = (l-p)+2\alpha+1+\delta$ $(0\le p\le l)$. We take a constant C such that $|x|^{\beta_\delta(p)}e^{-x}\le C$ for all $x\in\mathbf{R}_+$ and $0\le p\le l$. Then, since $2(m+\rho)\ge 1$ for all m, $(d/dx)^p((\phi\Delta)(x)x^n)\cdot e^{-2(m+\rho)tx}$ in (**??**) is estimated for $0<x<1$ as

$$\begin{aligned} &\left|\left(\frac{d}{dx}\right)^p((\phi\Delta)(x)x^n)\right|\cdot C|2(m+\rho)tx|^{-\beta_\delta(p)} \\ \le\ &cC|x|^{2M+2\alpha+1+n-p-\beta_\delta(p)}((m+\rho)t)^{-\beta_\delta(p)} \\ \le\ &cC|x|^{-\delta}((m+\rho)t)^{-\beta_\delta(p)}. \end{aligned}$$

Therefore, the integral over $[0,1]$ is finite and (**??**) is dominated by $t^{n-(2\alpha+1+\delta)}$ $(m+\rho)^{-(2\alpha+1+\delta)}$. Hence, we have

$$\left|\left(\frac{d}{d\lambda}\right)^n I_m(\lambda+i\rho)\right| \le ct^{n-(2\alpha+1+\delta)}(m+\rho)^{-(2\alpha+1+\delta)}(1+|t\lambda|)^{-2M}.$$

As for $I_m(-(\lambda+i\rho))$, we can repeat the exactly same process after replacing $e^{-2(m+\rho)tx}$ by e^{-2mtx}. However, when $m=0$, we have no exponential term in (**??**) and thus, we cannot take $\beta_\delta(p)$ for $I_0(-(\lambda+i\rho))$ as before. In this case we note that $I_0(-(\lambda+i\rho))$ is nothing but the Euclidean Fourier transform $(\phi\Delta)^{\sim}(t\lambda)$ of $(\phi\Delta)(x)$ and thus, it is rapidly decreasing with respect to $|t\lambda|$. Hence, we have for arbitrary $n' \ge 0$,

$$\begin{aligned} &\left|\left(\frac{d}{d\lambda}\right)^n I_m(\pm(\lambda+i\rho)\right| \\ \le\ & ct^n \begin{cases} t^{-(2\alpha+1+\delta)}(m+\rho)^{-(2\alpha+1+\delta)}(1+|t\lambda|)^{-2M} & \text{if } m>0 \\ (1+|t\lambda|)^{-n'} & \text{if } m=0. \end{cases} \end{aligned} \tag{27}$$

On the other hand, for $\xi = \pm(\lambda+i\rho)$, $\Gamma_m(\xi)$ and $C(\xi)$ satisfy the following estimates (see Proposition 8.2 in §8 and [**3**, Theorem 2]): For each $n \in \mathbf{N}$

$$\left|\left(\frac{d}{d\lambda}\right)^n \Gamma_m(\pm(\lambda+i\rho))\right| \le c(1+m)^{2\alpha+\delta/2}, \tag{28}$$

$$\left|\left(\frac{d}{d\lambda}\right)^n C(\pm(\lambda+i\rho))\right| \le c(1+|\lambda|)^{-(\alpha+1/2+n)}. \tag{29}$$

Therefore, letting $n' = 2M+\alpha+1/2$ and substituting (**??**), (**??**) and (**??**) into (**??**), we see that for $t>1$

$$\begin{aligned} \left|\left(\frac{d}{d\lambda}\right)^n I(\pm(\lambda+i\rho))\right| &\le ct^n(1+|t\lambda|)^{-(2M+\alpha+1/2)}\sum_{m=0}^{\infty}(1+m)^{-(1+\delta/2)} \\ &\le ct^n(1+|t\lambda|)^{-(2M+\alpha+1/2)}. \end{aligned}$$

This completes the proof of the lemma. ■

Lemma 3.6. *Let notations be as above.*

(1) $\hat{\phi}_t(\lambda+i\rho) \to 1$ as $|t\lambda| \to 0$,
(2) $|\hat{\phi}_t(\lambda+i\rho)| \ge 1/2$ if $0 \le |t\lambda| \le 2$.

Proof. (1) We note that $\hat{\phi}_t(\lambda+i\rho) = \int_0^1 \phi(x)\varphi_{\lambda+i\rho}(tx)\Delta(x)dx$ and $\varphi_{i\rho} \equiv 1$. Hence $\hat{\phi}_t(i\rho) = 1$ by (**??**) and it follows from [**3**, Lemma 11, Lemma 14] that

$$\begin{aligned} |\hat{\phi}_t(\lambda+i\rho) - 1| &= |\int_0^1 \phi(x)\Delta(x)\left(\varphi_{\lambda+i\rho}(tx) - \varphi_{i\rho}(tx)\right)dx| \\ &\leq |\lambda| \int_0^1 \phi(x)\Delta(x) \left| \left(\frac{d}{d\lambda}\right) \varphi_{\lambda+i\rho}(tx)|_{\lambda=\lambda_0} \right| dx, \ 0 \leq \lambda_0 \leq \lambda \\ &\leq |t\lambda| \int_0^1 \phi(x) x \Delta(x) dx. \end{aligned}$$

(2) We note that the last term is dominated by $|t\lambda|/4$ (see (**??**)). Therefore, if $0 \leq |t\lambda| \leq 2$, then $|\hat{\phi}_t(\lambda+i\rho)| = |(\hat{\phi}_t(\lambda+i\rho) - 1) + 1| \geq -1/2 + 1 = 1/2$. ■

We here put $\ell(t,\lambda) = \hat{\phi}_t(\lambda+i\rho) = W_+(\phi_t)^{\sim}(\lambda)$ and refer to §8.B. Then Lemmas 3.4, 3.5, and 3.6 imply that $\ell(t,\lambda)$ belongs to the class $\mathsf{A}_{N,2M}$ for all $N \in \mathbf{N}$ (see Definition 8.4) and, furthermore, Lemma 3.6 (2) implies that $\ell(t,\lambda)$ satisfies the assumption (**??**) in Theorem 8.6. Since $M_\ell = M_\phi^{\mathbf{R}}$ (see Definitions 3.2 and 8.5), Theorem 8.6 yields the following.

Theorem 3.7. *Let ϕ be as above and suppose that $M \geq 2$. Then $F \in H^1(\mathbf{R})$ if and only if $M_\phi^{\mathbf{R}} F \in L^1(\mathbf{R})$:*

$$\|F\|_{H^1(\mathbf{R})} \approx \|M_\phi^{\mathbf{R}} F\|_{L^1(\mathbf{R})}.$$

4. Real Hardy spaces on G.

Let ϕ be the same as in the previous section (see (**??**),(**??**)) and $M_\phi, M_\phi^{\mathbf{R}}$ the corresponding radial maximal operators on G and $\mathbf{R}$ respectively (see Definitions 3.1 and 3.2). In this section we shall define two real Hardy spaces $H_\phi^1(G//K)$ and $W_-(H^1(\mathbf{R}))$ on G and give a relation between them. Especially, we shall give a simple proof of Theorem A based on fractional calculus.

We introduce the real Hardy space $H_\phi^1(G//K)$ on G as follows.

Definition 4.1. *We define*

$$H_\phi^1(G//K) = \{f \in L^1_{\text{loc}}(G//K) \ ; \ M_\phi f \in L^1(G//K)\}$$

and $\|f\|_{H_\phi^1(G)} = \|M_\phi f\|_1$.

Since $\|f\|_1 \leq \|M_\phi f\|_1$, it follows that

$$H_\phi^1(G//K) \subset L^1(G//K).$$

Let M_s, $s \geq 0$, denote the Euclidean Fourier multiplier corresponding to $(\lambda + i\rho)^s$:

$$\mathsf{M}_s(F)^{\sim}(\lambda) = (\lambda + i\rho)^s F^{\sim}(\lambda).$$

We introduce a pull-back of $H^1(\mathbf{R})$ to G via W_+ (see (**??**)) and M_s:

Definition 4.2. *For $s \geq 0$, we define*

$$W_-(\mathsf{M}_{-s}(H^1(\mathbf{R}))) = \{f \in L^1_{\mathrm{loc}}(G//K) \ ; \mathsf{M}_s \circ W_+(f) \in H^1(\mathbf{R})\}$$

and give the norm by $\|\mathsf{M}_s \circ W_+(f)\|_{H^1(\mathbf{R})}$. We denote $W_-(\mathsf{M}_0(H^1(\mathbf{R})))$ by $W_-(H^1(\mathbf{R}))$.

An easy consequence of Proposition 3.3 and Theorem 3.7 is the following.

Corollary 4.3. *Let $M \geq 2$. There exists a positive constant c such that $\|W_+(f)\|_{H^1(\mathbf{R})} \leq c\|f\|_{H^1_\phi(G)}$ for all $f \in H^1_\phi(G//K)$ and thus,*

$$H^1_\phi(G//K) \subset W_-(H^1(\mathbf{R})).$$

We shall conversely control the H^1_ϕ-norm of f by using $W_+(f)$. Before starting the argument we shall obtain some basic properties of the Eucliden fractional integral operators $W^{\mathbf{R}}_\mu$ and $\tilde{W}^{\mathbf{R}}_\mu$ on $\mathbf{R}_+$, which correspond to the case of $\alpha = \beta = -1/2$ and $\sigma = 1$ in (**??**) and (**??**) respectively: For $\mu \in \mathbf{C}$ and $y > 0$,

$$W^{\mathbf{R}}_\mu(f)(y) = \frac{(-1)^n}{\Gamma(\mu + n)} \int_y^\infty \frac{d^n f(x)}{dx^n}(x - y)^{\mu+n-1} dx \tag{30}$$

and

$$\tilde{W}^{\mathbf{R}}_\mu(f)(y) = \frac{1}{\Gamma(\mu + n)} \frac{d^n}{dy^n} \int_0^y f(x)(x - y)^{\mu+n-1} dx, \tag{31}$$

where $n = 0$ if $\Re\mu > 0$ and $-n < \Re\mu \leq -n + 1$, $n = 0, 1, 2, \cdots$, if $\Re\mu \leq 0$. Let $0 \leq \mu < 1$ and let f be an integrable function on $\mathbf{R}_+$. Then, by changing the order of integration and by using integration by parts, we see that $W^{\mathbf{R}}_\mu \circ W^{\mathbf{R}}_{-\mu}(f) = f$ if $f'(x)x^{-\mu}$ is integrable, and $W^{\mathbf{R}}_{-\mu} \circ W^{\mathbf{R}}_\mu(f) = f$ if $f(x)x^{\mu-1}$ is integrable. Moreover, if f, g are integrable functions on $\mathbf{R}_+$ such that $f(x)x^{\mu-1}, g'$ are integrable and $g(0) = \lim_{x\to\infty} g'(x) = 0$, then

$$\langle f, g\rangle_{L^1(\mathbf{R}_+)} = \langle W^{\mathbf{R}}_\mu(f), \tilde{W}^{\mathbf{R}}_{-\mu}(g)\rangle. \tag{32}$$

Lemma 4.4 *Let f, f' be integrable on $x \geq 0$ and $0 \leq \mu < 1$.*

(1) *For* $\gamma \le 0$,

$$\int_x^\infty f'(s)s^\gamma(s-x)^{-\mu}ds = cW^{\mathbf{R}}_{-\mu}(f)(x)x^\gamma + \int_x^\infty W^{\mathbf{R}}_{-\mu}(f)(s)B_\gamma(x,s)ds,$$

where $B_\gamma(x,s)$ *is smooth on* $0 < x \le s$ *and there exists a constant* $c > 0$ *such that*

$$|B_\gamma(x,s)| \le cx^{\gamma-1} \quad \text{for all } 0 < x \le s. \tag{33}$$

(2) *For* $\gamma > 0$,

$$\int_x^\infty f'(s)e^{-\gamma s}(s-x)^{-\mu}ds = ce^{-\gamma x}\left(W^{\mathbf{R}}_{-\mu}(f)(x) + \int_x^\infty f(s)C(x,s)ds\right),$$

where $C(x,s)$ *is smooth on* $0 < x \le s$ *and there exists a constant* $c > 0$ *such that*

$$\int_0^s |C(x,s)|dx \le c \quad \text{for all } s > 0. \tag{34}$$

(3) *Let* g *be a smooth function.*

$$W^{\mathbf{R}}_{-\mu}(f\cdot g)(x) = cW^{\mathbf{R}}_{-\mu}(f)(x)\cdot g(x) + \int_x^\infty W^{\mathbf{R}}_{-\mu}(f)(s)D_g(x,s)ds,$$

where $D_g(x,s)$ *is smooth on* $0 < x \le s$ *and there exists a constant* $c > 0$ *such that*

$$|D_g(x,s)| \le c \sup_{x\le t\le s}|g'(t)| \quad \text{for all } 0 < x \le s.$$

Especially, if g *is supported on* $[0,1]$ *or* g *is constant on* $[1,\infty)$, *then* $D_g(x,s) = 0$ *for* $1 \le x \le s$.
(4) *For* $\gamma > 0$,

$$W^{\mathbf{R}}_{-\mu}(e^{-\gamma x}f)(x) = ce^{-\gamma x}\left(W^{\mathbf{R}}_{-\mu}(f)(x) + \int_x^\infty f(s)C(x,s)ds\right),$$

where $C(x,s)$ *is smooth on* $0 < x \le s$ *and satisfies* (**??**).

Proof. We note that

$$\begin{aligned}
&\int_x^\infty f'(s)s^\gamma(s-x)^{-\mu}ds \\
=\ &\int_x^\infty \left(f'(s)x^\gamma + f'(s)(s^\gamma - x^\gamma)\right)(s-x)^{-\mu}ds \\
=\ &cW^{\mathbf{R}}_{-\mu}(f)(x)x^\gamma - \int_x^\infty f(s)\left((s^\gamma - x^\gamma)(s-x)^{-\mu}\right)' ds \\
=\ &cW^{\mathbf{R}}_{-\mu}(f)(x)x^\gamma + \int_x^\infty W^{\mathbf{R}}_{-\mu}(f)(s)B_\gamma(x,s)ds,
\end{aligned}$$

where

$$\begin{aligned}B_\gamma(x,s) &= -\tilde{W}^{\mathbf{R}}_{\mu}\left(\left((s^\gamma - x^\gamma)(s-x)^{-\mu}\right)'\chi_{[x,\infty)}\right)(s)\\ &= -\int_x^s \left(\left(\gamma t^{\gamma-1}(t-x)^{-\mu} - \mu(t^\gamma - x^\gamma)(t-x)^{-\mu-1}\right)(s-t)^{\mu-1}dt.\end{aligned}$$

Since $\gamma - 1 < 0$ and $(t^\gamma - x^\gamma)/(t-x) \leq \gamma x^{\gamma-1}$ for $0 < x < t$, it follows that

$$|B_\gamma(x,s)| \leq c x^{\gamma-1}\int_x^s (t-x)^{-\mu}(s-t)^{\mu-1}dt \sim x^{\gamma-1}.$$

(2): We note that

$$\begin{aligned}&\int_x^\infty f'(s)e^{-\gamma s}(s-x)^{-\mu}ds\\ &= \int_x^\infty f'(s)e^{-\gamma x}\left(1 + (e^{-\gamma(s-x)} - 1)\right)(s-x)^{-\mu}ds\\ &= ce^{-\gamma x}\left(W^{\mathbf{R}}_{-\mu}(f)(x) + \int_x^\infty f(s)C(x,s)ds\right),\end{aligned}$$

where

$$C(x,s) = \left((e^{-\gamma(s-x)} - 1)(s-x)^{-\mu}\right)'\chi_{[x,\infty)}(s).$$

Therefore,

$$\begin{aligned}\int_0^s |C(x,s)|dx &= c\int_x^s \left|\left((e^{-\gamma(t-x)} - 1)(t-x)^{-\mu}\right)'\right|dt\\ &\leq c\sup_{x\leq t\leq s}|(e^{-\gamma(t-x)} - 1)(t-x)^{-\mu}| \leq c.\end{aligned}$$

(3): Since

$$\begin{aligned}(f(s)g(s))' &= f'(s)g(x) + f'(s)\left(g(s) - g(x)\right) + f(s)g'(s)\\ &= f'(s)g(x) + (f(s)(g(s) - g(x))',\end{aligned}$$

it follows that

$$\begin{aligned}&W^{\mathbf{R}}_{-\mu}(f\cdot g)(x)\\ &= \int_x^\infty (f(s)g(s))'(s-x)^{-\mu}ds\\ &= g(x)W^{\mathbf{R}}_{-\mu}(f) + \mu\int_x^\infty f(s)(g(s) - g(x))(s-x)^{-\mu-1}ds \qquad (35)\\ &= g(x)W^{\mathbf{R}}_{-\mu}(f) + \int_x^\infty W^{\mathbf{R}}_{-\mu}(f)(s)D_g(x,s)ds,\end{aligned}$$

where

$$\begin{aligned} D_g(x,s) &= c\tilde{W}^{\mathbf{R}}_{\mu}\left((g(t)-g(x))\,(t-x)^{-\mu-1}\chi_{[x,\infty)}(t)\right)(s) \\ &= c\int_x^s \left(\frac{g(t)-g(x)}{t-x}\right)(t-x)^{-\mu}(s-t)^{\mu-1}dt \\ &\leq c\sup_{x\leq t\leq s}|g'(t)|\int_x^s (t-x)^{-\mu}(s-t)^{\mu-1}dt. \end{aligned}$$

Then the desired result follows.
(4): We put $C(x,s)=e^{\gamma x}(e^{-\gamma s}-e^{-\gamma x})(s-x)^{-\delta-1}$ in (**??**). Since

$$\int_0^s |C(x,s)|dx = \int_0^s |e^{-\gamma x}-1|x^{-\mu-1}ds \leq \int_0^\infty (1-e^{-\gamma x})x^{-\mu-1}dx,$$

$C(x,s)$ satisfies (**??**). ■

We shall deduce the local and global forms of the Weyl type fractional operator W_- in (**??**). In what follows, for simplicity, we put

$$\begin{aligned} \alpha-\beta &= n+\delta, \qquad n=[\alpha-\beta], \\ \beta+1/2 &= n'+\delta', \qquad n'=[\beta-1/2]. \end{aligned} \tag{36}$$

Furthermore, we denote

$$\underline{n}=n+n', \quad \underline{\delta}=\delta+\delta', \quad s_\alpha=\alpha+1/2=\underline{n}+\underline{\delta} \tag{37}$$

and

$$\underline{D}=\{0,\delta,\delta',\delta+\delta'\}. \tag{38}$$

Clearly, from the explicit values of α,β (cf. [**5**, Table 1 in p.265]), we may suppose that $\delta=0$. However, for the sake of the Fourier-Jacobi analysis (cf. [**4**]) we dare not take $\delta=0$.

Proposition 4.5. (1) *If F is W_+-smooth and supported on $0<x\leq 1$, then*

$$\begin{aligned} &|W_-(F)(x)| \\ &\leq c\sum_{m,\xi}\left(x^{-2s_\alpha+m+\xi}W^{\mathbf{R}}_{-(m+\xi)}(F)(x)+\int_x^\infty |W^{\mathbf{R}}_{-(m+\xi)}(F)(s)|A^1_{m,\xi}(x,s)ds\right), \end{aligned}$$

where the sum is taken over $0\leq m\leq \underline{n}$ and $\xi\in\underline{D}$, and $A^1_{m,0}(x,s)\equiv 0$ and $A^1_{m,\xi}(x,s)$ satisfies

$$0\leq A^1_{m,\xi}(x,s)\leq x^{-2s_\alpha+m+\xi-1} \quad \textit{for all } 0<x\leq s. \tag{39}$$

(2) *If F is W_+-smooth, supported on $x \geq 1$ and F, F' are integrable, then*

$$
\begin{aligned}
& |W_-(F)(x)| \\
\leq\; & c \sum_{m,\xi} \Bigg(\Big(x^{-2s_\alpha+m+\xi} W^{\mathbf{R}}_{-(m+\xi)}(F)(x) + \int_x^\infty |W^{\mathbf{R}}_{-(m+\xi)}(F)(s)| A^2_{m,\xi}(x,s)ds \\
& + x^{-2s_\alpha+m+\xi} \int_x^\infty |W^{\mathbf{R}}_{-(m+\xi)}(F)(s)| A^3_{m,\xi}(x,s)ds \Big) \chi_{[0,1]}(x) \\
& + e^{-2\rho x} \Big(W^{\mathbf{R}}_{-(m+\xi)}(F)(x) + \int_x^\infty |W^{\mathbf{R}}_{-(m+\xi)}(F)(s)| A^4_{m,\xi}(x,s)ds \Big) \chi_{[1,\infty)}(x) \Bigg),
\end{aligned}
$$

*where $A^j_{m,0}(x,s) \equiv 0$ and $A^j_{m,\xi}(x,s) \geq 0$ for $j = 2,3,4$, and $A^2_{m,\xi}(x,s)$ satisfies (**??**) and there exists a positive constant c such that for $j = 3,4$,*

$$
\int_0^s A^j_{m,\xi}(x,s)dx \leq c \quad \textit{for all } s > 0. \tag{40}
$$

Proof. We shall consider the case of $0 < \delta, \delta' < 1$. Other cases easily follows from the same argument. In the following we use the same letters $B_\gamma(x,s)$ and $C(x,s)$ to denote different functions satisfying (**??**) and (**??**) respectively.

(1): Let F be differentiable and supported on $[0,1]$. For $\sigma \geq 0$, Lemma 4.4 (1) yields

$$
\begin{aligned}
W^\sigma_{-\delta}(F)(x) &= c \int_x^\infty \frac{dF}{d\mathrm{ch}\sigma x}(s)(\mathrm{ch}\sigma s - \mathrm{ch}\sigma x)^{-\delta} \mathrm{sh}\sigma s ds \\
&\sim \int_x^\infty F'(s)(s-x)^{-\delta}(s+x)^{-\delta} ds \\
&= c x^{-\delta} W^{\mathbf{R}}_{-\delta}(F)(x) + \int_x^\infty W^{\mathbf{R}}_{-\delta}(F)(s) B_{-\delta}(x,s) ds
\end{aligned}
$$

and for $p \in \mathbf{N}$, Lemma 4.4 (3) gives

$$
W^{\mathbf{R}}_{-\delta}(x^{-p}F)(x) = c x^{-p} W^{\mathbf{R}}_{-\delta}(F)(x) + \int_x^\infty W^{\mathbf{R}}_{-\delta}(F)(s) D_{x^{-p}}(x,s) ds.
$$

Therefore, combining these equations, we have

$$
\begin{aligned}
W^\sigma_{-\delta}(x^{-p}F)(x) \;\sim\; & x^{-(p+\delta)} W^{\mathbf{R}}_{-\delta}(F)(x) + x^{-\delta} \int_x^\infty W^{\mathbf{R}}_{-\delta}(F)(s) D_{x^{-p}}(x,s) ds \\
& + \int_x^\infty s^{-p} W^{\mathbf{R}}_{-\delta}(F)(s) B_{-\delta}(x,s) ds \\
& + \int_x^\infty \left(\int_s^\infty W^{\mathbf{R}}_{-\delta}(F)(t) D_{x^{-p}}(s,t) dt \right) B_{-\delta}(x,s) ds.
\end{aligned}
$$

Since $|D_{x^{-p}}(x,s)| \leq cx^{-p-1}$, $|B_{-\delta}(x,s)| \leq cx^{-\delta-1}$ and

$$|\int_x^t D_{x^{-p}}(s,t)B_{-\delta}(x,s)ds| \;\; \leq \;\; cx^{-\delta-1}\int_x^t s^{-p-1}ds \sim x^{-(p+\delta)-1}$$

for $0 < x \leq s$, it follows that

$$W^{\sigma}_{-\delta}(x^{-p}F)(x) \sim x^{-(p+\delta)}W^{\mathbf{R}}_{-\delta}(F)(x) + \int_x^{\infty} W^{\mathbf{R}}_{-\delta}(F)(s)B_{-(p+\delta)}(x,s)ds. \quad (41)$$

We here apply $W^{\sigma'}_{-\delta'}$ $(\sigma' \geq 0)$ to (**??**). Repeating the above argument, we see that

$$\begin{aligned} & W^{\sigma'}_{-\delta'}(x^{-(p+\delta)}W^{\mathbf{R}}_{-\delta}F)(x) \\ = \;\; & cx^{-(p+\delta+\delta')}W^{\mathbf{R}}_{-(\delta+\delta')}(F)(x) + \int_x^{\infty} W^{\mathbf{R}}_{-(\delta+\delta')}(F)(s)B_{x^{-(p+\delta+\delta')}}(x,s)ds \end{aligned}$$

and

$$\begin{aligned} & W^{\sigma'}_{-\delta'}\left(\int_x^{\infty} W^{\mathbf{R}}_{-\delta}(F)(s)B_{-(p+\delta)}(x,s)ds\right) \\ = \;\; & \int_x^{\infty} W^{\mathbf{R}}_{-\delta}(F)(s)\left(W^{\sigma'}_{-\delta'}B_{-(p+\delta)}\right)(x,s)ds \\ = \;\; & \int_x^{\infty} W^{\mathbf{R}}_{-(\delta+\delta')}(F)(s)\tilde{W}^{\mathbf{R}}_{\delta'}\left(W^{\sigma'}_{-\delta'}B_{-(p+\delta)}\right)(x,s)ds. \end{aligned}$$

Since

$$\left|\tilde{W}^{\mathbf{R}}_{\delta'}\left(W^{\sigma'}_{-\delta'}B_{-(p+\delta)}\right)(x,s)\right| \leq cx^{-(p+\delta)-1-2\delta'+\delta'} = x^{-(p+\delta+\delta')-1}$$

for $0 < x \leq s$ (see [**10**, Lemma 3.1, Lemma 3.4]), it follows that

$$\begin{aligned} W^{\sigma'}_{-\delta'} \circ W^{\sigma}_{-\delta}(x^{-p}F)(x) \;\; \sim \;\; & x^{-(p+\delta+\delta')}W^{\mathbf{R}}_{-(\delta+\delta')}(F)(x) \\ + \;\; & \int_x^{\infty} W^{\mathbf{R}}_{-(\delta+\delta')}(F)(s)B_{-(p+\delta+\delta')}(x,s)ds. \quad (42) \end{aligned}$$

Now let $\gamma = -(\alpha - \beta) = -(n + \delta)$ and $\gamma' = -(\beta + 1/2) = -(n' + \delta')$ (see (**??**)). Since $W^{\sigma}_{-1}(F)(x) = dF/d(\text{ch}\sigma x) \sim F'(x)/x$, it follows that $W^{\sigma}_{-n}(F)(x) \sim \sum_{m=0}^{n} c_m x^{-(2n-m)}F^{(m)}(x)$ and thus

$$W^{\sigma}_{-n}(e^{-\rho x}F)(x) \sim \sum_{m=0}^{n}\sum_{\ell=0}^{m} c_{m\ell}x^{-(2n-m)}F^{(\ell)}(x), \quad (43)$$

$$W^{\sigma}_{-n}(x^{-p}F)(x) \sim \sum_{m=0}^{n}\sum_{\ell=0}^{m} c_{m\ell}x^{-(2n-\ell+p)}F^{(\ell)}(x). \tag{44}$$

Here we note that

$$W^{\sigma'}_{\gamma'} \circ W^{\sigma}_{\gamma} = W^{\sigma'}_{-n'} \circ \left(W^{\sigma'}_{-\delta'} \circ W^{\sigma}_{-\delta}\right) \circ W^{\sigma}_{-n}. \tag{45}$$

Then, combining (??), (??) and (??) in this order, we finally obtain that

$$\begin{aligned}
& W^{\sigma'}_{\gamma'} \circ W^{\sigma}_{\gamma}(e^{-\rho x}F)(x) \\
\sim\ & \sum_{m=0}^{n}\sum_{m'=0}^{n'}\sum_{\ell=0}^{m}\sum_{\ell'=0}^{m'} \Bigg(x^{-(2n+2n'-m-\ell'+\underline{\delta})} W^{\mathbf{R}}_{-(\ell+\ell'+\underline{\delta})}(F)(x) \\
& + x^{-(2n'-m')}\int_x^{\infty} W^{\mathbf{R}}_{-(\ell+\underline{\delta})}(F)(s)B_{-(2n-m-m'+\underline{\delta})}(x,s)ds \Bigg) \\
\sim\ & \sum_{m=0}^{n}\sum_{m'=0}^{n'}\sum_{\ell=0}^{m}\sum_{\ell'=0}^{m'} \Bigg(x^{-2s_\alpha+\underline{\delta}+m+\ell'} W^{\mathbf{R}}_{-(\ell+\ell'+\underline{\delta})}(F)(x) \\
& + \int_x^{\infty} W^{\mathbf{R}}_{-(\ell+\underline{\delta})}(F)(s)B_{-2s_\alpha+\underline{\delta}+m}(x,s)ds \Bigg).
\end{aligned} \tag{46}$$

Since $0 \le x \le 1$, we can replace x^m and $B_{-2s_\alpha+\underline{\delta}+m}$ by x^ℓ and $B_{-2s_\alpha+\underline{\delta}+\ell}$ respectively, the desired result follows.

(2) Let F be differentiable and supported on $[1,\infty)$. We keep the notations in (1). It follows from Lemma 4.4 (2) that

$$\begin{aligned}
W^{\sigma}_{-\delta}(F)(x) &= c\int_x^{\infty} \frac{dF}{d\mathrm{ch}\sigma x}(s)(\mathrm{ch}\sigma s - \mathrm{ch}\sigma x)^{-\delta}\mathrm{sh}\sigma s ds \\
&\sim \int_x^{\infty} F'(x)e^{-\delta\sigma s}(1+(s-x)^{-\delta})ds \\
&= ce^{-\delta\sigma x}\left(W^{\mathbf{R}}_{-\delta}(F)(s) + \int_x^{\infty} F(s)C(x,s)ds + F(x)\right).
\end{aligned}$$

Then, by substituting F with $e^{-\xi x}F$, Lemma 4.4 (4) yields that for $\xi > 0$,

$$\begin{aligned}
& W^{\sigma}_{-\delta}(e^{-\xi x}F)(x) \\
=\ & ce^{-(\xi+\delta\sigma)x}\Bigg(W^{\mathbf{R}}_{-\delta}(F)(x) + \int_x^{\infty} F(s)C(x,s)ds \\
& + e^{\xi x}\int_x^{\infty} F(s)e^{-\xi s}C(x,s)ds + F(x)\Bigg) \\
=\ & ce^{-(\xi+\delta\sigma)x}\left(W^{\mathbf{R}}_{-\delta}(F)(x) + \int_x^{\infty} F(s)C(x,s)ds + F(x)\right).
\end{aligned} \tag{47}$$

Let $\Phi(x)$ be an even C^∞ function on $\mathbf{R}$ such that $0 \leq \Phi(x) \leq 1$, $\Phi(x) \equiv 1$ if $|x| \leq 1/2$, and $\Phi(x) \equiv 0$ if $|x| \geq 1$. We put

$$W^{\sigma}_{-\delta}(e^{-\xi x}F) = W^{\sigma}_{-\delta}(e^{-\xi x}F)(1-\Phi) + W^{\sigma}_{-\delta}(e^{-\xi x}F)\Phi. \tag{48}$$

We apply $W^{\sigma'}_{-\delta'}$ to each term in (??). We first substitute (??) into (??) and, to each resultant term we apply Lemma 4.4 (3) and (??) with σ, δ, ξ replaced by $\sigma', \delta', \xi + \delta\sigma$:

$$\begin{aligned} W^{\sigma'}_{-\delta'}\left(e^{-(\xi+\delta\sigma)x}W^{\mathbf{R}}_{-\delta}(F)(1-\Phi)\right) &\sim e^{-(\xi+\delta\sigma+\delta'\sigma')x}\Bigg(W^{\mathbf{R}}_{-(\delta+\delta')}(F)(x) \\ &+ \int_x^\infty W^{\mathbf{R}}_{-\delta}(F)(s)C(x,s)ds + W^{\mathbf{R}}_{-\delta}(F)(x)\Bigg)(1-\Phi) + K_1(x) \qquad (49) \\ \sim\ & e^{-(\xi+\delta\sigma+\delta'\sigma')x}\Bigg(W^{\mathbf{R}}_{-(\delta+\delta')}(F)(x) \\ &+ \int_x^\infty \left(W^{\mathbf{R}}_{-\delta}(F)(s) + W^{\mathbf{R}}_{-\underline{\delta}}(F)(s)\right)C(x,s)ds + W^{\mathbf{R}}_{-\delta}(F)(x)\Bigg), \end{aligned}$$

where, if we denote the first term in the right hand side of (??) by $I_1(x)(1-\Phi)$, then $K_1(x)$ is given by $K_1(x) = \displaystyle\int_x^s I_1(s)D_{1-\Phi}(x,s)ds$. Here, to deduce the last line, we used Lemma 4.4 (3) and the fact that $\displaystyle\int_x^t C(s,t)D_{1-\Phi}(x,s)ds$ satisfies (??). Similarly, we have

$$\begin{aligned} W^{\sigma'}_{-\delta}\left(e^{-(\xi+\delta\sigma)x}\int_x^\infty F(s)C(x,s)ds\cdot(1-\Phi)\right) &\sim e^{-(\xi+\delta\sigma+\delta'\sigma')x} \\ \times\ \Bigg(\int_x^\infty F(s)\left(W^{\mathbf{R}}_{-\delta'}C\right)(x,s)ds &+ \int_x^\infty\left(\int_s^\infty F(t)C(s,t)dt\right)C(x,s)ds \\ &+ \int_x^\infty F(s)C(x,s)ds\Bigg)(1-\Phi) + K_2(x) \qquad (50) \\ \sim\ e^{-(\xi+\delta\sigma+\delta'\sigma')x}\Bigg(\int_x^\infty W^{\mathbf{R}}_{-\delta'}(F)(s)C(x,s)ds &+ \int_x^\infty F(s)C(x,s)ds\Bigg), \end{aligned}$$

where, if we denote the first term in the right hand side of (??) by $I_2(x)(1-\Phi)$, then $K_2(x)$ is given by $K_2(x) = \displaystyle\int_x^s I_2(s)D_{1-\Phi}(x,s)ds$. In this case, to deduce the last line, we used Lemma 4.4 (3), (??) and the facts that $\tilde{W}^{\mathbf{R}}_{\delta'} \circ W^{\mathbf{R}}_{-\delta'}(C) \sim$

C (see [**10**, Lemma 3.1]) and $\displaystyle\int_x^t C(s,t)C(x,s)ds$ satisfies (**??**). Therefore, we see that

$$
\begin{aligned}
& W^{\sigma'}_{-\delta'}\left(W^{\sigma}_{-\delta}(e^{-\xi x}F)(1-\Phi)\right) \\
\sim\ & e^{-(\xi+\delta\sigma+\delta'\sigma')x}\Bigg(W^{\mathbf{R}}_{-(\delta+\delta')}(F)(x)+W^{\mathbf{R}}_{-\delta}(F)(x)+W^{\mathbf{R}}_{-\delta'}(F)(x) \qquad (51)\\
+\ & \int_x^\infty \left(W^{\mathbf{R}}_{-\delta}(F)(s)+W^{\mathbf{R}}_{-\delta'}(F)(s)+W^{\mathbf{R}}_{-\underline{\delta}}(F)(s)+F(s)\right)C(x,s)ds\Bigg).
\end{aligned}
$$

As for $W^{\sigma'}_{-\delta'}\left(W^{\sigma}_{-\delta}(e^{-\xi x}F)\Phi\right)$, we use (**??**) and note that $W^{\sigma}_{-\delta}(e^{-\xi x}F)\Phi$ is supported on $[0,1]$. Hence, (**??**) with γ,ρ replaced by $0,\xi+\delta\sigma$ respectively yields that

$$
\begin{aligned}
& W^{\sigma'}_{-\delta'}\left(W^{\sigma}_{-\delta}(e^{-\xi x}F)\Phi\right) \\
=\ & W^{\sigma'}_{-\delta'}\left(e^{-(\xi+\delta\sigma)x}\Big(W^{\mathbf{R}}_{-\delta}(F)(x)+\int_x^\infty F(s)C(x,s)ds+F(x)\Big)\Phi\right) \\
\sim\ & \Bigg(x^{-\delta'}W^{\mathbf{R}}_{-\underline{\delta}}(F)(x)+\int_x^\infty W^{\mathbf{R}}_{-\underline{\delta}}(F)(s)B_{-\delta'}(x,s)ds \\
& +x^{-\delta'}\int_x^\infty F(s)\left(W^{\mathbf{R}}_{-\delta'}C\right)(x,s)ds \\
& +\int_x^\infty\left(\int_s^\infty F(t)\left(W^{\mathbf{R}}_{-\delta'}C\right)(s,t)dt\right)B_{-\delta'}(x,s)ds \\
& +x^{-\delta'}W^{\mathbf{R}}_{-\delta'}(F)(x)+\int_x^\infty W^{\mathbf{R}}_{-\delta'}(F)(s)B_{-\delta'}(x,s)ds\Bigg)\Phi \\
& +K_3(x) \qquad (52)\\
\sim\ & x^{-\delta'}\Big(W^{\mathbf{R}}_{-\delta'}(F)(x)+W^{\mathbf{R}}_{-\underline{\delta}}(F)(x)+\int_x^\infty W^{\mathbf{R}}_{-\delta'}(F)(s)C(x,s)ds\Big)\Phi \\
& +\int_x^\infty\left(W^{\mathbf{R}}_{-\underline{\delta}}(F)+W^{\mathbf{R}}_{-\delta'}(F)\right)(s)B_{-\delta'}(x,s)ds\cdot\Phi, \qquad (53)
\end{aligned}
$$

where, if we denote the first term in the right hand side of (**??**) by $I_3(x)\Phi$, then $K_3(x)$ is given by $K_3(x)=\displaystyle\int_x^s I_3(s)D_\Phi(x,s)ds$. Here, to deduce the last line, we used Lemma 4.4 (3), $\tilde{W}^{\mathbf{R}}_{\delta'}\circ W^{\mathbf{R}}_{-\delta'}(C)\sim C$ and the fact that $\left|\displaystyle\int_x^s C(s,t)B_{-\delta'}(x,s)ds\right|\le cx^{-\delta'-1}$. Combining (**??**) and (**??**), we can finally deduce that

$$\begin{aligned}
& W^{\sigma'}_{-\delta'} \circ W^{\sigma}_{-\delta}(e^{-\xi x}F)(x) \\
\sim\ & e^{-(\xi+\delta\sigma+\delta'\sigma')x}\Bigg(F(x) + W^{\mathbf{R}}_{-\delta}(F)(x) + W^{\mathbf{R}}_{-\delta'}(F)(x) + W^{\mathbf{R}}_{-\underline{\delta}}(F)(x) \\
& + \int_x^\infty \left(F(s) + W^{\mathbf{R}}_{-\delta}(F)(s) + W^{\mathbf{R}}_{-\delta'}(F)(s) + W^{\mathbf{R}}_{-\underline{\delta}}(F)(s)\right) C(x,s)ds\Bigg) \\
& + x^{-\delta'}\left(W^{\mathbf{R}}_{-\delta'}(F)(x) + W^{\mathbf{R}}_{-\underline{\delta}}(F)(x) + \int_x^\infty W^{\mathbf{R}}_{-\delta'}(F)(s)C(x,s)ds\right)\Phi \\
& + \int_x^\infty \left(W^{\mathbf{R}}_{-\delta'}(F) + W^{\mathbf{R}}_{-\underline{\delta}}(F)\right)(s)B_{-\delta'}(x,s)ds \cdot \Phi.
\end{aligned}$$

We substitute this formula to (**??**) and note that

$$W^{\sigma}_{-n}(e^{-\rho x}F)(x) \sim e^{-(n\sigma+\rho)x}\sum_{m=0}^{n} c_m F^{(m)}(x)$$

for $x \geq 1$. Then desired result follows from the same argument used in the previous case. ■

Now let $f \in W_-(\mathsf{M}_{-s_\alpha}H^1(\mathbf{R}))$, where $s_\alpha = \alpha + 1/2 = \underline{n} + \underline{\delta}$ (see (**??**)). Since the Fourier multiplier M_{-s_α} satisfies the Hörmander condition (cf. [**16**, §5 in Chap.11]), it is bounded on $H^1(\mathbf{R})$ (cf. [**16**, Theorem 4.4 in Chap.14]). In particular, $F = W_+(f)$ belongs to $H^1(\mathbf{R})$. We note that

$$W^{\mathbf{R}}_{-\gamma}(F)^{\sim}(\lambda) = (i\lambda)^{\gamma}F^{\sim}(\lambda) \tag{54}$$

(cf [**13**, (4.39')]) and thus, $\mathsf{M}^{-1}_{s_\alpha} \circ W^{\mathbf{R}}_{-\gamma}$, $0 \leq \gamma \leq s_\alpha$, is the the Fourier multiplier corresponding to $(i\lambda)^\gamma/(\lambda + i\rho)^{s_\alpha}$. Since it satisfies the Hörmander condition, $\mathsf{M}^{-1}_{s_\alpha}\circ W^{\mathbf{R}}_{-\gamma}$ is bounded on $H^1(\mathbf{R})$. Hence, each $W^{\mathbf{R}}_{-\gamma}(F)$ also belongs to $H^1(\mathbf{R})$. Therefore, the condition that $f \in W_-(\mathsf{M}_{-s_\alpha}H^1(\mathbf{R}))$ guarantees that for $0 \leq \gamma \leq s_\alpha$,

$$\|W^{\mathbf{R}}_{-\gamma}(F)\|_{H^1(\mathbf{R})} = \|M^{\mathbf{R}}_{\phi}(W^{\mathbf{R}}_{-\gamma}(F))\|_{L^1(\mathbf{R})} \leq c\|\mathsf{M}_{s_\alpha}(F)\|_{H^1(\mathbf{R})}.$$

Comparing $M^{\mathbf{R}}_{\phi}(W^{\mathbf{R}}_{-\gamma}(F))$ with $M_\phi f$, we have the following inequalities.

Theorem 4.6 *Let ϕ be as in §3 and $M \geq 2$. For $f \in W_-(\mathsf{M}_{-s_\alpha}H^1(\mathbf{R}))$ we put $F = W_+(f)$. Then there exist c_1, c_2 such that for all $0 \leq \gamma \leq s_\alpha$,*

$$\begin{aligned}
c_1\|M^{\mathbf{R}}_{\phi} \circ W^{\mathbf{R}}_{-\gamma}(F)(x)(\mathrm{th}x)^{\gamma}\|_{L^1(\mathbf{R})} &\leq \|f\|_{H^1_\phi(G)} \\
\leq c_2\sum_{m=0}^{\underline{n}}\sum_{\xi\in\underline{D}} &\|M^{\mathbf{R}}_{\phi} \circ W^{\mathbf{R}}_{-(m+\xi)}(F)(x)(\mathrm{th}x)^{m+\xi}\|_{L^1(\mathbf{R})}.
\end{aligned}$$

Especially,

$$\begin{aligned}
\|f\|_{H^1_\phi(G)} &\approx \sum_{m=0}^{n}\sum_{\xi\in\underline{D}} \|M^{\mathbf{R}}_\phi \circ W^{\mathbf{R}}_{-(m+\xi)}(F)(x)(\mathrm{th}x)^{m+\xi}\|_{L^1(\mathbf{R})} \\
&\le c\sum_{m=0}^{n}\sum_{\xi\in\underline{D}} \|W^{\mathbf{R}}_{-(m+\xi)}(F)\|_{H^1(\mathbf{R})} \\
&\le c\|\mathsf{M}_{s_\alpha}(F)\|_{H^1(\mathbf{R})}
\end{aligned}$$

and thus, $W_-(\mathsf{M}_{-s_\alpha}H^1(\mathbf{R})) \subset H^1_\phi(G//K)$.

Proof. We denote $\psi_t = W_+(\phi_t)$, where ψ_t is not a dilation of ψ, however, as shown in the previous section, $W_+(\phi_t)$ has the same properties as a dilation. Therefore, we use this notation to abbrebiate $W_+(\phi_t)$. Let $\Phi(x)$ be the same as in the proof of Proposition 4.5. We decompose $f * \phi_t$ as

$$\begin{aligned}
f*\phi_t &= W_-(W_+(f*\phi_t)) = W_-(F*\psi_t) \\
&= W_-(F*\psi_t\cdot\Phi) + W_-(F*\psi_t\cdot(1-\Phi))
\end{aligned}$$

(see (**??**)). We apply Proposition 4.5 (1) to $F*\psi_t\cdot\Phi$. Then Lemma 4.4 (3) and the same argument used in the proof of Proposition 4.5 yield that

$$\begin{aligned}
\sup_{0<t<\infty}|W_-(F*\psi_t\cdot\Phi)(x)|\Delta(x) \le c\sum_{m=0}^{n}\sum_{\xi\in\underline{D}}\Bigg(x^{m+\xi}M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-(m+\xi)}(F)(x) \\
+\int_x^\infty M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^1_{m,\xi}(x,s)ds\Delta(x)\Bigg)
\end{aligned}$$

Similarly, applying Proposition 4.5 (2) to $F*\psi_t\cdot(1-\Phi)$, we have

$$\begin{aligned}
&\sup_{0<t<\infty}|W_-(F*\psi_t\cdot(1-\Phi))(x)|\Delta(x) \\
\le\ & c\sum_{m=0}^{n}\sum_{\xi\in\underline{D}}\Big(x^{m+\xi}M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-(m+\xi)}(F)(x) \\
&+\int_x^\infty M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^2_{m,\xi}(x,s)ds\Delta(x) \\
&+x^{m+\xi}\int_x^\infty M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^3_{m,\xi}(x,s)ds\Big)\chi_{[0,1]}(x) \\
+\ & c\sum_{m=0}^{n}\sum_{\xi\in\underline{D}}\Big(M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-(m+\xi)}(F)(x) \\
&+\int_x^\infty M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^4_{m,\xi}(x,s)ds\Big)\chi_{[1,\infty)}(x).
\end{aligned}$$

We recall that $A^j_{m,\xi}(x,s)$, $j=3,4$, satisfy (??), and for $j=1,2$ and $0<x\le s\le 1$, $A^j_{m,\xi}(x,s)\Delta(x)\le cx^{m+\xi-1}$ (see (??), (??)) and thus,

$$\int_0^s A^j_{m,\xi}(x,s)\Delta(x)dx \le c\int_0^s x^{m+\xi-1}dx \le cs^{m+\xi}, \quad 0<s\le 1.$$

Therefore, we can deduce that

$$\begin{aligned} \|f\|_{H^1(G)} = \|M_\phi f\|_1 &= \int_0^\infty \sup_{0<t<\infty} |W_-(F*\psi_t)(x)|\Delta(x)dx \\ &\le \sum_{m=0}^{\underline{n}} \sum_{\xi\in\underline{D}} \|M^{\mathbf{R}}_\phi \circ W^{\mathbf{R}}_{-(m+\xi)}(F)(\text{th}x)^{m+\xi}\|_{L^1(\mathbf{R})}. \end{aligned}$$

As for the first inequality in the theorem we recall that

$$\begin{aligned} F*\psi_t(x) &= W_+(f*\phi_t)(x) \\ &= e^{\rho x}W^1_{\alpha-\beta}\circ W^2_{\beta+1/2}(f*\phi_t)(x) \\ &= ce^{\rho x}\int_x^\infty f*\phi_t(s)A(x,s)ds \end{aligned}$$

and for $0\le\gamma\le s_\alpha$ and $0<x\le s$,

$$|W^{\mathbf{R}}_{-\gamma}A(x,s)| \le ce^{(\rho-2)s}(\text{sh}2s)(\text{th}s)^{2\alpha-1-\gamma} \le ce^{\rho s}(\text{th}s)^{2\alpha-\gamma}$$

(see (??), (??)). Hence,

$$|W^{\mathbf{R}}_{-\gamma}(F)*\psi_t(x)| = |W^{\mathbf{R}}_{-\gamma}(F*\psi_t)(x)| \le ce^{\rho x}\int_x^\infty |f*\phi_t(s)|e^{\rho s}(\text{th}s)^{2\alpha-\gamma}ds.$$

We take the supremum over $0<t<\infty$. Since $2\alpha-\gamma\ge 2\alpha-s_\alpha=\alpha-1/2>-1$, we can deduce that,

$$\begin{aligned} &\int_0^\infty M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-\gamma}(F)(x)(\text{th}x)^\gamma dx \\ \le\ & c\int_0^\infty e^{\rho x}\left(\int_x^\infty M_\phi f(s)e^{\rho s}(\text{th}s)^{2\alpha-\gamma}ds\right)(\text{th}x)^\gamma dx \\ \le\ & c\int_0^\infty M_\phi f(s)e^{2\rho s}(\text{th}s)^{2\alpha+1}ds \sim \|M_\phi f\|_1 = \|f\|_{H^1(G)}. \end{aligned}$$

This completes the proof of the theorem. ■

Remark 4.7. (1) Since

$$C(-(\lambda+i\rho)) \sim (1+|\lambda|)^{-(\alpha+1/2)},\ \lambda\in\mathbf{R}$$

(see [**3**, Theorem 2]), Theorem 4.6 means that

$$W_{-}(\mathsf{M}_{C_\rho}(H^1(\mathbf{R}))) = W_{-}(\mathsf{M}_{-s_\alpha} H^1(\mathbf{R})) \subset H^1_\phi(G//K).$$

Theorem A in §1 follows from this relation and the one in Corollary 4.3.

(2) Since $(\mathrm{th}x)^{m+\xi}$ is bounded, it is easy to see that $M_\phi^{\mathbf{R}}$ in Theorem 4.6 can be replaced by M_ℓ, $\ell \in \mathsf{A}_{N,2M}$ (see Definitions 8.4 and 8.5, and cf. Theorem 8.6). Especially, $H^1_\phi(G//K)$ does not depend on an individual ϕ. We shall skip writing ϕ and denote simply as

$$H^1(G//K) = H^1_\phi(G//K), \quad \|f\|_{H^1(G)} = \|f\|_{H^1_\phi(G)}.$$

We keep the notations in the proof of Theorem 4.6 and we shall suppose that $t \geq 1$. Since we can transfer the Fourier multiplier $W^{\mathbf{R}}_{-\gamma}$ as

$$W^{\mathbf{R}}_{-\gamma}(F * \psi_t) = W^{\mathbf{R}}_{-\gamma}(F) * \psi_t = F * W^{\mathbf{R}}_{-\gamma}(\psi_t),$$

it follows from the proof of Theorem 4.6 that for each $C \geq 0$,

$$\begin{aligned} & \int_0^\infty \sup_{t\geq 1} |f * \phi_t(x)| \Delta(x) dx \\ \leq\ & c \sum_{m=0}^{\underline{n}} \sum_{\xi \in \underline{D}} \int_0^\infty \sup_{t \geq 1} |F * W^{\mathbf{R}}_{-(m+\xi)}(\psi_t)(x)| dx \\ \leq\ & c \sum_{m,\xi,m+\xi\neq 0} \int_0^\infty \sup_{t\geq 1} |F * \left(W^{\mathbf{R}}_{-(m+\xi)}(\psi_t) - C\psi_t\right)(x)| dx \\ +\ & d \int_0^\infty \sup_{t\geq 1} |F * \psi_t(x)| dx, \end{aligned}$$

where $d = c(1 + (4\underline{n} - 1)C)$. We note that

$$\begin{aligned} W^{\mathbf{R}}_{-\gamma}(\psi_t)^{\sim}(\lambda) &= W_{+}(\phi_t)^{\sim}(\lambda)(i\lambda)^\gamma \\ &= \hat{\phi}_t(\lambda + i\rho)(it\lambda)^\gamma \cdot t^{-\gamma}. \end{aligned} \tag{55}$$

Since $t \geq 1$, Lemma 3.5 implies that each $W^{\mathbf{R}}_{-\gamma}(\psi_t)^{\sim}(\lambda)$, $0 \leq \gamma \leq s_\alpha = \alpha + 1/2$, belongs to $\mathsf{A}_{N,2M}$ (see Definition 8.4). Moreover, if $\gamma > 0$, then $W^{\mathbf{R}}_{-\gamma}(\psi_t)^{\sim}(\lambda) \to 0$ as $|t\lambda| \to 0$, because it has the term $(t\lambda)^\gamma$ in (**??**). Since $(\psi_t)^{\sim}(\lambda) = \hat{\phi}_t(\lambda + i\rho)$ satisfies Lemma 3.6 (2), we may suppose that for a sufficiently large $C > 0$, each $\ell_{m+\xi}(t,\lambda) = W^{\mathbf{R}}_{-(m+\xi)}(\psi_t)^{\sim}(\lambda) - C\psi_t^{\sim}(\lambda)$ also belongs to $\mathsf{A}_{N,2M}$ and satisfies the assumption (**??**) in Theorem 8.6. Hence, Proposition 3.7 and Theorem 8.6 yield that, if $M \geq 2$ in (**??**), then

$$\begin{aligned} &\int_0^\infty \sup_{t\geq 1} |f * \phi_t(x)| \Delta(x) dx \\ \leq\ & c \int_0^\infty \left(\sum_{m,\xi,m+\xi\neq 0} \sup_{t\geq 1} |(\mathsf{M}_{\ell_{m+\xi}(t,\cdot)} F)(x)| + M_\phi^{\mathbf{R}} F(x) \right) dx \\ \leq\ & c \sum_{m,\xi,m+\xi\neq 0} \|M_{\ell_{m+\xi}} F\|_{L^1(\mathbf{R})} + c \|M_\phi^{\mathbf{R}} F\|_{L^1(\mathbf{R})} \leq c\|F\|_{H^1(\mathbf{R})} \end{aligned} \tag{56}$$

(see §8.B for the definitions of $\mathsf{M}_{\ell_{m+\xi}(t,\cdot)}$ and $M_{\ell_{m+\xi}}$). We here define a truncated maximal operator M_ϕ^{loc} on G as follows.

Definition 4.8. *For $f \in L^1_{\mathrm{loc}}(G//K)$,*

$$(M_\phi^{\mathrm{loc}} f)(g) = \sup_{0<t<1} |(f * \phi_t)(g)|, \quad g \in G.$$

Then (**??**) implies that $\|f\|_{H^1(G)} \leq \|M_\phi^{\mathrm{loc}} f\|_1 + c\|F\|_{H^1(\mathbf{R})}$. Hence, Corollary 4.3 yields the following.

Theorem 4.9. *Let $M \geq 2$. For $f \in H^1(G//K)$ we put $F = W_+(f)$. Then*

$$\|f\|_{H^1(G)} \approx \|M_\phi^{\mathrm{loc}} f\|_1 + \|F\|_{H^1(\mathbf{R})}.$$

5. Atomic Hardy spaces on G. In this section we shall introduce some K-bi-invariant atoms on G and define the corresponding atomic Hardy spaces on G, on which the radial maximal operator M_ϕ is bounded to $L^1(G)$.

For $x \in \mathbf{R}$ and $r > 0$, let $R(x,r)$ denote the interval on $\mathbf{R}$ centered at x with radius r and $|R(x,r)|$ its volume with respect to $\Delta(x)dx$:

$$R(x,r) = [x-r, x+r] \quad \text{and} \quad |R(x,r)| = \int_{x-r}^{x+r} \Delta(s) ds. \tag{57}$$

For $x \in G$ and $r > 0$, we also denote the annulus $\{y \in G; |\sigma(x) - \sigma(y)| \leq r\}$ $= K\{a_s; s \in R(\sigma(x), r)\}K$ on G by the same notation $R(x,r)$. Clearly, $R(x,r) = R(a_{\sigma(x)}, r)$ for $x \in G$. We put $B(r) = R(e,r)$ if $x = e$. Obviously, if $\sigma(x) \geq r$, then the volume $|R(x,r)|$ of $R(x,r)$ with respect to dg coincides with $|R(\sigma(x), r)|$ in (**??**). Moreover, if $\sigma(x) \leq r$, then $R(x,r) = B(\sigma(x)+r)$.

Definition 5.1. *We say that a K-bi-invariant function a on G is a $(1,\infty,0)$-atom provided that there exist $x \in G$ and $0 < r \leq \sigma(x)$ such that*

$$\begin{aligned} &(i) \quad supp(a) \subset R(x,r) \\ &(ii) \quad \|a\|_\infty \leq |R(x,r)|^{-1} \\ &(iii) \quad \int_G a(g)dg = 0. \end{aligned} \tag{58}$$

For $\epsilon \geq 0$, we say that a is a $(1,\infty,0,\epsilon)$-atom if we replace (ii) by

$$(ii)_\epsilon \quad \|a\|_\infty \leq |R(x,r)|^{-1}(1+r)^{-\epsilon}, \tag{59}$$

and a $(1,\infty,+)$-atom if we replace (iii) by

$$(iii)_+ \quad \int_G a(g)dg = 0 \quad \text{if } r < 1. \tag{60}$$

Moreover, if $x = 0$, we call a a centered atom.

We introduce atomic Hardy spaces on G as follows.

Definition 5.2. *We define*

$$\begin{aligned} H^1_{\infty,0}(G//K) \quad = \quad &\{f = \sum_i \lambda_i a_i \ ; \\ &a_i \text{ is a } (1,\infty,0)\text{-atom on } G \text{ and } \textstyle\sum_i |\lambda_i| < \infty\} \end{aligned}$$

and $\|f\|_{H^1_{\infty,0}(G)} = \inf \sum_i |\lambda_i|$, where the infimum is taken over all such representations $f = \sum_i \lambda_i a_i$. Similarly, replacing $(1,\infty,0)$-atoms by $(1,\infty,0,\epsilon)$-atoms and $(1,\infty,+)$-atoms, we define $H^{1,\epsilon}_{\infty,0}(G//K)$ and $H^{1,+}_{\infty,0}(G//K)$ respectively.

Definition 5.3. *We define the small Hardy space $h^1_{\infty,0}(G//K)$ on G by restricting $(1,\infty,0)$-atoms in the above definition of $H^1_{\infty,0}(G//K)$ to ones with radius ≤ 1.*

Clearly, we have

$$h^1_{\infty,0}(G//K) \subset H^{1,\epsilon}_{\infty,0}(G//K) \subset H^1_{\infty,0}(G//K) \subset H^{1,+}_{\infty,0}(G//K).$$

In what follows we shall characterize the difference between $h^1_{\infty,0}(G//K)$ and $H^{1,+}_{\infty,0}(G//K)$, and then we shall obtain a relation between $H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ and $H^1_\phi(G//K)$. For $x_0 \in G$ and $r > 0$, let $\chi_{x_0,r}$ denote the characteristic function of $R(x_0,r)$ on G. For $x, y \in G$, we define

$$I_{x_0}(x,r,y) = \int_K \chi_{x_0,r}(x^{-1}ky)dk. \tag{61}$$

Then $I_{x_0}(x,r,y)$ is K-bi-invariant with respect to x, y and, as a function of y, it is supported on $R(x, \sigma(x_0)+r)$, where $R(x,\sigma(x_0)+r) = B(\sigma(x)+\sigma(x_0)+r)$ if $\sigma(x) \le \sigma(x_0)+r$. For simplicity, when $x_0 = e$, we skip writing the suffices $x_0 = e$:

$$\chi_r = \chi_{e,r}, \quad I(x,r,y) = I_e(x,r,y).$$

Lemma 5.4. *Let $x, y \in G$ and $\sigma(x) > r$. Then*

(1) $I(x,r,y) \le I(x,r,x)$,
(2) $|B(r)||R(x,r)|^{-1} \le I(x,r,x) \le |B(2r)||R(x,r)|^{-1}$
(3) $I_{x_0}(x,r,y) \le |B(2r)||R(x,r)|^{-1}$ *if* $r \ge 1$, $\sigma(x) \ge 1$, $\sigma(x_0) \ge r+1$.

Proof. We regard $I(x,r,y)$ as a function on $\mathbf{R}_+ \times \mathbf{R}_+$. For a fixed x, as a function of y, it is supported on $|x-y| \le r$ and $I(x,r,y)$ is increasing on $x - r \le y \le x$. Hence (1) is obvious. As for (2), let f be an arbitrary function in $L^\infty(G//K)$. Then

$$\begin{aligned} \chi_r * f(x) &= \int_G \chi_r(xg^{-1}) f(g) dg \\ &= \int_{x-r}^{x+r} I(x,r,s) f(s) \Delta(s) ds. \end{aligned}$$

Therefore, letting $f \equiv 1$, we see from (1) that

$$\chi_r * f(x) = |B(r)| \le I(x,r,x)|R(x,r)|.$$

Similarly, letting f be the characteristic function of $R(x,|x-y|)$, we see that

$$\chi_r * f(x) \ge I(x,r,y)|R(x,|x-y|)|$$

and thus, $I(x,r,y) \le \|\chi_r\|_1 \|f\|_\infty |R(x,|x-y|)|^{-1} \le |B(r)||R(x,|x-y|)|^{-1}$. Hence

$$\begin{aligned} I(x,r,x) &\le I(x-r, 2r, x-r) \\ &\le I(x, 2r, x-r) \\ &\le |B(2r)||R(x,r)|^{-1}. \end{aligned}$$

As for (3), we may suppose that $x_0, x, y \in A_+ \cong \mathbf{R}_+$ and we use the kernel form of the product of spherical functions:

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty K(x,y,z)\varphi_\lambda(z)\Delta(z)dz$$

(see [**4**, (4.2)]). Since $x \geq 1, x_0 - r \geq 1$, it follows that

$$\begin{aligned} I_{x_0}(x,r,y) &= \int_{x_0-r}^{x_0+r} K(x,y,z)\Delta(z)dz \\ &\leq ce^{-\rho x}e^{-\rho(x_0+x-r)}\int_{x_0-r}^{x_0+r} e^{\rho z}dz \\ &\leq ce^{-2\rho x}e^{2\rho r} \sim |B(2r)||R(x,r)|^{-1}. \quad \blacksquare \end{aligned}$$

We set

$$\theta(g) = |B(1)|^{-1}\chi_1(g), \quad g \in G, \tag{62}$$

where χ_1 is the characteristic function of $B(1)$, and for each (not necessary K-bi-invariant) function f on G, we define a K-bi-invariant function $f_x^\flat$, $x \in G$, as

$$f_x^\flat(g) = \int_K \int_K f(x^{-1}kgk')dkdk' \quad (g \in G). \tag{63}$$

Proposition 5.5. *For $f \in H^{1,+}_{\infty,0}(G//K)$ there exist $f_0 \in h^1_{\infty,0}(G//K)$ and $x_i \in G$, $\lambda_i \in \mathbf{R}$ such that*

$$f = f_0 + \sum_i \lambda_i \theta^\flat_{x_i},$$

where $\|f_0\|_{h^1_{\infty,0}(G)}$ and $\sum_i |\lambda_i|$ are both bounded by $\|f\|_{H^{1,+}_{\infty,0}(G)}$.

Proof. By Definitions 5.2 and 5.3, it is enough to obtain the above decomposition for a $(1,\infty,+)$-atom a on G with radius $r \geq 1$. Let $B(a_{\sigma(x)}, r)$, $x \in \mathbf{R}_+$, denote the support of a. We identify K-bi-invariant functions on G with functions on $\mathbf{R}_+$, so a is supported on $R(x,r)$. Without loss of generality we may suppose that $x, r \in \mathbf{N}$. Actually, since $r \geq 1$, $|R(x,r)| \sim |R([x],[r])|$ uniformly on $x \geq 0$ and $r > 1$. We decompose $R(x,r)$ as

$$R(x,r) \subset \bigcup_{k=0}^{r} I_k, \quad I_k = R(x - r + 2k, 1),$$

and set $a_k = |I_k|^{-1}|R(x,r)| \cdot a|_{I_k}$. Then, a_k is supported on I_k, $\|a_k\|_\infty \leq |I_k|^{-1}$ and

$$a = \sum_{k=0}^{2r} \mu_k a_k, \;\; \mu_k = |I_k||R(x,r)|^{-1}.$$

Here $\sum_k \mu_k \sim 1$. Let $\theta_k = \theta^\flat_{a_{x-r+k}}$ $(0 \leq k \leq r)$, where a_{x-r+k} is the element in A_+ corresponding to $x - r + k \in \mathbf{R}_+$ (see (**??**), (**??**), (**??**)). Clearly, θ_k is supported on I_k, $\int_G \theta_k(g)dg = \int_G \theta(g)dg = 1$, and $\|\theta_k\|_\infty \le |I_k|^{-1}$ by Lemma 5.4 (1), (2). Let $m_k = \int_G a_k(g)dg$ and $b_k = (a_k - m_k\theta_k)/2$. Then $|m_k| \le 1$, b_k is supported on I_k, $\|b_k\|_\infty \le |I_k|^{-1}$, and $\int_G b_k(g)dg = 0$. This means that b_k is a $(1, \infty, 0)$-atom with radius 1. Therefore, letting

$$a = \sum_{k=0}^{2r} 2\mu_k b_k + \sum_{k=0}^{2r} \mu_k m_k \theta_k,$$

we have the desired decomposition of a $(1, \infty, +)$-atom a with $r \ge 1$. ■

We set

$$\langle\theta\rangle = \{\sum_i \lambda_i \theta^\flat_{x_i} \ ; \ \sum_i |\lambda_i| < \infty, x_i \in G\}.$$

Then Proposition 5.5 means that

$$H^{1,+}_{\infty,0}(G//K) = h^1_{\infty,0}(G//K) + \langle\theta\rangle. \tag{64}$$

Next we shall restrict our attention to *centered* atoms a and consider a linear combination of their averaged translations $a^\flat_x$, $x \in G$ (see (**??**)).

Definition 5.6. *We define*

$$\mathsf{H}^1_{\infty,0}(G//K) = \{f = \sum_i \lambda_i a^\flat_{i,x_i} \ ;$$
$$a_i \text{ is a } (1,\infty,0)\text{-centered atom on } G,\ x_i \in G, \text{ and } \textstyle\sum_i |\lambda_i| < \infty\}$$

and $\|f\|_{\mathsf{H}^1_{\infty,0}(G)} = \inf \sum_i |\lambda_i|$, *where the infimum is taken over all such representations* $f = \sum_i \lambda_i a^\flat_{i,x_i}$. *Similarly, replacing centered* $(1, \infty, 0)$*-atoms* a_i *by centered* $(1, \infty, 0, \epsilon)$*-atoms, centered* $(1, \infty, +)$*-atoms and centered atoms with radius* ≤ 1 *respectively, we define* $\mathsf{H}^{1,\epsilon}_{\infty,0}(G//K)$, $\mathsf{H}^{1,+}_{\infty,0}(G//K)$, $\mathsf{h}^1_{\infty,0}(G//K)$.

In this definition each atom a_i is K-bi-invariant (see Definition 5.1). We shall consider non-K-bi-invariant cases. Let a_i be a centered function on G satisfying (**??**) to (**??**), not necessary K-bi-invariant. Even if a_i is not K-bi-invariant, $a^\flat_{i,x_i}$ is K-bi-invariant. Hence, by using such a_i, let $\mathsf{H}^1_{\infty,0}(G//K)^\natural$ denote the space of all $f = \sum_i \lambda_i a^\flat_{i,x_i}$ with $x_i \in G$ and $\sum_i |\lambda_i| < \infty$. Similarly, we define $\mathsf{H}^{1,\epsilon}_{\infty,0}(G//K)^\natural$, $\mathsf{H}^{1,+}_{\infty,0}(G//K)^\natural$ and $\mathsf{h}^1_{\infty,0}(G//K)^\natural$ respectively as in Definition 5.6.

Proposition 5.7. *Let* $\epsilon > 0$. *We have the following inclusions:*

$$\mathsf{h}^1_{\infty,0}(G//K)^\natural \subset \mathsf{H}^{1,\epsilon}_{\infty,0}(G//K)^\natural \subset \mathsf{H}^1_{\infty,0}(G//K)^\natural \subset \mathsf{H}^{1,+}_{\infty,0}(G//K)^\natural$$

$$
\begin{array}{ccccccc}
\| & & \cup & & \cup & & \cup \\
h^1_{\infty,0}(G//K) & \subset & H^{1,\epsilon}_{\infty,0}(G//K) & \subset & H^1_{\infty,0}(G//K) & \subset & H^{1,+}_{\infty,0}(G//K) \\
\cup & & \cup & & \cup & & \cup \\
\mathsf{h}^1_{\infty,0}(G//K) & \subset & \mathsf{H}^{1,\epsilon}_{\infty,0}(G//K) & \subset & \mathsf{H}^1_{\infty,0}(G//K) & \subset & \mathsf{H}^{1,+}_{\infty,0}(G//K).
\end{array}
$$

Proof. The horizontal inclusions are clear from the definitions. We first prove that each space in the first line contains the bottom one in the second line.

Lemma 5.8. *Let a be a K-bi-invariant function on G supported on $R(x,r)$ and $\|a\|_\infty \leq |R(x,r)|^{-1}$. Let $r_0 = 2r$ if $r \leq 1$ and $r_0 = r+1$ if $r > 1$. Then there exists a constant $c > 0$ such that*

$$\|a/I(x,r_0,\cdot)\|_\infty \leq c|B(r)|^{-1},$$

where c is independent of x, r.

Proof. If $r \leq 1$, then Lemma 5.4 implies that for $|x-y| < r$,

$$
\begin{aligned}
I(x,r_0,y) &\geq I(x,r_0,x-r) \\
&\geq I(x-r,r,x-r) \\
&\geq |B(r)||R(x-r,r)|^{-1} \\
&\geq |B(r)||R(x,r)|^{-1}.
\end{aligned}
$$

Similarly, if $r > 1$, then $I(x,r_0,y) \geq I(x-r,1,x-r) \geq |B(1)||R(x-r,1)|^{-1} \sim e^{-2\rho(x-r)} \geq |B(r)||R(x,r)|^{-1}$. Thereby, $\|a/I(x,r_0,\cdot)\|_\infty \leq c|B(r)|^{-1}$. ■

We recall the relation $H^{1,\sharp}_{\infty,0}(G) = H^{1,\flat}_{\infty,0}(G)$ obtained in [**9**, Theorem 5.5]. Here $H^{1,\flat}_{\infty,0}(G)$ is nothing but $\mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp$ in this paper and $H^{1,\sharp}_{\infty,0}(G)$ contains $H^{1,+}_{\infty,0}(G//K)$ by Lemma 5.8. Then, it is easy to see that each space in the first line contains the bottom one in the second line. Furthermore, since $I(x,r,y) \leq c|R(x,r)|^{-1}$ if $r \leq 1$ by Lemma 5.4, it follows that $\mathsf{h}^1_{\infty,0}(G//K)^\natural \subset h^1_{\infty,0}(G//K)$.

Next we shall prove that each space in the second line contains the bottom one in the third line.

$\mathsf{H}^1_{\infty,0}(G//K) \subset H^1_{\infty,0}(G//K)$: Let a be a centered $(1,\infty,0)$-atom on G supported on $B(r)$ and $x \in G$. Then $a^\flat_x$ is supported on $R(x,r)$, $\displaystyle\int_G a^\flat_x(g)dg = \int_G a(g)dg = 0$, and $\|a^\flat_x\|_\infty \leq |R(x,r)|^{-1}\ |B(2r)|\ |B(r)|^{-1}$ by (**??**), (**??**) and Lemma 5.4 (2). Therefore, if $r \leq 1$, then $\|a^\flat_x\|_\infty \leq c|R(x,r)|^{-1}$. This means that $c^{-1}a_x^\flat$ is a $(1,\infty,0)$-atom on G. Let $r>1$ and $\sigma(x)\le 2$. We may regard that $a_x^\flat$ is supported on $B(\sigma(x)+r)$ and we note that $\|a_x^\flat\|_\infty \le |B(r)|^{-1} \le c|R(\sigma(x)+r)|^{-1}$. Therefore, $c^{-1}a_x^\flat$ is a $(1,\infty,0)$-atom on G. Let $r>1$ and $\sigma(x)>2$. In this case, since a is centered, a has an L^1 non-increasing denominator $|B(r)|^{-1}\chi_{B(r)}(x)$ (see [**7**, 4.4]). Then a can be decomposed as $a=\sum_i \lambda_i a_i$, where $\sum_i |\lambda_i| \sim 1$ and each a_i is a $(1,\infty,0)$-atom supported on $R(x_i,1)$, $x_i\in \mathbf{N}$ and $0<x_i\le r$ (see the proof of [**7**, Theorem 4.5]). Hence $a_{i,x}^\flat$ is supported on $R(x,x_i+1)$ and $\|a_{i,x}^\flat\|_\infty \le |R(x_i,1)|^{-1}|B(2)||R(x,1)|^{-1} \le c|R(x,x_i+1)|^{-1}$ by (**??**), (**??**) and Lemma 5.4 (3). Therefore, each $c^{-1}a_{i,x_i}^\flat$ is a $(1,\infty,0)$-atom on G. These observations imply that $a\in H^1_{\infty,0}(G//K)$ and its norm is bounded by a constant independent of a. Hence the desired inclusion follows.

$\mathsf{h}^1_{\infty,0}(G//K)\subset h^1_{\infty,0}(G//K)$: This is clear from the case of $r\le 1$ in the above argument.

$\mathsf{H}^{1,\epsilon}_{\infty,0}(G//K)\subset H^{1,\epsilon}_{\infty,0}(G//K)$: Let $r>1$ and a be a centered $(1,\infty,0,\epsilon)$-atom on G. We repeat the above argument. Since $\|a\|_\infty$ has an extra decay $r^{-\epsilon}$, $\|a_x^\flat\|_\infty$ in $\sigma(x)\le 2$ and $\|a_{i,x}^\flat\|_\infty$ in $\sigma(x)>2$ also have the same extra decay. Since $r^{-\epsilon}\le c(\sigma(x)+r)^{-\epsilon}$ in $\sigma(x)\le 2$ and $r^{-\epsilon}\le x_i^{-\epsilon}\le c(x_i+1)^{-\epsilon}$ in $\sigma(x)>2$, it follows that $a_x^\flat$ and $a_{i,x}^\flat$ are $(1,\infty,0,\epsilon)$-atoms on G up to a constant multiplication. Hence the desired inclusion follows.

$\mathsf{H}^{1,+}_{\infty,0}(G//K)\subset H^{1,+}_{\infty,0}(G//K)$: Let $r>1$ and a be a centered $(1,\infty,+)$-atom on G. Then Proposition 5.5 means that a is decomposed as $a=\sum_i \lambda_i a_i$, where $\sum_i |\lambda_i|\sim 1$ and each a_i is a $(1,\infty,+)$-atom supported on $R(x_i,1)$, $x_i\in\mathbf{N}$ and $|x_i|\le r$. As in the previous case, since $R(x_i,1)$ has the radius 1, each $a_{i,x}$, $x\in G$, is also a $(1,\infty,+)$-atom on G up to constant multiplication. Hence the desired inclusion follows. ∎

Let ϕ be the K-bi-invariant function on G introduced in §3 and M_ϕ^{loc} the truncated maximal operator in Definition 4.8. Since $H^{1,\flat}_{\infty,0}(G)$ in [**9**] coincides with $\mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp$, it follows from [**9**, Theorem 5.3] that

Proposition 5.9. *Let $M\ge 2$. M_ϕ is bounded from $\mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp \cap W_-(H^1(\mathbf{R}))$ to $L^1(G//K)$, that is, there exists a constant $c>0$ such that*

$$\|M_\phi f\|_1 \le c\left(\|f\|_{\mathsf{H}^{1,+}_{\infty,0}(G)^\sharp} + \|W_+(f)\|_{H^1(\mathbf{R})}\right)$$

for all $f\in \mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp\cap W_-(H^1(\mathbf{R}))$ and thus,

$$\mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp\cap W_-(H^1(\mathbf{R}))\subset H^1(G//K).$$

Theorem 5.10. *Let $M \geq 2$. M_ϕ is bounded from $H^{1,1}_{\infty,0}(G//K)$ to $L^1(G//K)$, that is, there exists a constant $c > 0$ such that*

$$\|M_\phi f\|_1 \leq c\|f\|_{H^{1,1}_{\infty,0}(G)}$$

for all $f \in H^{1,1}_{\infty,0}(G//K)$ and thus,

$$H^{1,1}_{\infty,0}(G//K) \subset H^1(G//K).$$

Proof. Since $H^{1,1}_{\infty,0}(G//K) \subset \mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp$ (see Proposition 5.7), it is enough to prove that $H^{1,1}_{\infty,0}(G//K) \subset W_-(H^1(\mathbf{R}))$ (see Proposition 5.9). We shall prove that, for all $(1,\infty,0,1)$-atoms a on G, $\|W_+(a)\|_{H^1(\mathbf{R})} \leq c$, where c is independent of a.

Lemma 5.11. *Let $\gamma \in \mathbf{R}$ and $\sigma > 0$. If F is supported on $R(x_0, r)$ and smooth if $\gamma < 0$, then $W^\sigma_\gamma(F)(|x|)$ is also supported on $R(x_0.r)$.*

Proof. When $\gamma = -n$, $n \in \mathbf{N}$, W^σ_{-n} is a differential operator $(d/d\text{ch}\sigma x)^n$ and thus, the desired result is obvious. Hence, it is enough to consider the case of $\gamma > 0$. We denote $F_0(x) = F(x + x_0)$. Then it follows from (**??**) that

$$\begin{aligned} W^\sigma_{-\delta}(F)(|x|) &= c\int_{|x|}^\infty F(s)(\text{ch}\sigma s - \text{ch}\sigma x)^{\gamma-1}\text{sh}s ds \\ &= c\int_{|x-x_0|}^\infty F_0(s)(\text{ch}\sigma(s+x_0) - \text{ch}\sigma x)^{\gamma-1}\text{sh}(s+x_0)ds \\ &= c\int_{|x-x_0|}^\infty F_0(s)\Big((\text{ch}\sigma s - \text{ch}\sigma(x-x_0))\text{ch}x_0 \\ &\quad +(\text{sh}\sigma s - \text{sh}\sigma(x-x_0))\text{sh}x_0\Big)^{\gamma-1}\Big(\text{sh}s\text{ch}x_0 + \text{ch}s\text{sh}x_0\Big)ds \\ &= G_0(x-x_0)\text{ch}x_0 + G_1(x-x_0)\text{sh}x_0, \end{aligned}$$

where

$$G_0(x) = c\int_{|x|}^\infty F_0(s)\Big((\text{ch}\sigma s - \text{ch}\sigma x)\text{ch}x_0 + (\text{sh}\sigma s - \text{sh}\sigma x)\text{sh}x_0\Big)^{\gamma-1}\text{sh}s ds$$

and $G_1(x)$ is defined by replacing $\text{sh}s ds$ with $\text{ch}s ds$. Since F_0 is supported on $B(r)$, clearly G_0 is also supported on $B(r)$. Therefore, $W^\sigma_{-\delta}(F)$ is supported on $R(x_0, r)$. ■

Let a be a $(1,\infty,0,1)$-atom supported on $R(x_0, r)$. By (**??**) and (**??**), $\|a\|_\infty \leq |R(x_0,r)|^{-1}(1+r)^{-1}$ and $\int_G a(g)dg = 0$. We put $A = W_+(a)$. Then

A is supported on $R(x_0, r)$ by Lemma 5.11 and

$$\int_{-\infty}^{\infty} A(x)dx = A^{\sim}(0) = \hat{a}(i\rho) = \int_G a(g)dg = 0.$$

Moreover, we recall that $|A(x)| \leq ce^{2\rho x}\text{th}(x_0+r)^{2s_\alpha}\|a\|_\infty$ (see (**??**), (**??**) and cf. [**10**], Lemma 3.4).

Case I: $x_0 - r \geq 1$. Since A is supported on $R(x_0, r)$ and

$$|R(x_0, r)| \sim \int_{x_0-r}^{x_0+r} e^{2\rho x}dx \sim e^{2\rho x_0}\text{sh}r,$$

it follows that $|A(x)| \leq ce^{2\rho(x_0+r)}(e^{2\rho x_0}\text{sh}r)^{-1}(1+r)^{-1} \leq cr^{-1}$.

Case II: $x_0 - r < 1$ and $r \geq 1$. Since $x_0 + r \geq 1$,

$$|R(x_0, r)| \geq c\int_{1}^{x_0+r} e^{2\rho x}dx \sim e^{2\rho(x_0+r)}.$$

Therefore, as in Case I, we have $\|A\|_\infty \leq cr^{-1}$.

Case III: $x_0 - r < 1$, $r < 1$ and $x_0 > 2r$. Since $x_0 > 2r$, it follows that $x_0 + r \geq 3$ and thus

$$|R(x_0, r)| \sim \int_{x_0-r}^{x_0+r} x^{2s_\alpha}dx \leq c(x_0 - r)^{s_\alpha}r.$$

Then, since $(x_0+r)/(x_0-r) \leq 3$, $|A(x)| \leq c\text{th}(x_0+r)^{s_\alpha}((x_0-r)^{s_\alpha}r)^{-1} \leq cr^{-1}$.

Case IV: $x_0 - r < 1$, $r < 1$ and $x_0 \leq 2r$. Since $x_0 + r \leq 3r < 3$ and $|R(x_0, r)| \geq |B(r)| \sim |B(3r)|$, we may suppose that a is a centered atom supported on $B(3r)$. Then $|A(x)| \leq c(\text{th}3r)^{s_\alpha}|B(3r)|^{-1} \leq cr^{-1}$.

These four cases imply that cA is a $(1, \infty, 0)$-atom on $\mathbf{R}$ and c is independent of a. ■

Theorem 5.12. *Let $M \geq 2$. Then M_ϕ is bounded from $\mathsf{H}^1_{\infty,0}(G//K)$ to $L^1(G//K)$, that is, there exists a constant $c > 0$ such that*

$$\|M_\phi f\|_1 \leq c\|f\|_{\mathsf{H}^1_{\infty,0}(G)}$$

for all $f \in \mathsf{H}^1_{\infty,0}(G//K)$ and thus,

$$\mathsf{H}^1_{\infty,0}(G//K) \subset H^1(G//K).$$

Proof. Since M_ϕ is sublinear and $\|M_\phi f^\flat_x\|_1 \leq \|M_\phi f\|_1$, in order to obtain the $\mathsf{H}^1_{\infty,0}$-L^1 boundedness of M_ϕ, it is enough to show that there exists a constant $c > 0$ such that $\|M_\phi a\|_1 \le c$ for all centered $(1,\infty,0)$-atoms a on G. Let $B(r)$ denote the support of a. Here we recall Theorem 4.5 in [**7**] and the proof. Since $|B(r)|^{-1}\chi_{B(r)}(x)$ is an L^1 non-increasing denominator of a, a has a $(1,\infty,0)$-atomic decomposition $a = \sum_i \lambda_i a_i$ on G such that $\sum_i |\lambda_i| \le c$, where each atom a_i has radius $r_i \le 1$. This means that a_i is a $(1,\infty,0,1)$-atom on G. Hence, it follows from Theorem 5.10 that $\|M_\phi a_i\| \le c$ and thus, $\|M_\phi a\|_1 \le c$, where c is independent of a. ■

6. Atomic decomposition of $H^1(G//K)$. We shall prove that each function f in $H^1(G//K)$ has a $(1,\infty,+)$-atomic decomposition on G. This means that $H^1(G//K) \subset H^{1,+}_{\infty,0}(G//K)$ and then our main Theorem C in §1 follows. In the following, first we shall introduce a space $W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$ and give a $(1,\infty,+)$-atomic decomposition on G for this space. Then, we shall prove that $H^1(G//K) \subset W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$ and obtain the desired $(1,\infty,+)$-atomic decomposition for $H^1(G//K)$.

We set

$$d_\alpha(x_0, r) = \int_{x_0-r}^{x_0+r} |\mathrm{th} x|^{s_\alpha}\, dx \tag{65}$$

and define $H^{1,+}_{\infty,0}(\mathbf{R})_\alpha$ as the space of all $F = \sum_i \lambda_i A_i$ such that $\sum_i |\lambda_i| < \infty$ and each A_i satisfies

$$\begin{aligned} &(i) \quad \mathrm{supp}(A_i) \subset R(x_i, r_i) \\ &(ii) \quad \|W^{\mathbf{R}}_{-s_\alpha}(A_i)\|_\infty \le d_\alpha(x_i, r_i)^{-1} \\ &(iii) \quad \int_{-\infty}^{\infty} A_i(x)dx = 0 \text{ if } r_i < 1. \end{aligned} \tag{66}$$

Definition 6.1. *We define*

$$W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha) = \{f \in L^1_{\mathrm{loc}}(G//K)\ ;\ W_+(f) \in H^{1,+}_{\infty,0}(\mathbf{R})_\alpha\}.$$

Proposition 6.2. *Functions in $W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$ have $(1,\infty,+)$-atomic decompositions, that is, $W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha) \subset H^1_{\infty,0}(G//K)$.*

Proof. Let $f \in W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$ and $F = W_+(f) = \sum_i \lambda_i A_i$ the decomposition of F given by (**??**). By using the same argument as in the proof of Proposition 5.5 (see (**??**)), we may suppose that $r_i \le 1$ in (**??**). More precisely, when $r_i > 1$, we decompose the support of A_i by using a smooth decomposition of 1, where each piece is supported in the interval with radius ≤ 1. Then $A_i = \sum_j A_{ij}$ and each A_{ij} satisfies (**??**) with radius ≤ 1 by

Lemma 6.3 below and Lemma 4.4 (3). Hence, we may suppose $r_i \leq 1$ to begin with.

When $|x_i| \leq 2r_i$, A_i is supported on $B(r_i')$, $r_i' = |x_i| + r_i \leq 3r_i$, and satisfies (ii) and (iii) with $r_i' \leq 3$, because

$$d_\alpha(x_i, r_i) \geq d_\alpha(0, r_i) \sim r_i^{s_\alpha+1} \sim d_\alpha(0, 3r_i).$$

Hence, we may suppose that $x_i = 0$ with $r_i \leq 3$ or $|x_i| > 2r_i$. We recall that $e^{-\rho x}W_+(f)(x) = F_f^0(x)$ is an even function on $\mathbf{R}$ (cf. [**9**, (3.6)]). Therefore, in the decomposition $e^{-\rho x}F = \sum_i \lambda_i e^{-\rho x}A_i$, the counterparts $e^{\rho x}A_i(-x)$ must appear in the decomposition. When $x_i = 0$ with $r_i \leq 3$, we may suppose that $e^{-\rho x}A_i(x)$ is even. Actually, we let

$$\big(e^{-\rho x}A_i(x) + e^{\rho x}A_i(-x)\big) = 2(1 + e^{2\rho r_i}) \cdot e^{-\rho x}B_i(x).$$

Then B_i is supported on $B(r_i)$ and $e^{-\rho x}B_i(x)$ is even. Moreover, if $r_i < 1$, then $\displaystyle\int B_i(x)dx = 0$, because A_i satisfies (iii) originally and the even property of $e^{-\rho x}F(x)$ implies that $\displaystyle\int_{-\infty}^{\infty} e^{2\rho x}A_i(-x)dx = 0$. Since $r_i \leq 3$, B_i also satisfies (ii) and $2(1+e^{2\rho r_i}) \sim 1$. Therefore, replacing the left hand side with the right one, we may suppose that $e^{-\rho x}A_i(x)$ is even if $x_i = 0$ with $r_i \leq 3$. Thereby, we can rearrange the decomposition of F as

$$F = \sum_i \lambda_i A_i + \sum_j \mu_j B_j + \sum_k \gamma_k E_k,$$

where each A_i satisfies (i), (ii) with $x_i = 0$, $r_i \leq 3$, $\displaystyle\int A_i(x)dx = 0$; each B_j satisfies (i) to (iii) with $|x_j| \geq 2r_j$, $r_j < 1$; each E_k satisfies (i), (ii) with $|x_k| \geq 2r_k$, $r_k \geq 1$, and moreover, $\sum_i |\lambda_i| + \sum_j |\mu_j| + \sum_k |\gamma_k| < \infty$. Since F is W_+-smooth, finally, we have

$$f = \sum_i \lambda_i a_i + \sum_j \mu_j b_j + \sum_k \gamma e_k, \tag{67}$$

where $a_i = W_-(A_i)$, $b_j = W_-(B_j)$ and $e_k = W_-(E_k)$. Lemma 5.11 implies that each a_i, b_j, e_k have the same supports as A_i, B_j, E_k respectively.

Now we apply fractional calculus in [**10**] to estimate each a_i, b_j, e_k. For simplicity, we skip writing the suffices i, j, k and denote the supports of a, b, e by $R(x_0, r)$. Without loss of generality, we may suppose that $x_0 \geq 0$.

As for e, since e is supported on $R(x_0, 1)$ and $x_0 \geq 2$, it follows that $x_0 - 1 \geq 1$ and thus, $d_\alpha(x_0, 1) \sim 1$. Thereby, (ii) and [**10**, Lemma 3.3] [1]

[1]If $f(x) = O(x^{2\tau})$ around $x = 0$, then $W_{-\mu}^\sigma(f)(x) = O(x^{2(\tau - \Re\mu)})$. However, under the assumption (ii) of (**??**), it follows that $W_{-\mu}^\sigma(f)(x) = O(x^{-\Re\mu})$.

imply that on the support of e

$$|e(x)| \leq c(\text{th}x)^{-(\alpha+1/2)} e^{-2\rho x} \leq c e^{-2\rho x} \leq c|R(x,1)|^{-1}.$$

This means that $c^{-1}e$ is a $(1,\infty,+)$-atom on G.

As for b, we recall that $x_0 = 0$ or $x_0 > 2r$.

Case I. $x_0 - r \geq 1$: Since $x_0 - r \geq 1$, $d_\alpha(x_0, r) \sim r$. Thereby, (ii) and [**10**, Lemma 3.3] imply that on the support of b

$$|b(x)| \leq c(\text{th}x)^{-(\alpha+1/2)} e^{-2\rho x} r^{-1} \leq c e^{-2\rho x} r^{-1} \leq c|R(x,r)|^{-1}.$$

This means that $c^{-1}b$ is a $(1,\infty,0)$-atom on G.

Case II. $x_0 - r < 1$: Since $r < 1$ and $x_0 > 2r$, it follows that $x_0 < r + 1 < 2$, $x_0 - r > x_0/2$, and $x_0 + r < 3x_0/2 < 3$. Therefore, $d_\alpha(x_0, r) \leq c(x_0 - r)^{s_\alpha} r$ and thus, on the support of b

$$|b(x)| \leq c(\text{th}x)^{-(\alpha+1/2)} e^{-2\rho x} r^{-1} (x_0 - r)^{-s_\alpha} \leq c(x_0 - r)^{-(2\alpha+1)} r^{-1}.$$

Since $(x_0 + r)/(x_0 - r) \leq 3$, it follows that

$$|R(x_0, r)| \leq c(x_0 + r)^{2\alpha+1} r \leq c(x_0 - r)^{2\alpha+1} r.$$

Therefore, $|b(x)| \leq c|R(x_0, r)|^{-1}$ on the support. This means that $c^{-1}b$ is a $(1,\infty,0)$-atom on G.

As for a, since $x_0 = 0$ and $r < 1$, it follows that $d_\alpha(0, r) \sim r^{s_\alpha+1}$ and

$$|a(x)| \leq c(\text{th}x)^{-(\alpha+1/2)} e^{-2\rho x} r^{-1} r^{-(s_\alpha+1)} \leq c\Delta(x)^{-1} r^{-1}. \tag{68}$$

We put

$$a_+(x) = c\Delta(x)^{-1} r^{-1} \chi_{[0,r]}(x), \ x > 0.$$

Clearly, $|a(x)| \leq a_+(x)$ and a_+ is a non-increasing function on $\mathbf{R}_+$ with finite L^1-norm:

$$\|a_+\|_{L^1(\Delta)} = \int_0^\infty a_+(x)\Delta(x)dx \leq c_0.$$

Since a is supported on $B(r)$ and $\int_G a(g)dg = \int_{-\infty}^\infty A(x)dx = 0$, it follows that $|B(s)|^{-1} \int_s^\infty a(x)\Delta(x)dx$ is also supported on $B(r)$ and

$$\frac{1}{|B(s)|} \int_s^\infty a(x)\Delta(x)dx = \frac{1}{|B(s)|} \int_0^s a(x)\Delta(x)dx \leq c\Delta(s)^{-1} r^{-1}. \tag{69}$$

Here we used (**??**) and $|B(s)| \sim \Delta(s)s$ if $s \le r \le 1$ (see (**??**)). Hence,

$$\frac{1}{|B(s)|}\int_s^\infty a(x)\Delta(x)dx \le a_+(s). \tag{70}$$

This means that ca_+ is an L^1 non-increasing denominator of a satisfying (**??**). Then [**7**, Theorem 4.5] yields that a has a centered $(1,\infty,0)$-atomic decomposition $a = \sum_j \gamma_j a_j$ on G such that $\sum_j |\gamma_j| \le c\|a_+\|_{L^1(\Delta)} \le cc_0$. Especially, $a \in H^1_{\infty,0}(G//K)$ and $\|a\|_{H^1_{\infty,0}(G)} \le cc_0$.

These three cases imply that all a_i, b_j, e_k in (**??**), and thus f also, belong to $H^{1,+}_{\infty,0}(G//K)$. This completes the proof of the proposition. ■

Theorem 6.4. *Let notations be as above. Then*

$$\begin{aligned} H^1(G//K) &= \mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp \cap W_-(H^1(\mathbf{R})) \\ &= H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R})). \end{aligned}$$

Proof. Because of Proposition 5.9 and Corollary 4.3, it is enough to prove that $H^1(G//K) \subset H^{1,+}_{\infty,0}(G//K)$. First we shall consider the case that $\delta = \delta' = 0$, that is, s_α is an integer (see (**??**)).

Let $f \in H^1(G//K)$ and put $F = W_+(f)$. Then it follows from Theorem 4.6 that $\|M^{\mathbf{R}}_\phi \circ W^{\mathbf{R}}_{-s_\alpha}(F)(x)(\mathrm{th}x)^{s_\alpha}\|_{L^1(\mathbf{R})} < \infty$. We here recall the constructive proof of the atomic decomposition of $H^1(\mathbf{R})$: For example, we refer to [**12**]. Then, measuring the volume of the dyadic cubes (intervals in the one-dimensional case) Q^k_j appeared in the Whitney decomposition theorem by (**??**) and keeping the convolution and the dilation, we can easily deduce that $W^{\mathbf{R}}_{-s_\alpha}(F)$ has a $(1,\infty,s_\alpha)$-atomic decomposition with respect to $|\mathrm{th}x|^{s_\alpha}dx$:

$$W^{\mathbf{R}}_{-s_\alpha}(F) = \sum_i \lambda_i B_i,$$

where B_i is supported on $R(x_i,r_i)$, $\displaystyle\int_{-\infty}^\infty B_i(x)x^k dx = 0$, $\le k \le s_\alpha$, $\|B_i\|_\infty \le d_\alpha(x_i,r_i)^{-1}$, and $\sum_i |\lambda_i| < \infty$. We set

$$F = \sum_i \lambda_i W^{\mathbf{R}}_{s_\alpha}(B_i) = \sum_i \lambda_i A_i.$$

Since s_α is an integer and each B_i satisfies the s_α-th moment condition, it follows that A_i is supported on $R(x_i,r_i)$ and $\displaystyle\int_{-\infty}^\infty A_i(x)dx = 0$. Moreover, $\|W^{\mathbf{R}}_{-s_\alpha}(A_i)\|_\infty = \|B_i\|_\infty \le d_\alpha(x_i,r_i)^{-1}$. Therefore, A_i satisfies (**??**) and thus, $F \in H^{1,+}_{\infty,0}(\mathbf{R})_\alpha$. Hence Proposition 6.2 implies that $f \in H^{1,+}_{\infty,0}(G//K)$.

We shall remove the assumption that $\delta = \delta' = 0$ and consider a general case of $0 \le \delta, \delta' < 1$. We recall the Fourier-Jacobi analysis (cf. [**4**]) and note that all results obtained in the previous sections can be extended to the Fourier-Jacobi analysis, that is, for arbitrary $\alpha, \beta \ge -1/2$. Especially, we may denote $W_\pm = W_\pm^{\alpha,\beta}$ and replace "$G//K$" as "(α, β)" such as

$$H^1(G//K) = H^1(\alpha, \beta).$$

For $f \in H^1(\alpha, \beta)$, we set

$$\begin{aligned} F = W_+^{\alpha,\beta}(f) &= e^{\rho x} W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(f) \\ &= e^{\rho x} W^1_\delta \circ W^1_n \circ W^2_{n'} \circ W^2_{\delta'}(f) \\ &= W_+^{n,n'}(S_{\delta,\delta'}(f)), \end{aligned}$$

where $\rho = \alpha + \beta + 1$ and

$$S_{\delta,\delta'}(f) = W_-^{n,n'} \circ W_+^{\alpha,\beta}(f).$$

We note that Theorem 4.6 means that

$$\sum_{m=0}^{n} \| M_\phi^{\mathbf{R}} \circ W_{-m}^{\mathbf{R}}(F) \|_{L^1(\mathbf{R})} < \infty$$

and thus, $S_{\delta,\delta'}(f)$ belongs to $H^1(n, n')$. Then, since n, n' are integers (see (**??**)), $S_{\delta,\delta'}(f)$ belongs to $H^{1,+}_{\infty,0}(n, n')$ and it has a $(1, \infty, +)$-atomic decomposition with respect to (n, n'):

$$S_{\delta,\delta'}(f) = \sum_i \lambda_i a_i,$$

where $\sum_i |\lambda_i| < \infty$. Here

$$F = \sum_i \lambda_i A_i, \quad A_i = W_+^{n,n'}(a_i)$$

and each A_i satisfies (**??**) with respect to (n, n'). We here recall that F is originally W_+-smooth. Hence, from a constructive proof of atomic decompositions $W^{\mathbf{R}}_{-(n+n')}(F) = \sum_i \lambda_i B_i$ (cf. [**4**, B in Chap. 3]) it follows that A_i satisfies (**??**) with respect to (α, β). We let $b_i = W_-(W_+^{n,n'}(a_i))$. Each b_i has the same support as a_i by Lemma 5.11 and $\int_G b_i(g)dg = A_i^\sim(0) = 0$ if $r_i < 1$. Moreover, we see from [**8**, Lemma 3.4] (or the same arguments in the case of

$\delta = \delta' = 0$) that $\|b_i\|_\infty \le c|R(x_i, r_i)|^{-1}$. Thereby, we see that $f = \sum_i \lambda_i b_i$ is a $(1, \infty, +)$-atomic decomposition of f. ■

Theorem 6.5. *Let* $\epsilon \ge 0$. *Then* $H^{1,\epsilon}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ *is dense in* $W_-(H^1(\mathbf{R}))$. *Especially,* $H^1(G//K)$ *is dense in* $W_-(H^1(\mathbf{R}))$.

Proof. Let $f \in W_-(H^1(\mathbf{R}))$. We approximate f with a rapidly decreasing function in $W_-(H^1(\mathbf{R}))$. For example, let us take $f_1 = f * \phi_t$ (see §3). Since $W_+(f_1) = W_+(f) * W_+(\phi_t)$ and $W_+(\phi_t)$ satisfies Lemma 3.6, it is easy to see that $F_1 = W_+(f_1) \in H^1(\mathbf{R})$ and $\|f - f_1\|_{H^1(G)} = \|F - F_1\|_{H^1(\mathbf{R})} = \|F - F * W_+(\phi_t)\|_{H^1(\mathbf{R})} \to 0$ as $t \to 0$ (cf. [**15**, Chap. 3, 5.1 (d)). We fix a sufficiently small $t > 0$ and let $F_1 = W_+(f * \phi_t) = \sum_{i \in I} \lambda_i A_i$ denote a $(1, \infty, 0)$-atomic decomposition of F_1. For a sufficiently small $\kappa > 0$ and a finite large index set $J \subset I$, we put

$$F_0 = \sum_{i \in J_0} \lambda_i A_i, \;\; J_0 = \{j \in J; r_i \ge \kappa\}, \tag{71}$$

where we take κ and J for which $e^{-\rho x} F_0(x)$ is even. We here recall a constructive proof of the atomic decomposition of $H^1(\mathbf{R})$ (cf. [**12**] and [**6**, B in Chap. 3]). Since F_1 is smooth, each A_i is also smooth and if $i \in J_0, r_i \le 1$, then

$$\|W^{\mathbf{R}}_{-s_\alpha}(A_i)\|_\infty \le c r_i^{-(s_\alpha+1)} \le c\kappa^{-s_\alpha} r_i^{-1}$$

and if $i \in J_0, r_i \ge 1$, then

$$\|W^{\mathbf{R}}_{-s_\alpha} A_i\|_\infty \le c r_i^{-1}. \tag{72}$$

In particular, $\|W^{\mathbf{R}}_{-s_\alpha} A_i\|_\infty \le R(x_i, r_i)^{-1}$ if $\mathrm{supp}(A_i) = R(x_i, r_i)$. Hence, each A_i, $i \in J_0, r_i \le 1$, satisfies (**??**) and thus, $F_0 \in H^{1,+}_{\infty,0}(\mathbf{R})_\alpha$. Then, by Proposition 6.2, we can define $f_0 \in H^{1,+}_{\infty,0}(G//K)$ such that $W_+(f_0) = F_0$. Since $F_0 \in H^1(\mathbf{R})$, it follows that $f_0 \in H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ and $\|f_1 - f_0\|_{H^1(G)} = \|F_1 - F_0\|_{H^1(\mathbf{R})} \to 0$ if $\kappa \to 0$ and $J \to I$. This means that $H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ is dense in $W_-(H^1(\mathbf{R}))$. Since

$$H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R})) \subset H^1(G//K) \subset W_-(H^1(\mathbf{R}))$$

(see Proposition 5.9 and Corollary 4.3), in order to obtain the desired density it is enough to show that f_0 belongs to $H^{1,\epsilon}_{\infty,0}(G//K)$.

We keep the atomic decomposition of F_0 in (**??**). Let $I_i = R(x_i, r_i)$ denote the support of A_i. As in the proof of Proposition 6.2, we may suppose that $x_i = 0$ or $x_i > 2r_i$. We recall that $e^{-\rho x} F_0$ is even and $I_i \cap -I_i = \emptyset$ if $0 \notin I_i$. When $x_i = 0$, we put

$$(e^{-\rho x} A_i(x) + e^{\rho x} A_i(-x)) = 2(1 + e^{2\rho r_i}) \cdot e^{-\rho x} B_i(x).$$

Then B_i is a $(1,\infty,0)$-atom on $\mathbf{R}$ and $e^{-\rho x}B_i(x)$ is even. Since the index set J_0 is finite, changing the coefficient λ_i by $2\lambda_i e^{2\rho r_i}$ and replacing the left hand side with the right one, we may suppose that $e^{-\rho x}A_i(x)$ is even if $x_i = 0$. Since each A_i is piecewise W_+-smooth, we can define a K-bi-invariant function $a_i = W_-(A_i)$ on G such that

$$f_0 = \sum_{i\in J_0} \lambda_i a_i.$$

Case I. $x_i > 2r_i$ and $r_i < 1$: As in the proof of Proposition 6.2, each a_i is a $(1,\infty,0)$-atom on G up to a constant multiplication.
Case II. $x_i > 2r_i$ and $r_i \geq 1$: Since $x_i > 2r_i$, it follows that $x_i + r_i \geq 3$ and $x_i - r_i \geq 1$. Thereby, $|R(x_i, r_i)| \sim e^{2\rho(x_i+r_i)}$. Since J_0 is finite, we may replace $\lambda_i a_i$ as

$$\lambda_i a_i = \lambda_i e^{4\rho r_i} r_i^{\epsilon} \cdot b_i,\ \ b_i = e^{-4\rho r_i} r_i^{-\epsilon} a_i.$$

Then, similarly as for (**??**), we can deduce from (**??**) that

$$\|b_i\|_\infty \leq c e^{-4\rho r_i} r_i^{-\epsilon} \Delta(x_i - r_i)^{-1} r_i^{-1} \leq c|R(x_i, r_i)|^{-1} r_i^{-\epsilon}.$$

This means that a is a $(1,\infty,0,\epsilon)$-atom on G up to a constant multiplication.
Case III. $x_i = 0$ and $r_i \leq 1$: As in the proof of Proposition 6.2, each a_i has a $(1,\infty,0)$-atomic decomposition.
Case IV. $x_i = 0$ and $r_i > 1$: Since J_0 is finite, we may replace $\lambda_i a_i$ as

$$\lambda_i a_i = \lambda_i r_i \cdot b_i,\ \ b_i = r_i^{-1} a_i.$$

Then, similarly as for (**??**), we can deduce from (**??**) that

$$|b_i(x)| \leq c(\mathrm{th} x)^{-(2\alpha+1)} e^{-2\rho x} r_i^{-2} \leq c\Delta(x) r_i^{-2}. \tag{73}$$

We set $b_+(x) = \Delta(x)^{-1} r_i^{-1}$. As in (**??**), using the moment condition, we see that

$$\frac{1}{|B(s)|}\int_s^\infty b_i(x)\Delta(x)dx \leq \frac{s}{|B(s)|} r_i^{-2} \leq b_+(s),\ \ 0 < s \leq r_i. \tag{74}$$

Therefore, b_+ is an L^1 non-increasing denominator of b_i satisfying (**??**). Hence, b_i has a $(1,\infty,0)$-atomic decomposition (see [**7**, Theorem 4.5]).

These four cases yield that $f_0 \in H^{1,\epsilon}_{\infty,0}(G//K)$. ■

Remark 6.6. As in the proof of Theorem 6.4, we set

$$S_{\gamma,\gamma'}(f) = W_-^{\alpha-\gamma,\beta-\gamma'} \circ W_+^{\alpha,\beta}(f)$$

for $0 \le \gamma \le \alpha + 1/2$, $0 \le \gamma' \le \beta + 1/2$. Clearly, $S_{0,0}$ is the identity operator and $S_{\alpha+1/2,\beta+1/2} = W_+^{\alpha,\beta}$. Then it follows that

$$H^1(\alpha, \beta) \subset S_{\gamma,\gamma'}^{-1}\Big(H^1(\alpha - \gamma, \beta - \gamma')\Big).$$

When $\gamma = \alpha + 1/2$ and $\gamma' = \beta + 1/2$, this relation is nothing but the one in Corollary 4.3, because $H^1(-1/2, -1/2) = H^1(\mathbf{R})$.

7. Other operators. We shall consider (H^1, L^1)-boundedness of singular integrals, heat and Poisson maximal operators $M_{\mathbf{H}}$ and $M_{\mathbf{P}}$, and the Riesz transform R on G.

Let $T_m^{\mathbf{R}} = \mathsf{M}_m$ be a Fourier multiplier corresponding to a bounded function m and K the distribution kernel of M_m, that is, $K^{\sim}(\lambda) = m(\lambda)$ and $\mathsf{M}_m(F) = K * F$ in a distribution sense (cf. [**15**, Chapter 1, §6]). We put $k = W_-(K)$ and define an operator T_m^G on G as $T_m^G(f) = k * f$. Then it is easy to see from (**??**), Definition 4.2, Theorem 4.6 that $\|T_m^G(f)\|_{W_-(H^1(\mathbf{R}))} = \|W_+(k * f)\|_{H^1(\mathbf{R})} = \|T_m^{\mathbf{R}}(W_+(f))\|_{H^1(\mathbf{R})}$ and $\|T_m^G(f)\|_{H^1(G)} \le c\|\mathsf{M}_{s_\alpha} W_+(T_m^G(f))\|_{H^1(\mathbf{R})} = \|\mathsf{M}_{s_\alpha} \circ T_m^{\mathbf{R}}(W_+(f))\|_{H^1(\mathbf{R})}$. Therefore, we have the following.

Theorem 7.1. *Let notation be as above. If $T_m^{\mathbf{R}}$ is bounded on $H^1(\mathbf{R})$, then T_m^G is bounded on $W_-(H^1(\mathbf{R}))$. Moreover, if $\mathsf{M}_{s_\alpha} \circ T_m^{\mathbf{R}}$ is bounded on $H^1(\mathbf{R})$, then T_m^G is bounded from $W_-(H^1(\mathbf{R}))$ to $H^1(G//K)$, especially, it is boundend on $H^1(G//K)$.*

Let $h_t(g)$ and $p_t(g)$, $g \in G, t > 0$, denote the heat and Poisson kernels on G. They are K-bi-invariant functions on G whose spherical Fourier transforms are respectively given by

$$\hat{h}_t(\lambda) = e^{-t(\lambda^2+\rho^2)} \quad \text{and} \quad \hat{p}_t(\lambda) = e^{-t\sqrt{\lambda^2+\rho^2}}.$$

Definition 7.2. *For $\epsilon \ge 0$ we define the modified heat and Poisson maximal operators $M_{\mathbf{H}}^\epsilon$ and $M_{\mathbf{P}}^\epsilon$ on G as follows.*

$$(M_{\mathbf{H}}^\epsilon f)(g) = \sup_{0<t<\infty} (1+t)^{-\epsilon} |(f * h_t)(g)|,$$
$$(M_{\mathbf{P}}^\epsilon f)(g) = \sup_{0<t<\infty} (1+t)^{-\epsilon} |(f * p_t)(g)|.$$

For simplicity, we denote $M_{\mathbf{H}}^0$ and $M_{\mathbf{P}}^0$ by $M_{\mathbf{H}}$ and $M_{\mathbf{P}}$ respectively.

It is well-known that $M_{\mathbf{H}}$ and $M_{\mathbf{P}}$ satisfy the maximal theorem (see [**14**, p.73 and p.48], [**1**, §3 and §6]). Moreover, if we define their truncated maximal operators $M_{\mathbf{H}}^{\text{loc}}$ and $M_{\mathbf{P}}^{\text{loc}}$ by restricting the range of the supremum as $0 < t < 1$ (cf. Definition 4.8), then they are bounded from $\mathsf{H}_{\infty,0}^{1,+}(G//K)^\sharp$ to $L^1(G//K)$ (see [**8**, Theorem 6.1, Remark 6.5]).

We here introduce a modified atomic Hardy space $H^{1,\epsilon}_{\infty,N}(\mathbf{R})$, $\epsilon \geq 0$, $N = 0, 1, 2, \cdots$, on $\mathbf{R}$ by replacing the norm condition $\|A\|_\infty \leq r^{-1}$ of a $(1, \infty, N)$-atom A on $\mathbf{R}$ with

$$\|A\|_\infty \leq r^{-1}(1+r)^{-\epsilon} \tag{75}$$

(cf. Definition 5.1 on G).

Definition 7.3. *We define*

$$W_-(L^1(\mathbf{R})) = \{f \in L^1_{\rm loc}(G//K) \ ; \ W_+(f) \in L^1(\mathbf{R})\}$$

and

$$W_-(H^{1,\epsilon}_{\infty,N}(\mathbf{R})) = \{f \in L^1_{\rm loc}(G//K) \ ; \ W_+(f) \in H^{1,\epsilon}_{\infty,N}(\mathbf{R})\}.$$

We give their norms by $\|W_+(f)\|_{L^1(\mathbf{R})}$ and $\|W_+(f)\|_{H^1(\mathbf{R})}$ respectively.

Clearly,

$$H^{1,+}_{\infty,0}(G//K)^\natural \subset L^1(G//K) \subset W_-(L^1(\mathbf{R}))$$

(see (18) and Definition 5.6). Since $H^{1,\epsilon}_{\infty,N}(\mathbf{R}) \subset H^1_{\infty,N}(\mathbf{R}) = H^1(\mathbf{R})$ (cf. [**6**, (3.30)]), it follows that $W_-(H^{1,\epsilon}_{\infty,N}(\mathbf{R})) \subset W_-(H^1(\mathbf{R})) \subset W_-(L^1(\mathbf{R}))$ and hence, Proposition 5.7 and Theorem 6.4 imply that

$$\begin{aligned} & H^1(G//K) \cap W_-(H^{1,\epsilon}_{\infty,N}(\mathbf{R})) \\ = \ & H^{1,+}_{\infty,0}(G//K) \cap W_-(H^{1,\epsilon}_{\infty,N}(\mathbf{R})) \\ = \ & \mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp \cap W_-(H^{1,\epsilon}_{\infty,N}(\mathbf{R})). \end{aligned} \tag{76}$$

Theorem 7.4. *Let notations be as above. Then $M^\epsilon_{\mathbf{H}}$ is bounded from*

$$\begin{cases} \mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp & \text{if } \epsilon > 1/2 \\ H^1(G//K) \cap W_-(H^{1,1-\epsilon}_{\infty,0}(\mathbf{R})) & \text{if } 0 < \epsilon \leq 1/2 \\ H^1(G//K) \cap W_-(H^{1,3/2}_{\infty,1}(\mathbf{R})) & \text{if } \epsilon = 0 \end{cases}$$

to $L^1(G//K)$.

Proof. Since $M^{\rm loc}_{\mathbf{H}}$ is bounded from $\mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp$ to $L^1(G//K)$, we may suppose that $t > 1$ in the definition of $M^\epsilon_{\mathbf{H}}$. Let $M^{\epsilon,\mathbf{R}}_{\mathbf{H},\gamma}$, $\gamma \in \mathbf{R}$, denote the maximal operator on $\mathbf{R}$ associated to the Fourier multiplier corresponding to $t^{-\epsilon}\hat{h}_t(\lambda + i\rho)(i\lambda)^\gamma$:

$$M^{\epsilon,\mathbf{R}}_{\mathbf{H},\gamma}(F) = \sup_{t>1} \frac{1}{t^\epsilon} \int_{-\infty}^{\infty} \hat{h}_t(\lambda + i\rho)(i\lambda)^\gamma F^\sim(\lambda) e^{i\lambda x} d\lambda.$$

For $f \in L^1(G//K)$ we put $F = W_+(f)$. Then the same argument as that used in the proof of Theorem 4.6 and (**??**) yield that

$$\begin{aligned}\|M^{\epsilon}_{\mathbf{H}} f\|_{L^1(G)} &\le c\sum_{m=0}^{n}\sum_{\xi\in\underline{D}} \|M^{\epsilon,\mathbf{R}}_{\mathbf{H},0}\circ W^{\mathbf{R}}_{-(m+\xi)}(F)\|_{L^1(\mathbf{R})}\\ &= c\sum_{m=0}^{n}\sum_{\xi\in\underline{D}} \|M^{\epsilon,\mathbf{R}}_{\mathbf{H},m+\xi}(F)\|_{L^1(\mathbf{R})}, \qquad (77)\end{aligned}$$

if the both sides exist. Let H_t denote the heat kernel on $\mathbf{R}$:

$$H_t(x) = (2t)^{-1/2} e^{-x^2/4t}. \qquad (78)$$

Since $\hat{H}_t(\lambda) = e^{-t\lambda^2}$, it follows that

$$\hat{h}_t(\lambda + i\rho) = cK_t^{\sim}(\lambda), \quad K_t(x) = H_t(x - 2\rho t). \qquad (79)$$

Hence, (**??**) can be rewritten as

$$\|M^{\epsilon}_{\mathbf{H}} f\|_{L^1(G)} \le c\sum_{m=0}^{n}\sum_{\xi\in\underline{D}} \|\sup_{t>1} t^{-\epsilon}|W^{\mathbf{R}}_{-(m+\xi)}(K_t) * F|\|_{L^1(\mathbf{R})}. \qquad (80)$$

Lemma 7.5. *Let $t \ge 1$ and $x, \mu \ge 0$. Then for each $\gamma \ge 0$, there exists a constant c such that*

$$|W^{\mathbf{R}}_{-\mu}(H_t)(x)| \le \frac{c}{\sqrt{t}}\left(1 + \frac{x}{\sqrt{t}}\right)^{-\gamma} t^{-\mu/2} \qquad (81)$$

and

$$|W^{\mathbf{R}}_{-\mu}(K_t)(x)| \le \frac{c}{\sqrt{t}}\left(1 + \left|\frac{x}{t} - 2\rho\right|\right)^{-\gamma} t^{-\mu/2}.$$

Proof. Let $0 \le \delta < 1$ and $x \ge 0$. We note that

$$\begin{aligned}W^{\mathbf{R}}_{-\mu}(H_t)(x) &= \frac{c}{\sqrt{t}}\int_x^{\infty} e^{-s^2/4t}\frac{s}{t}(s-x)^{-\mu} ds\\ &\le \frac{c}{\sqrt{t}} e^{-x^2/4t}\int_0^{\infty} e^{-s^2/4t}\frac{s+x}{t} s^{-\mu} ds\\ &\le \frac{c}{\sqrt{t}} e^{-x^2/4t}\left(1 + \frac{x}{\sqrt{t}}\right) t^{-\mu/2}.\end{aligned}$$

Then the desired result for H_t follows. Since $t \ge 1$, by replacing $x/\sqrt{t}$ by x/t in (**??**), the one for K_t follows from (**??**). ■

We return to the proof of the theorem.

Case I: $\epsilon > 1/2$. Since $t \geq 1$, Lemma 7.5 with $\gamma = 1/2 + \epsilon$ means that, for $0 \leq m \leq \underline{n}$, and $\xi \in \underline{D}$, $|t^{-\epsilon} W^{\mathbf{R}}_{-(m+\xi)}(K_t)(x)| \leq ct^{-(\epsilon+1/2)}(1 + ||x|/t - 2\rho|)^{-(1/2+\epsilon)}$. When $|x|/t > 4\rho$, $t^{-(\epsilon+1/2)}(1 + ||x|/t - 2\rho|)^{-(\epsilon+1/2)} \leq ct^{-(\epsilon+1/2)}(1 + |x|/t|)^{-(\epsilon+1/2)} \leq c(1 + |x|)^{-(\epsilon+1/2)}$, because $t \geq 1$, and when $|x|/t \leq 4\rho$, $t^{-(\epsilon+1/2)}(1 + ||x|/t - 2\rho|)^{-(\epsilon+1/2)} \leq ct^{-(\epsilon+1/2)} \leq c(1 + |x|)^{-(\epsilon+1/2)}$. Hence, $t^{-\epsilon}|W^{\mathbf{R}}_{-(m+\delta_\alpha)}(K_t) * F| \leq c\Phi * |F|$, where $\Phi(x) = (1+|x|)^{-(1/2+\epsilon)}$. Since Φ belongs to $L^1(\mathbf{R})$, the right hand side of (**??**) is dominated by $\|F\|_{L^1(\mathbf{R})}$. This means that $M^{\epsilon}_{\mathbf{H}}$, $t > 1$ and $\epsilon > 1/2$, is bounded from $W_-(L^1(\mathbf{R}))$ to $L^1(G//K)$. Since $\mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp \subset W_-(L^1(\mathbf{R}))$ (see (**??**)), the desired result follows.

Case II: $0 < \epsilon \leq 1/2$. Since $t^{-\epsilon} W^{\mathbf{R}}_{-(m+\xi)}(K_t)$ is a convolution operator, to obtain the desired L^1-boundedness on $W_-(H^{1,1-\epsilon}_{\infty,0}(\mathbf{R}))$, it is enough to prove that there exists a constant $C > 0$ such that for each centered $(1, \infty, 0, 1-\epsilon)$-atom A on $\mathbf{R}$, $0 \leq m \leq \underline{n}$, and $\xi \in \underline{D}$,

$$\| \sup_{t>1} t^{-\epsilon} |W^{\mathbf{R}}_{-(m+\xi)}(K_t) * A| \|_{L^1(\mathbf{R})} \leq C.$$

Let $[-r, r]$ denote the support of A and, for simplicity, we put

$$K^{\epsilon}_{t,m+\xi}(x) = t^{-\epsilon} W^{\mathbf{R}}_{-(m+\xi)}(K_t)(x).$$

First, we shall prove that

$$K^{\epsilon}_{t,m+\xi}(x) \leq c(1 + |x|^{1/2-\epsilon})\Psi_t(x),$$

where Ψ_t is a Euclidean dilation of $\Psi(x) = (1 + |x - 2\rho|)^{-2}$. Actually, if $\rho t \geq 2|x|$, then $(|x| - 2\rho t)^2 = |x|^2 + 4\rho t(\rho t - |x|) \geq |x|^2 + 2\rho^2 t^2$ and thus, it follows from (**??**), (**??**) that $K_t(x) \leq e^{-(|x|^2+2\rho^2 t^2)/4t} \leq e^{-(|x|/t)^2} e^{-\rho^2 t/2} \leq ct^{-1}\ (1 + |x|/t)^{-2}$, because $t \geq 1$. Thereby, the above estimate is clear. If $\rho t \leq 2|x|$, then, by letting $\gamma = 2$ in Lemma 7.5, it follows that $|K^{\epsilon}_{t,m+\xi}(x)| \leq ct^{-(\epsilon+1/2)}(1 + ||x|/t - 2\rho|)^{-2} \leq ct^{-(\epsilon-1/2)} t^{-1}(1 + ||x|/t - 2\rho|)^{-2} \leq cx^{-(\epsilon-1/2)} t^{-1}$ $(1 + ||x|/t - 2\rho|)^{-2}$ for all $0 \leq m \leq \underline{n}$ and $\xi \in \underline{D}$. Hence we have the desired estimate. By using this estimate, we see that

$$\begin{aligned} |K^{\epsilon}_{t,m+\xi} * A(x)| &\leq c \int_{x-r}^{x+r} (1 + |y|^{1/2-\epsilon})\Psi_t(y)|A(x - y)|dy \\ &\leq c(1 + |x + r|^{1/2-\epsilon})\Psi_t * |A|(x) \\ &\leq c(1 + |x + r|^{1/2-\epsilon}) M^{\mathbf{R}}_{\Psi}(|A|)(x), \end{aligned}$$

where $M_\Psi^{\mathbf{R}}(F) = \sup_{0<t<\infty} |\Psi_t * F(x)|$. Since $M_\Psi^{\mathbf{R}}$ is bounded on $L^\infty(\mathbf{R})$ and $\|A\|_\infty \le r^{-1}(1+r)^{-(1-\epsilon)}$, it follows that, on $|x| \le 2r$,

$$\begin{aligned}
\int_{|x|\le 2r} \sup_{t>1} |K^\epsilon_{t,m+\xi} * A(x)| dx &\le c(1+r^{1/2-\epsilon}) \int_{|x|\le 2r} M_\Psi^{\mathbf{R}}(|A|)(x) dx \\
&\le c(1+r^{1/2-\epsilon}) r \|A\|_\infty \\
&\le c(1+r^{1/2-\epsilon})(1+r)^{-(1-\epsilon)} \le C. \qquad (82)
\end{aligned}$$

On the other hand, on $|x| > 2r$, it follows from the moment condition of A that

$$\begin{aligned}
K^\epsilon_{t,m+\xi} * A(x) &= \int_{-\infty}^{\infty} K^\epsilon_{t,m+\xi}(y) A(x-y) dy \\
&= \int_{x-r}^{x+r} (K^\epsilon_{t,m+\xi})'(y) \left(\int_{-\infty}^{y} A(x-s) ds \right) dy.
\end{aligned}$$

Lemma 7.5 with $\gamma = 1+\epsilon$, $\epsilon > 0$, and $\mu = 1$ yields that $|(K^\epsilon_{t,m+\xi})'(y)| \le ct^{-(1+\epsilon)}(1 + ||y|/t - 2\rho|)^{-(1+\epsilon)} \le c(1+|y|)^{-(1+\epsilon)}$. Since $\|A\|_1 \le r\|A\|_\infty \le (1+r)^{-(1-\epsilon)}$, it follows that

$$\sup_{t>1} |K^\epsilon_{t,m+\xi} * A(x)| \le c(1+|x-r|)^{-(1+\epsilon)} \cdot r \cdot (1+r)^{-(1-\epsilon)}. \qquad (83)$$

Therefore,

$$\int_{|x|\ge 2r} \sup_{t>1} |K^\epsilon_{t,m+\xi} * A(x)| dx \le \int_{|x|\ge 2r} \frac{(1+r)^\epsilon}{(1+|x-r|)^{1+\epsilon}} dx \le C.$$

Hence we see that $\| \sup_{t>1} K^\epsilon_{t,m+\xi} * A\|_{L^1(\mathbf{R})} \le C$.

Case III: $\epsilon = 0$. Let A be a centered $(1, \infty, 1, 3/2)$-atom on $\mathbf{R}$. On $|x| \le 2r$, similarly as for (**??**), we see that

$$\int_{|x|\le 2r} \sup_{t>1} |K^0_{t,m+\xi} * A(x)| dx \le C.$$

Moreover, the moment condition of A yields that $\int_{-\infty}^{y} A(x-s) ds$ is supported on $[x-r, x+r]$ and

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{t} A(x-s) ds \right) dt = 0.$$

Then

$$K^0_{t,m+\xi} * A(x) = \int_{x-r}^{x+r} (K^0_{t,m+\xi})''(y) \left(\int_{-\infty}^{y} \left(\int_{-\infty}^{y} A(x-s)ds \right) dt \right) dy.$$

Since $\|A\|_\infty \le r^{-1}(1+r)^{-3/2}$, Lemma 7.5 with $\gamma = 3/2$ and $\mu = 2$ similarly yields that

$$|K^0_{t,m+\xi} * A(x)| \le c(1+|x-r|)^{-3/2} r \cdot r \cdot (1+r)^{-3/2}. \tag{84}$$

Therefore, on $|x| \ge 2r$, it follows that

$$\int_{|x|\ge 2r} \sup_{t>1} |K^0_{t,m+\xi} * A(x)| dx \le \int_{|x|\ge 2r} \frac{(1+r)^{1/2}}{(1+|x-r|)^{3/2}} dx \le C.$$

Hence we see that $\| \sup_{t>1} K^0_{t,m+\xi} * A\|_{L^1(\mathbf{R})} \le C$. ■

Remark 7.6. Let $\epsilon = 1/2$. Then $M^{1/2}_{\mathbf{H}}$ is bounded from $\mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp \cap W_-(H^{1,1/2}_{\infty,0}(\mathbf{R}))$ to $L^1(G//K)$ by Theorem 7.4 (2). In the proof we use a $(1,\infty,0,1/2)$-atom A, whose L^∞-norm has a decay $r^{-1}(1+r)^{-1/2}$. This extra decay $(1+r)^{-1/2}$ is only used to deduce (**??**). Clearly, if the first derivative of K_t creates a decay t^{-1}, same as the one for Ψ_t of a Euclidean dilation of Ψ, then we do not need use the extra decay. Actually, we may apply a common argument used to prove that a radial maximal operator M_Φ on $\mathbf{R}$ is bounded from $H^1_{\infty,0}(\mathbf{R})$ to $L^1(\mathbf{R})$ (cf. [**4**, §3]). However, as shown in Lemma 7.5 (2), the first derivative of K_t creates only $t^{-1/2}$ decay. Therefore, we need the modification of the L^∞-norm of A. This situation is the same in the case of $\epsilon = 0$. The second derivative of K_t creates a decay t^{-1}, not t^{-2} (see Lemma 7.5 (2)) and thus, we need an extra decay $(1+r)^{-3/2}$ (see (**??**)).

Theorem 7.7. *$M_{\mathbf{P}}$ is bounded from $H^1(G//K)$ to $L^1(G//K)$.*

Proof. Since $M^{\rm loc}_{\mathbf{P}}$ is bounded from $\mathsf{H}^{1,+}_{\infty,0}(G//K)^\sharp$ to $L^1(G//K)$, we may suppose that $t > 1$ in the definition of $M_{\mathbf{P}}$ (see Theorem 6.4). Let $M^{\mathbf{R}}_{\mathbf{P},\gamma}$, $\gamma \in \mathbf{R}$, denote the maximal operator on $\mathbf{R}$ associated to the Fourier multiplier corresponding to $\hat{p}_t(\lambda + i\rho)(i\lambda)^\gamma = W_+(p_t)^\sim(\lambda)(i\lambda)^\gamma$:

$$M^{\mathbf{R}}_{\mathbf{P},\gamma}(F) = \sup_{t>1} | \int_{-\infty}^{\infty} \hat{p}_t(\lambda + i\rho)(i\lambda)^\gamma F^\sim(\lambda) e^{i\lambda x} d\lambda |.$$

Then for $f \in H^1(G//K)$, we have

$$\begin{aligned} \|M_{\mathbf{P}} f\|_{L^1(G)} &\le c \sum_{m=0}^{\underline{n}} \sum_{\xi \in \underline{D}} \|M^{\mathbf{R}}_{\mathbf{P},0} \circ W^{\mathbf{R}}_{-(m+\xi)}(F)\|_{L^1(\mathbf{R})} \\ &\le c \sum_{m=0}^{\underline{n}} \sum_{\xi \in \underline{D}} \|M^{\mathbf{R}}_{\mathbf{P},m+\xi}(F)\|_{L^1(\mathbf{R})}, \end{aligned}$$

where $F = W_+(f)$. Since $F \in W_-(H^1(\mathbf{R}))$ (see Theorem 6.4), F has a $(1,\infty,1)$-atomic decomposition $F = \sum_i \lambda_i A_i$. Hence, it is enough to show that $\|M^{\mathbf{R}}_{\mathbf{P},m+\xi}(A)\|_{L^1(\mathbf{R})} \leq c$ for all $(1,\infty,2)$-atom A. Clearly, we may suppose that A is centered, that is, supported in $[-r,r]$. We recall that

$$\hat{p}_t(\lambda + \imath\rho) = K_t^{\sim}(\lambda), \quad K_t(x) = \frac{ct}{(t^2+x^2)^{1/2}}\mathsf{K}_t(\rho(t^2+x^2)^{1/2})e^{\rho x},$$

where K_t is the modified Bessel function (see [**1**, §6]).

Lemma 7.8. *Let notations be as above and $M^{\mathbf{R}}_{HL}$ be the Hardy-Littlewood maximal function on $\mathbf{R}$. Then for all $F \in L^1(\mathbf{R})$ and $\gamma \geq 0$,*

$$M^{\mathbf{R}}_{\mathbf{P},\gamma}(F)(x) \leq c M^{\mathbf{R}}_{HL}(F)(x).$$

In particular, $M^{\mathbf{R}}_{\mathbf{P},\gamma}$ satisfies the maximal theorem.

Proof. We note that $|W^{\mathbf{R}}_{\gamma} K_t(x)| \leq c K_t(x)$ (see [**1**, p.289]) and $\mathsf{K}_t(x) \leq c|x|^{-1/2}\, e^{-t|x|}$. Hence we see that

$$M^{\mathbf{R}}_{\mathbf{P},\gamma}(F)(x) \leq c\int_{-\infty}^{\infty} |F(y)| \frac{t}{(t^2+(x-y)^2)^{3/4}} e^{-\rho(t^2+(x-y)^2)^{1/2}} e^{\rho|x-y|}dy.$$

Here we divide the integral as $\displaystyle\int_{I_0} + \sum_{k=1}^{\infty}\int_{I_k}$, where $I_0 = \{y; |x-y| \leq t^2\}$ and $I_k = \{y; 2^k t^2 < |x-y| \leq 2^{k+1}t^2\}, k = 1, 2, \cdots$. Since $t^3 e^{-\rho(t^2+x^2)^{1/2}}$ has the maximum at $t \sim (t^2+x^2)^{1/4}$, it follows that

$$\begin{aligned}
&\int_{I_0} |F(y)| \frac{t}{(t^2+(x-y)^2)^{3/4}} e^{-\rho(t^2+(x-y)^2)^{1/2}} e^{\rho|x-y|}dy \\
\leq\ & c\frac{1}{t^2}\int_{|x-y|\leq t^2} |F(y)|dy \leq c M^{\mathbf{R}}_{HL}(F)(x).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_{I_k} |F(y)| \frac{t}{(t^2+(x-y)^2)^{3/4}} e^{-\rho(t^2+(x-y)^2)^{1/2}} e^{\rho|x-y|}dy \\
\leq\ & c\int_{|x-y|\leq 2^{k+1}t^2} |F(y)| \frac{t}{(2^k t^2)^{3/2}}dy \\
\leq\ & c2^{-k/2}\frac{1}{2^{k+1}t^2}\int_{|x-y|\leq 2^{k+1}t^2} |F(y)|dy \\
\leq\ & c2^{-k/2} M^{\mathbf{R}}_{HL}(F)(x).
\end{aligned}$$

Hence, the desired result follows. ■

Lemma 7.9. *Let A be a $(1,\infty,2)$-atom on $\mathbf{R}$ supported on $[-r,r]$. Then for $x \geq 2r$,*

$$M^{\mathbf{R}}_{\mathbf{P},\gamma}(A)(x) \leq c \max\left(\frac{r^{1/2}}{|x-r|^{3/2}}, \frac{r}{x^2}\right).$$

Proof. Since $\|A\|_{L^1(\mathbf{R})} \leq 1$ and $|x-y| \geq |x-r|$ if $|y| \leq r$, it follows that

$$M^{\mathbf{R}}_{\mathbf{P},\gamma}(A)(x) \leq c \int_{-\infty}^{\infty} |A(y)| \frac{t}{(t^2+(x-y)^2)^{3/4}} dy \leq c \frac{t}{|x-r|^{3/2}}.$$

Especially,

$$M^{\mathbf{R}}_{\mathbf{P},\gamma}(A)(x) \leq c \frac{r^{1/2}}{|x-r|^{3/2}} \quad \text{if } r \geq t^2.$$

On the other hand, we note that

$$A * W^{\mathbf{R}}_{\gamma}(K_t)(x) = c \int_{-\infty}^{\infty} \hat{p}_t(\lambda + i\rho)\tilde{A}(\lambda)\lambda^{\gamma} e^{-i\lambda x} d\lambda.$$

Since $\lambda(\lambda+2i\rho) = |\lambda||\lambda+2i\rho|e^{i\theta}$, $\tan\theta = 2\rho/\lambda$, and thus, $\Re((\lambda+i\rho)^2+\rho^2))^{1/2} = |\lambda|^{1/2}|\lambda+2i\rho|^{1/2}\cos(\theta/2) \geq c|\lambda|^{1/2}$ with $c > 0$, it is easy to see that

$$\begin{aligned}
&|\hat{p}_t(\lambda+i\rho)| \leq c e^{-ct|\lambda|^{\frac{1}{2}}},\\
&|(\frac{d}{d\lambda})\hat{p}_t(\lambda+i\rho)| \leq ct\left(\frac{1+|\lambda|}{|\lambda|}\right)^{\frac{1}{2}} e^{-ct|\lambda|^{\frac{1}{2}}},\\
&|(\frac{d}{d\lambda})^2\hat{p}_t(\lambda+i\rho)| \leq c\left(t^2\frac{1+|\lambda|}{|\lambda|} + t\frac{1}{|\lambda|}\left(\frac{1+|\lambda|}{|\lambda|}\right)^{\frac{1}{2}}\right) e^{-ct|\lambda|^{\frac{1}{2}}}.
\end{aligned}$$

Since A is a $(1,\infty,2)$-atom on $\mathbf{R}$ supported on $[-r,r]$, it follows that $|(d/d\lambda)^n \hat{A}(\lambda)| \leq r^n$ $(n \in \mathbf{N})$ and $|(d/d\lambda)^n\hat{A}(\lambda)| \leq r^{n+1}|\lambda|$, $n = 0,1,2$, by the moment condition of A. Moreover, since $(d/d\lambda)^{n-k}\hat{A}(\lambda)$ is the k-fold integral of $(d/d\lambda)^n\hat{A}(\lambda)$ over $[0,\lambda]$, it follows that $|(d/d\lambda)^{n-k}\hat{A}(\lambda)| \leq r^{n+1}|\lambda|^{k+1}$, $0 \leq k \leq n = 0,1,2$. Therefore,

$$\begin{aligned}
&|x^2 A * W^{\mathbf{R}}_{\gamma}(K_t)(x)|\\
= \;& c\left|\int_{-\infty}^{\infty} \frac{d^2}{d\lambda^2}\left(\hat{p}_t(\lambda+i\rho)\tilde{A}(\lambda)\lambda^{\gamma}\right) e^{-i\lambda x} d\lambda\right|\\
\leq \;& c\int_{-\infty}^{\infty}\left(\left(t^2\frac{1+|\lambda|}{|\lambda|} + t\frac{1}{|\lambda|}\left(\frac{1+|\lambda|}{|\lambda|}\right)^{\frac{1}{2}}\right) r^2|\lambda|^{\gamma+2} + r^2|\lambda|^{\gamma}\right) e^{-ct|\lambda|^{\frac{1}{2}}} d\lambda\\
\leq \;& c\frac{r^2}{t^{2+2\gamma}}.
\end{aligned}$$

This means that

$$|A * W_\gamma^{\mathbf{R}}(K_t)(x)| \le c\frac{r}{x^2}\frac{r}{t^{2+2\gamma}},$$

and thus, since $t \ge 1$, it follows that

$$M_{\mathbf{P},\gamma}^{\mathbf{R}}(A)(x) \le c\frac{r}{x^2} \quad \text{if } r \le t^2.$$

This completes the proof of the lemma. ■

We return to the proof of Theorem 7.7. Since $M_{\mathbf{P},\gamma}^{\mathbf{R}}$ is bounded on $L^2(\mathbf{R})$ (see Lemma 7.8) and $\|A\|_2 \le r^{-1/2}$, it follows that

$$\int_{|x|\le 2r} M_{\mathbf{P},\gamma}^{\mathbf{R}}(A)(x)dx \le r^{1/2}\|A\|_2 \le c.$$

Moreover, it follows from Lemma 7.9 that

$$\int_{|x|>2r} M_{\mathbf{P},\gamma}^{\mathbf{R}}(A)(x)dx \le \int_{|x|>2r}\left(\frac{r^{1/2}}{|x-r|^{3/2}} + \frac{r}{x^2}\right)dx \le c.$$

Hence we obtain that $\|M_{\mathbf{P},\gamma}^{\mathbf{R}}(A)\|_{L^1(\mathbf{R})} \le c$ for all $(1,\infty,2)$-atom A. ■

We last treat the Riesz transform R on G. Under the standard notation in [**14**] we denote the covariant differentiation on G by ∇: $|\nabla|^2(f) = \Omega(f^2)$ $-2\Omega f \cdot f$ for $f \in C^\infty(G)$. Then the Riesz transform R on G is defined as follows.

Definition 7.10. *For* $f \in C_c^\infty(G//K)$

$$(\mathsf{R}f)(g) = |\nabla| \circ (-\Omega)^{-1/2}(f)(g) \quad (g \in G).$$

This operator R also satisfies the maximal theorem (see [**1**, Corollary 5.2]). We recall that $|\nabla f|^2(g) = \sum_{m=1}^n |X_i f|^2(g)$ $(g \in G)$, here $\{X_i; 1 \le i \le n\}$ is denoted as an orthonormal basis of the Lie algebra $\mathfrak{g}$ of G and each X_i is regarded as a left (or right) invariant differential operator on G. Especially, since f is K-bi-invariant on G, $|\nabla f|^2(g)$ is simply expressed as $c|(d/dx)f(a_x)|^2$ provided $\sigma(g) = x$ (cf. [**2**, §2]).

Now let $\ell(\lambda) = i\lambda/\sqrt{\lambda(\lambda + 2i\rho)}$ and M_ℓ the corresponding Fourier multiplier. Then, since $(-\Omega)^{-1/2}$ is the Fourier multiplier corresponding to $(\lambda^2 + \rho^2)^{-1/2}$, applying the similar argument used in the proof of Theorem 4.6, we can deduce that

$$\|\mathsf{R}f\|_{L^1(G)} \le c\sum_{m=0}^{\underline{n}}\sum_{\xi \in \underline{D}} \|\mathsf{M}_\ell \circ W_{-(m+\xi)}^{\mathbf{R}}(F)\|_{L^1(\mathbf{R})}.$$

Since $\ell(\lambda)$ satisfies the Hörmander condition and $m+\xi \leq \alpha+1/2$, Remark 4.7 yields the following.

Theorem 7.11. *R is bounded from $W_{-}(\mathsf{M}_{C_\rho}(H^1(\mathbf{R})))$ to $L^1(G//K)$.*

8. Appendix. A. Estimate of Γ_m: In [**8**, Proposition 8.3] we estimate derivatives of $\Gamma_m(\lambda)$ when $\Im\lambda \geq \rho$ and show that $\Gamma_m(\xi+i\rho)$ ($\xi \in \mathbf{R}$) satisfies Hörmander's condition. Here we shall obtain an estimate of $\Gamma_m(\lambda)$ for $\Im\lambda \geq -\rho$. We refer to the notations and the proof in [**3**, Lemma 7] and we denote Γ_{2m} by Γ_m. Then the recursive definition of $\Gamma_m(\lambda)$ yields that

$$|\Gamma_m(\lambda)| \leq \frac{\rho|\rho-i\lambda|}{m|m-i\lambda|}\prod_{k=1}^{m-1}\left(1+\frac{r_k(\lambda)}{c_k(\lambda)}\right). \tag{85}$$

Here

$$c_k(\lambda) = 4k|k-i\lambda| \quad \text{and} \quad r_k(\lambda) = 4\kappa|2k-i\lambda+\rho|,$$

where $\kappa = \alpha-\beta$ if $k \equiv m+1 \pmod 2$ and $\kappa = \rho$ if $k \equiv m \pmod 2$.

Lemma 8.1. *For each $\delta > 0$, there exists a positive constant c such that for all $m \in \mathbf{N}$ and $\lambda = \xi - i\rho$ ($\xi \in \mathbf{R}$),*

$$|\Gamma_m(\lambda)| \leq cm^{2\alpha+\delta}.$$

Proof. For each $\epsilon > 0$ we take $n_0 \in \mathbf{N}$ such that $4(\rho+n_0)^2/n_0^2 \leq 4(1+\epsilon)^2$. We first estimate each $k^2\theta^{-2}r_k(\lambda)^2c_k(\lambda)^{-2} = |2k-i\lambda+\rho|^2/|k-i\lambda|^2$.

If $k \geq \rho+n_0$, then

$$\frac{|2k+\rho-i\lambda|^2}{|k-i\lambda|^2} = \frac{4k^2+\xi^2}{(k-\rho)^2+\xi^2} \leq \frac{4k^2}{(k-\rho)^2} \leq \frac{4(\rho+n_0)^2}{n_0^2} \leq 4(1+\epsilon)^2. \tag{86}$$

If $k < \rho+n_0$ and $\rho \notin \mathbf{N}$, then

$$\frac{|2k+\rho-i\lambda|^2}{|k-i\lambda|^2} \leq \frac{4k^2}{(k-\rho)^2} \leq \frac{4(\rho+n_0)^2}{(1/2)^2} \leq c.$$

If $k < \rho+n_0$, $\rho \in \mathbf{N}$, and $k \neq \rho$, then

$$\frac{|2k+\rho-i\lambda|^2}{|k-i\lambda|^2} \leq \frac{4k^2}{(k-\rho)^2} \leq 4(\rho+n_0)^2 \leq c.$$

If $k < \rho+n_0$ and $k = \rho$, then

$$\frac{|2k+\rho-i\lambda|^2}{|k-i\lambda|^2} = \frac{4\rho^2+\xi^2}{\xi^2} \leq c\left(\frac{1+|\xi|}{|\xi|}\right)^2.$$

Therefore, substituting these estimates into (**??**), we have the following.

Case I. $\rho \notin \mathbf{N}$: Since

$$\left(k\frac{r_k(\lambda)}{c_k(\lambda)}\right)^2 \leq \begin{cases} (2(1+\epsilon)\theta)^2 & \text{if } k \geq \rho + n_0 \\ c & \text{if } k < \rho + n_0, \end{cases}$$

it follows that, if $\rho + n_0 \leq m-1$, then

$$\begin{aligned} & \prod_{k=1}^{m-1}\left(1+\frac{r_k(\lambda)}{c_k(\lambda)}\right) \\ \leq \ & \exp\left(\sum_{k=1}^{m-1}\frac{r_k(\lambda)}{c_k(\lambda)}\right) = \exp\left(\sum_{k=1}^{\rho+n_0-1}\frac{r_k(\lambda)}{c_k(\lambda)} + \sum_{k=\rho+n_0}^{m-1}\frac{r_k(\lambda)}{c_k(\lambda)}\right) \\ \leq \ & \exp\Bigg(c(\rho+n_0-1) + \rho \sum_{\substack{k\equiv m \\ 1\leq k\leq m-1}} 2(1+\epsilon)k^{-1} \\ + \ & (\alpha-\beta)\sum_{\substack{k\equiv m+1 \\ 1\leq k\leq m-1}} 2(1+\epsilon)k^{-1}\Bigg) \\ \leq \ & c\exp\left((1+\epsilon)(\rho+\alpha-\beta)\log m\right) \leq cm^{(1+\epsilon)(2\alpha+1)}. \end{aligned}$$

Therefore, we see that

$$|\Gamma_m(\lambda)| \leq b_m(\lambda) \leq c\frac{\rho}{m}\cdot\frac{|\xi|}{|m-\rho|+|\xi|}\cdot m^{(1+\epsilon)(2\alpha+1)} \leq cm^{(1+\epsilon)(2\alpha+1)-1}.$$

Clearly, if $\rho + n_0 > m-1$, then $|\Gamma_m(\lambda)| \leq c$.

Case II. $\rho \in \mathbf{N}$: In this case there is a possibility that the term corresponding to $k = \rho$ will appear in the product $\prod_{k=1}^{m-1}$ when $\rho \leq m-1$. Therefore, the above estimate is changed as

$$|\Gamma_m(\lambda)| \leq c\frac{\rho}{m}\cdot\frac{|\xi|}{|m-\rho|+|\xi|}\cdot\left(1+\frac{1+|\xi|}{|\xi|}\right)\cdot m^{(1+\epsilon)(2\alpha+1)-1} \leq cm^{(1+\epsilon)(2\alpha+1)-1}.$$

In both cases, letting ϵ sufficiently small so as to satisfy $(1+\epsilon)(2\alpha+1)-1 \leq 2\alpha+\delta$, we can obtain the desired estimate. ■

Let $k \in \mathbf{N}$ and $k \neq \rho$ if $\rho \in \mathbf{N}$. Applying the above argument for $\lambda = \xi - ik$ ($\xi \in \mathbf{R}$), we see that each $\Gamma_m(\lambda)$, $k \leq m-1$, has a pole at $\lambda = -ik$. On the other hand, if $\Im\lambda \neq -k$, we can deduce the same estimate. Therefore, the estimate obtained in Lemma 8.1 also holds on the tube domain $\mathsf{F}(a,b) = \{\lambda \in \mathbf{C}; a < \Im\lambda < b\}$ where $[a,b]$ does not contain $-k$ ($k \in \mathbf{N}$) except $-\rho$ if $\rho \in \mathbf{N}$. Hence, Cauchy's integral formula yields the following.

Proposition 8.2. *Suppose that the real interval* $[a, b]$ *does not contain* $-k$ $(k \in \mathbf{N})$ *except* $-\rho$ *if* $\rho \in \mathbf{N}$. *Then for each* $\delta > 0$ *and* $n \in \mathbf{N}$, *there exists a positive constant* c *such that for all* $m \in \mathbf{N}$ *and* $\lambda \in \mathsf{F}(a, b)$,

$$\left| \left(\frac{d}{d\lambda} \right)^n \Gamma_m(\lambda) \right| \le c m^{2\alpha+\delta}. \tag{87}$$

Remark 8.3. *When* $\Im\lambda \ge \rho$, *we can replace* $4(1+\epsilon)^2$ *in* (**??**) *by* 4. *Thereby, we can replace the power* $2\alpha + \delta$ *in* (**??**) *by* 2α *(see* [**7**, *Proposition* 8.3]*).*

B. Maximal Functions on R: In this section ϕ denotes a compactly supported C^∞ function on $\mathbf{R}$ with $\int_{-\infty}^{\infty} \phi(x)dx = 1$, and ϕ_t a Euclidean dilation:

$$\phi_t(x) = \frac{1}{t} \phi\left(\frac{x}{t}\right), \quad t > 0. \tag{88}$$

We define the radial maximal function $M_\phi(F)$ on $\mathbf{R}$ as

$$(M_\phi F)(x) = \sup_{0<t<\infty} |F * \phi_t(x)|. \tag{89}$$

Then $F \in L^1_{\mathrm{loc}}(\mathbf{R})$ belongs to $H^1(\mathbf{R})$, by the definition, if $M_\phi F$ belongs to $L^1(\mathbf{R})$ (cf. [**15**, p. 87]). We shall consider another characterization of $H^1(\mathbf{R})$ by using a maximal function associated to a Fourier multiplier $\mathsf{M}_{m(t,\cdot)}$ corresponding to a function $m(t, \lambda)$ on $\mathbf{R}_+ \times \mathbf{R}$:

$$\mathsf{M}_{m(t,\cdot)}(F)^\sim(\lambda) = m(t, \lambda) F^\sim(\lambda).$$

Definition 8.4. *We say that* $m(t, \lambda)$ *is in the class* $\mathsf{A}_{N,M}$ $(N, M \in \mathbf{N})$ *if there exists a constant* c *such that*

$$\left| \left(\frac{d}{d\lambda} \right)^n m(t, \lambda) \right| \le c t^n (1 + |t\lambda|)^{-M} \quad (0 \le n \le N).$$

We define related maximal operators M_m, $M_{\mathsf{A}_{N,M}}$, $M^{**}_{m,L}$ $(L \ge 0)$ and M^*_m on $\mathbf{R}$ as follows.

Definition 8.5. *Let* $N, M, L \in \mathbf{N}$ *and suppose that* $M > 1$. *For* $F \in$

$L^1_{\mathrm{loc}}(\mathbf{R})$ *and* $m(t,\lambda) \in \mathsf{A}_{N,M}$, *we define*

$$\begin{aligned} M_m F(x) &= \sup_{0<t<\infty} |\mathsf{M}_{m(t,\cdot)} F(x)|, \\ M_{\mathsf{A}_{N,M}} F(x) &= \sup_{m \in \mathsf{A}_{N,M}} M_m F(x), \\ M^{**}_{m,L} F(x) &= \sup_{0<t<\infty} \sup_{y \in \mathbf{R}} |F * m_t(x-y)| \left(1 + \frac{|y|}{t}\right)^{-L}, \\ M^{*}_{m} F(x) &= \sup_{|x-y|<t} |F * m_t(y)|, \end{aligned}$$

where $m_t(x)$ *is the inverse Fourier transform of* $m(t,\lambda)$.

Clearly, since $M > 1$, $m_t(x)$ is well-defined and $M_m F(x) \le M^*_m F(x) \le 2LM^{**}_{m,L}F(x)$ holds pointwise.

Theorem 8.6. *Let* $M, N \ge 4$. *Let* $\ell(t,\lambda) \in \mathsf{A}_{N,M}$ *and suppose that there exists* $c > 0$ *such that*

$$|\ell(t,\lambda)| \ge c|t\lambda| \quad (0 \le |t\lambda| \le 2). \tag{90}$$

Then $\|F\|_{H^1(\mathbf{R})} \approx \|M_\ell F\|_{L^1(\mathbf{R})}$.

Proof. $\|M_\ell F\|_{L^1(\mathbf{R})} \le c\|F\|_{H^1(\mathbf{R})}$: By using the atomic characterization of $H^1(\mathbf{R})$, it is enough to show that $\|M_\ell A\|_{L^1(\mathbf{R})} \le C$ for all $(1,\infty,0)$-atom A on $\mathbf{R}$. We may suppose that A is supported on $[-r,r]$. We first note that, since $\ell(t,\lambda) \in \mathsf{A}_{N,M}$ with $M \ge 4$, the inverse Fourier transform of $\ell(t,\lambda)$ is dominated by $t^{-1}(1+|x/t|^2)^{-1}$ and hence, M_ℓ satisfies the maximal theorem (cf. [**8**, Lemma 5.1]). Next we note that the Fourier transform $A^\sim(\lambda)$ satisfies

$$\left| \left(\frac{d}{d\lambda}\right)^{\ell} A^\sim(\lambda) \right| \le r^\ell \ (\ell \in \mathbf{N}).$$

and

$$|A^\sim(\lambda)| = \left| \int_0^\lambda A'(s)ds \right| \le cr|\lambda|,$$

because $A^\sim(0) = 0$ by the moment condition. Therefore, it follows that

$$\begin{aligned} |x^2 M_{\ell(t,\cdot)}(A)| &= \left| \int_{-\infty}^{\infty} A^\sim(\lambda)\ell(t,\lambda) \cdot \left(\frac{d}{d\lambda}\right)^2 e^{i\lambda x} d\lambda \right| \\ &\le \int_{-\infty}^{\infty} \left| \left(\frac{d}{d\lambda}\right)^2 (A^\sim(\lambda)\ell(t,\lambda)) \right| d\lambda \\ &\le c|M_{\ell(t,\cdot)}(x^2 A)| + c\int_{-\infty}^{\infty} \frac{rt + r|\lambda|t^2}{(1+|t\lambda|)^4} d\lambda \\ &\le cr, \end{aligned}$$

because $\|M_{\ell(t,\cdot)}(x^2A)\|_\infty \le c\|x^2A\|_\infty \le cr$. This means that $|M_{\ell(t,\cdot)}(A)(x)| \le r|x|^{-2}$. Finally, we can deduce that

$$\|M_\ell(A)\|_{L^1(\mathbf{R})} \le c\int_{|x|<2r}\|A\|_\infty dx + c\int_{|x|\ge 2r}\frac{r}{|x|^2}dx \le C.$$

$\|F\|_{H^1(\mathbf{R})} \le c\|M_\ell F\|_{L^1(\mathbf{R})}$: The proof is quite similar as the one for [**15**, Theorem 1 in Chap. 3]. We shall give an outline for a necessary modification of the proof.

Step 1. $\|F\|_{H^1(\mathbf{R})} \le c\|M_{\mathsf{A}_{N,M}}F\|_{L^1(\mathbf{R})}$: Let $\psi \in C_c^\infty(\mathbf{R})$ and $\int_{-\infty}^{\infty}\psi(x)dx = 1$. Since $\mathsf{A}_{N,M}$ contains a function $m(t,\lambda) = \hat{\psi}_t(\lambda) = \hat{\psi}(t\lambda)$, the desired result follows from [**15**, Theorem 1 in Chap. 3].

Step 2. $M_{\mathsf{A}_{N,M}}F(x) \le cM_{\ell,L}^{**}F(x)$: We refer to [**15**, Lemma 2 in p.93 and 1.4 in p.95]. We take $L \in \mathbf{N}$ such that $M, N > L+1$ and $L > 1$. Let $\psi_0 \in \mathsf{S}(\mathbf{R})$ satisfy $\hat{\psi}_0(\lambda) = 1$ if $|\lambda| \le 1$ and $\hat{\psi}_0(\lambda) = 0$ if $|\lambda| \ge 2$. Then there exists a decomposition of 1 of the form: $1 = \sum_{k=0}^\infty \hat{\psi}_k(\lambda)$, where each $\hat{\psi}_k$ $(k>0)$ is supported on $2^{k-1} \le |\lambda| \le 2^{k+1}$ and $|(d/d\lambda)^n\hat{\psi}_k(\lambda)| \le c2^{-kn}$. For each $m(t,\lambda) \in \mathsf{A}_{N,M}$ we let

$$\begin{aligned} m(t,\lambda) &= \sum_{k=0}^\infty \hat{\psi}_k(t\lambda)m(t,\lambda) \\ &= \sum_{k=0}^\infty \frac{\hat{\psi}_k(t\lambda)m(t,\lambda)}{\ell(2^{-k}t,\lambda)}\cdot\ell(2^{-k}t,\lambda) \\ &= \sum_{k=0}^\infty \eta_k(t,\lambda)\ell(2^{-k}t,\lambda). \end{aligned}$$

By the assumption (**??**) it follows that $\ell(2^{-k}t,\lambda) \ge c2^{-k}|t\lambda| \ge c2^{-1}$ if $t\lambda \in [2^{k-1},2^{k+1}]$ and moreover, $|(d/d\lambda)^n\ell(t,\lambda)| \le ct^n$ $(0 \le n \le N)$. Thereby, we easily see that

$$\left|\left(\frac{d}{d\lambda}\right)^n\eta_k(t,\lambda)\right| \le ct^n(1+|t\lambda|)^{-M}$$

and, if we denote the inverse Fourier transform of $\eta_k(t,\lambda)$ by $\eta_{k,t}(y)$, then

$$\begin{aligned} |y^n\eta_{k,t}(y)| &\le \left|\int_{2^{k-1}/t}^{2^{k+1}/t}\eta_k(t,\lambda)\cdot\left(\frac{d}{d\lambda}\right)^n e^{i\lambda y}d\lambda\right| \\ &\le ct^n\int_{2^{k-1}/t}^{2^{k+1}/t}(1+|t\lambda|)^{-M}d\lambda \\ &\le ct^{n-1}2^{-k(M-1)}. \end{aligned}$$

Since $N, M > L+1$, it follows that

$$\begin{aligned}(M_m F)(x) &\leq \sup_{t>0} \sum_{k=0}^{\infty} |(F * \eta_{k,t} * \ell_{2^{-k}t})(x)| \\ &\leq \sup_{t>0} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} |F * \ell_{2^{-k}t}(x-y)||\eta_{k,t}(y)| dy \\ &\leq cM_{\ell,L}^{**}F(x) \cdot \sup_{0<t<\infty} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \left(1+\frac{|y|}{2^{-k}t}\right)^{L} \left(1+\frac{|y|}{t}\right)^{-N} \\ &\qquad \times t^{-1} \cdot 2^{-k(M-1)} dy \\ &\leq cM_{\ell,L}^{**}F(x) \cdot \int_{-\infty}^{\infty} (1+|y|)^{L-N} dy \cdot \sum_{k=0}^{\infty} 2^{-k(M-L-1)} \\ &\leq cM_{\ell,L}^{**}F(x).\end{aligned}$$

Step 3. $\|M_{\ell,L}^{**}F\|_{L^1(\mathbf{R})} \leq c\|M_\ell^* F\|_{L^1(\mathbf{R})}$: See [**15**, Lemma 1 in p.93].

Step 4. $\|M_\ell^* F\|_{L^1(\mathbf{R})} \leq c\|M_\ell F\|_{L^1(\mathbf{R})}$: We refer to [**15**, 1.5 in p.95]. We put $f(x,t) = F * \ell_t(x)$ where $\ell_t(x)$ is the inverse Fourier transform of $\ell(t, \lambda)$. Then, for $|x'-y| \leq rt$, there exists $x' < z < y$ such that

$$|f(x',t) - f(y,t)| \leq rt \left| \left(\frac{d}{dx}\right) f(z,t) \right|.$$

We here note that $t(d/dx)f = F * (t(d/dx)\ell_t)$ and $(t(d/dx)\ell_t)^\wedge(\lambda) = \ell(t,\lambda) \cdot (it\lambda)$. Therefore, $(t(d/dx)\ \ell_t)^\wedge$ belongs to $\mathsf{A}_{N,M-1}$. The rest of the proof is same as in [**15**] if we replace F by $\mathsf{A}_{L,M}$.

These four steps complete the proof of the desired inequality. ■

C. Dual Spaces of $W_-(H^1(\mathbf{R}))$: We shall introduce a dual space of $W_-(H^1(\mathbf{R}))$. For $f \in C_c^\infty(G//K)$ we define the complex Fourier-Jacobi transform $\tilde{f}$ as

$$\tilde{f}(\lambda) = \int_0^\infty f(x)\Phi(\lambda, x)\Delta(x) dx \tag{91}$$

(see (**??**), [**10**, (2)]). Then for real valued functions $f, g \in C_c^\infty(G//K)$, we can deduce the following Plancherel formula (cf. [**10**, Theorem 5.1]):

$$\langle f, g\rangle_{L^2(\mathbf{R}_+, \Delta dx)} = c\int_{-\infty}^{\infty} \hat{f}(\lambda + i\rho)\tilde{g}(\lambda+i\rho) C(-\lambda - i\rho)^{-1} d\lambda.$$

We suppose that $f \in W_-(H^1(\mathbf{R}))$. Since $\hat{f}(\lambda + i\rho) = W_+(f)^\sim(\lambda)$ (see (**??**)) and $W_+(f) \in H^1(\mathbf{R})$ by the definition, a dual space of $W_-(H^1(\mathbf{R}))$ should be defined as the space consisting of g such that $\tilde{g}(\lambda+i\rho)C(-\lambda-i\rho)^{-1}$ is the Fourier transform of a function in the dual space of $H^1(\mathbf{R})$, that is, $BMO(\mathbf{R})$ (cf. [**15**, Chap.4]). We shall paraphrase this definition by using fractional calculus on G. First we introduce a generalized Riemann-Liouville type fractional integral operator $\tilde{W}^{\sigma}_{\mu}$: For $\sigma > 0$ and $\mu \in \mathbf{C}$, $\tilde{W}^{\sigma}_{\mu}(F)(y)$, $y > 0$, is defined by

$$\tilde{W}^{\sigma}_{\mu}(f)(y) = \frac{\sigma^{-1}}{\Gamma(\mu+n)}\frac{d^n}{d(\mathrm{ch}\sigma y)^n}\int_0^y f(x)(\mathrm{ch}\sigma y - \mathrm{ch}\sigma x)^{\mu+n-1}dx \cdot \mathrm{sh}\sigma y, \quad (92)$$

where $n = 0$ if $\Re\mu > 0$ and $-n < \Re\mu \le -n+1$, $n = 0, 1, 2, \cdots$, if $\Re\mu \le 0$. Similarly as in (16), if we set

$$\tilde{W}_+(f) = \tilde{W}^2_{\beta+1/2} \circ \tilde{W}^1_{\alpha-\beta}(e^{\rho x} f),$$

then we see from [**11**, §3] and [**10**, Theorem 5.1] that

$$\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta dx)} = \langle W_+(f), \tilde{W}_+^{-1}(g\Delta)\rangle_{L^2(\mathbf{R},dx)}$$

(cf. (**??**) for $\mathbf{R}_+$) and equivalently,

$$\tilde{g}(\lambda+i\rho)C(-\lambda-i\rho)^{-1} = c\left(\tilde{W}_+^{-1}(g\Delta)\right)^{\sim}(-\lambda),$$

where the symbol "$\sim$" in the left (resp. right) hand side denotes the complex Fourier-Jacobi (resp. the Euclidean Fourier) transform. Hence the following definition of a dual space of $W_-(H^1(\mathbf{R}))$ is quite natural.

Definition 8.7. *We define*

$$W_-(H^1(\mathbf{R}))^* = \{f \in L^1_{\mathrm{loc}}(G//K)\ ;\ \tilde{W}_+^{-1}(f\Delta) \in BMO(\mathbf{R})\}.$$

and $\|f\|_{W_-(H^1(\mathbf{R}))^*} = \|\tilde{W}_+^{-1}(f\Delta)\|_{BMO(\mathbf{R})}$.
In this definition $f \in W_-(H^1(\mathbf{R}))^*$ is required to be W_+-smooth.

Proposition 8.8. *Let notation be as above.*
(1) *For* $f \in W_-(H^1(\mathbf{R}))$ *and* $h \in W_-(H^1(\mathbf{R}))^*$,

$$\left|\int_G f(g)h(g)dg\right| \le c\|f\|_{W_-(H^1(\mathbf{R}))}\|h\|_{W_-(H^1(\mathbf{R}))^*}.$$

(2) *All K-bi-invariant W_+-smooth bounded functions belong to* $W_-(H^1(\mathbf{R}))^*$.

Proof. (1) is clear from the above observation and the dual inequality between $H^1(\mathbf{R})$ and $BMO(\mathbf{R})$ (see [**15**. p.146]). As for (2), let f be a K-bi-invariant $\tilde{W}_+$-smooth bounded function on G. Then $\tilde{W}_+^{-1}(f\Delta) \in L^\infty(\mathbf{R})$ by [**10**, Lemma 3.3]. Since $L^\infty(\mathbf{R}) \subset BMO(\mathbf{R})$, f belongs to $W_-(H^1(\mathbf{R}))^*$. ■

D. Real Hardy Spaces: In Definition 7.2 we introduced $W_-(H^{1,\epsilon}_{\infty,0}(\mathbf{R}))$ as a subspace of $W_-(H^1(\mathbf{R}))$. Here we shall define a subspace $W_-(\mathsf{H}^{1,\epsilon}_{\infty,0}(\mathbf{R}))$ of $W_-(H^{1,\epsilon}_{\infty,0}(\mathbf{R}))$, which corresponds to $\mathsf{H}^{1,\epsilon}_{\infty,0}(G//K)$. Let A be a centered $(1,\infty,0,\epsilon)$-atom on $\mathbf{R}$ and suppose that A is even. For $x_0 \geq 0$, we define

$$A_{x_0}(x) = \tilde{W}_+ \left(A(x-s)\right)(x_0)\Delta(x_0)^{-1}, \quad x > 0, \tag{93}$$

where $\tilde{W}_+$ acts on s and $\tilde{W}_+(F)(x_0)\Delta(x_0)^{-1} = F$ if $x_0 = 0$. Then, it follows from [**10**, Lemma 3.4] that A_{x_0} is also a $(1,\infty,0,\epsilon)$-atom on $\mathbf{R}$. We introduce a modified atomic Hardy space $\mathsf{H}^{1,\epsilon}_{\infty,0}(\mathbf{R})$ on $\mathbf{R}$ as the space of all $\sum_i \lambda_i(A_{i,x_i}(x) + A_{i,x_i}(-x))$ such that $\sum |\lambda_i| < \infty$, $x_i \geq 0$ and each A_i is an even centered $(1,\infty,0,\epsilon)$-atom on $\mathbf{R}$. Moreover, we define $\mathsf{H}^{1,\epsilon}_{\infty,0}(\mathbf{R})_\alpha$ if each A_i furthermore satisfies

$$\|W^{\mathbf{R}}_{-s_\alpha}(A)\|_\infty \leq r^{-1}(1+r)^{-\epsilon}. \tag{94}$$

Since each A_{i,x_i} is a $(1,\infty,0,\epsilon)$-atom on $\mathbf{R}$, it follows that

$$\mathsf{H}^{1,\epsilon}_{\infty,0}(\mathbf{R})_\alpha \subset \mathsf{H}^{1,\epsilon}_{\infty,0}(\mathbf{R}) \subset H^{1,\epsilon}_{\infty,0}(\mathbf{R}) \subset H^1(\mathbf{R}).$$

Definition 8.9. *We define*

$$W_-(H^{1,\epsilon}_{\infty,0}(\mathbf{R})) \;=\; \{f \in L^1_{\mathrm{loc}}(G//K) \;;\; W_+(f) \in \mathsf{H}^{1,\epsilon}_{\infty,0}(\mathbf{R})\}$$

and give their norms by $\|W_+(f)\|_{H^1(\mathbf{R})}$. *Similarly, we define* $W_-(H^{1,\epsilon}_{\infty,0}(\mathbf{R})_\alpha)$.

Proposition 8.10. *Let notation be as above. Then*

$$\mathsf{H}^{1,1}_{\infty,0}(G//K) \subset W_-(\mathsf{H}^1_{\infty,0}(\mathbf{R})).$$

Proof. Let a be a centered $(1,\infty,0,1)$-atom on G supported on $B(r)$. We put $A = W_+(a)$. Since A is also supported on $B(r)$ (see Lemma 5.11), it follows that

$$\begin{aligned}|W_+(A)(x)| &\leq ce^{2\rho x}(\mathrm{th} r)^{s_\alpha}\|a\|_\infty \\ &\leq ce^{2\rho r}(\mathrm{th} r)^{s_\alpha}|B(r)|^{-1}(1+r)^{-1} \leq cr^{-1}.\end{aligned}$$

Moreover, A satisifies the moment condition, because $A^\sim(0) = \hat{a}(i\rho) = 0$. Therefore, A is a $(1,\infty,0)$-atom on $\mathbf{R}$. We here note the following.

Lemma 8.11. $W_+(a^\flat_{x_0}) = W_+(a)_{x_0}$ *on* $\mathbf{R}_+$, *that is,*

$$W_+(a^\flat_{x_0})(x) = \tilde{W}_+\left(W_+(a)(y-s)\right)(x_0)\Delta(x_0)^{-1}, \quad x > 0,$$

where $\tilde{W}_+$ acts on s.

Proof. Let $B = W_+(a^\flat_{x_0})$. Since $e^{-\rho y}A(y)$, $e^{-\rho y}B(y)$ are even, we see that

$$\begin{aligned}\left(e^{-\rho y}B(y)\right)^{\sim}(\lambda) &= \hat{a}_{x_0}(\lambda)\\ &= \hat{a}(\lambda)\varphi_\lambda(x_0)\\ &= (e^{-\rho y}A(y))^{\sim}(\lambda)\tilde{W}_+(e^{-\rho s}\cos\lambda s)(x_0)\Delta(x_0)^{-1}\\ &= \tilde{W}_+((e^{-\rho y}A(y-s))^{\sim}(\lambda))(x_0)\Delta(x_0)^{-1}.\end{aligned}$$

Hence, $e^{-\rho y}B(y) = \tilde{W}_+(e^{-\rho y}A(y-s))(x_0)\Delta(x_0)^{-1}$ and the desired relation follows. ■

Lemma 8.11 and the fact shown before the lemma yield that each $a^\flat_{x_0}$ belongs to $W_-(\mathsf{H}^1_{\infty,0}(\mathbf{R}))$ and the norm is bounded by a constant independent of a, x_0. Hence, the desired result follows. ■

Proposition 8.12. *Let notation be as above. Then*

$$W_-(\mathsf{H}^{1,1}_{\infty,0}(\mathbf{R})_\alpha) \subset \mathsf{H}^1_{\infty,0}(G//K).$$

Proof. Let A be an even centered $(1,\infty,0,1)$-atom on $\mathbf{R}$ supported on $B(r)$ and moreover, satisfying (**??**). Let $a = W_-(A)$. Then Lemma 8.11 implies that $A_{x_0}(x) + A_{x_0}(-x) = W_+(a^\flat_{x_0})(x)$. Therefore, it is enough to show that a has a decomposition $a = \sum_i \lambda_i a_i$ such that $\sum_i |\lambda_i| \le c$ and each a_i is a centered $(1,\infty,0)$-atom on G, where c is independent of A. Since A is a $(1,\infty,0,1)$-atom supported on $B(r)$ and satisfying (**??**), as in the proof of Proposition 6.2, it follows that

$$|a(x)| \le c\Delta(x)^{-1}r^{-1}(1+r)^{-1}$$

We put

$$a_+(x) = c\Delta(x)^{-1}r^{-1}\chi_{[0,r]}(x),\ x > 0.$$

Then $|a(x)| \le a_+(x)$ and a_+ is a non-increasing function on $\mathbf{R}_+$ with finite L^1-norm. Since a is supported on $B(r)$, $|B(s)|^{-1}\int_s^\infty a_0(x)\Delta(x)dx$ is also supported on $B(r)$. For $0 < s < r$, it follows from the moment condition that

$$\frac{1}{|B(s)|}\int_s^\infty a_0(x)\Delta(x)dx = \frac{1}{|B(s)|}\int_0^s a_0(x)\Delta(x)dx \le c|B(s)|^{-1}sr^{-1}(1+r)^{-1}.$$

Since $|B(s)| \sim \Delta(s)s$ if $s \le 1$ and $|B(s)| \sim \Delta(s)$ if $s \ge 1$, it follows that

$$\frac{1}{|B(s)|}\int_s^\infty a_0(x)\Delta(x)dx \le a_+(s). \tag{95}$$

This means that a_+ is an L^1 non-increasing denominator of a satisfying (**??**). Then [**7**, Theorem 4.5] yields that a has a centered $(1, \infty, 0)$-atomic decomposition $a = \sum_i \lambda_i a_i$ on G such that $\sum_i |\lambda_i| \leq \|a_+\|_{L^1(\Delta dx)}$. ■

Main relationship among real Hardy spaces defined in this paper (see Definitions 4.1, 4.2, 5.2, 5.3, 5.8, 7.2, and 8.9) are summarized in the following diagram, where we abbreviate "$G//K$" as G. The relation between $H^1(G//K)$ and $H^1_{\infty,0}(G//K)$ is still open.

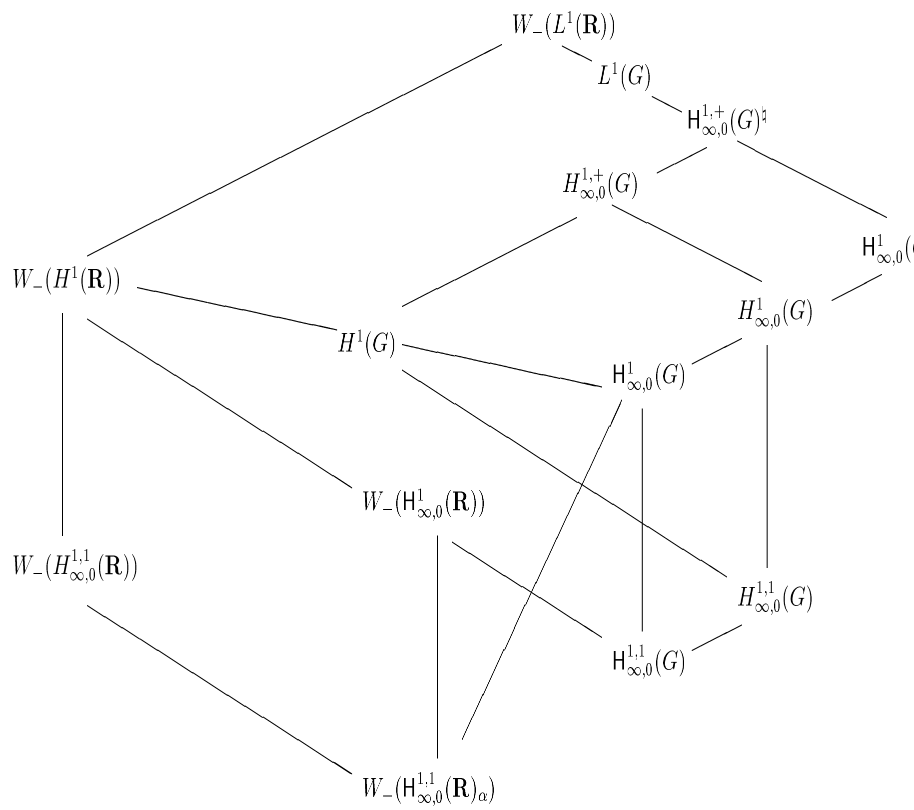

Figure 1: Real Hardy Spaces

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Real Hardy spaces on real rank 1 semisimple Lie groups

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Abstract

Let G be a real rank one connected semisimple Lie group with finite center. We introduce a real Hardy space $H^1(G//K)$ on G as the space consisting of all K-bi-invariant functions f on G whose radial maximal functions $M_\phi f$ are integrable on G. We shall obtain a relation between $H^1(G//K)$ and $H^1(\mathbf{R})$, the real Hardy space on the real line $\mathbf{R}$, via the Abel transform on G and give a characterization of $H^1(G//K)$.

1 Introduction

The study of the classical Hardy spaces on the unit disk and the upper half plane was originated during the 1910's by the complex variable method. In the 1970's, considering their boundary values, the Hardy spaces were completely characterized by various maximal functions and also by atoms, without using the complex variable method. This is a significant breakthrough in harmonic analysis. Nowadays, the spaces defined by the real variable method – maximal functions and atoms – called *real* Hardy spaces and a fruitful theory of real Hardy spaces has been extended to the spaces of homogeneous type: A topological space X with measure μ and distance d is of homogeneous type if there exists a constant $c > 0$ such that for all $x \in X$ and $r > 0$

$$\mu(B(x, 2r)) \leq c\mu(B(x, r)),$$

where $B(x, r)$ is the ball defined by $\{y \in X \mid d(x, y) < r\}$ and $\mu(B(x, r))$ the volume of the ball (cf. [**1**, §1]). However, when the

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space X is not of homogeneous type, little work on real Hardy spaces on X has been done. Hence, looking at the example of a semisimple Lie group G as a space of non-homogeneous type, we shall introduce a real Hardy space $H^1(G//K)$ by using a radial maximal function on G. In this article we shall overview some results obtained in the previous papers [**5**], [**6**], [**7**] and announce a new characterization of $H^1(G//K)$, which gives a relation between $H^1(G//K)$ and the real Hardy space $H^1(\mathbf{R})$ on $\mathbf{R}$ via the Abel transform on G.

2 Notation

Let G be a real rank one connected semisimple Lie group with finite center, $G = KAN = KAK$ Iwasawa and Cartan decompositions of G. Let $dg = dkdadn = \Delta(a)dkdadk'$ denote the corresponding decompositions of a Haar measure dg on G. In what follows we shall treat only K-bi-invariant functions on G. Since A is identified with $\mathbf{R}$ as $A = \{a_x; x \in \mathbf{R}\}$, all K-bi-invariant functions can be identified with even functions on $\mathbf{R}$ as

$$f(g) = f(a_{\sigma(g)}) = f(\sigma(g)) = f(-\sigma(g))$$

and $\Delta(a_x)$ is given by

$$\Delta(x) = c(\mathrm{sh}x)^{2\alpha+1}(\mathrm{sh}2x)^{2\beta+1}, \tag{1}$$

where $\alpha = (m_1 + m_2 - 1)/2$, $\beta = (m_2 - 1)/2$ and m_1, m_2 the multiplicities of a simple root γ of (G, A) and 2γ respectively. We note that the one dimensional space $\mathbf{R}$ with normal distance and weighted measure $\Delta(x)dx$ is not of homogeneous type, because $\Delta(x) \sim e^{2\rho x}$ with $\rho = \alpha + \beta + 1 > 0$ as $x \to \infty$. Let $L^p(G//K)$ denote the space of all K-bi-invariant functions on G with finite L^p-norm and $L^1_{\mathrm{loc}}(G//K)$ the space of all locally integrable, K-bi-invariant functions on G.

Let F be the dual space of the Lie algebra of A and for $\lambda \in \mathsf{F}$, φ_λ the normalized zonal spherical function on G which is explicitly given by

$$\varphi_\lambda(x) = {}_2F_1\left((\rho + i\lambda)/2, (\rho - i\lambda)/2; \alpha + 1; -\mathrm{sh}^2 x\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function. We recall that, if $\lambda \notin \mathbf{Z}$, then $\varphi_\lambda(x)$ has the so-called Harish-Chandra expansion:

$$\varphi_\lambda(x) = e^{-\rho x}\left(\Phi(\lambda, x)C(\lambda)e^{i\lambda x} + \Phi(-\lambda, x)C(-\lambda)e^{-i\lambda x}\right), \tag{2}$$

where $C(\lambda)$ is Harish-Chandra's C-function. For some basic properties of $\varphi_\lambda(x)$, $\Phi(\lambda, x)$, and $C(\lambda)$ we refer to [**2**, §2, §3] and [**12**, 9.1.4, 9.1.5].

For $f \in L^1(G//K)$ the spherical Fourier transform $\hat{f}(\lambda)$, $\lambda \in \mathsf{F}$, of f is defined by

$$\hat{f}(\lambda) = \int_G f(g)\varphi_\lambda(g)dg = \int_0^\infty f(x)\varphi_\lambda(x)\Delta(x)dx.$$

Since $\varphi_\lambda(x)$ is even with respect to λ, x and uniformly bounded on x if λ is in the tube domain $\mathsf{F}(\rho) = \{\lambda \in \mathsf{F}_c; |\Im\lambda| \leq \rho\}$, it follows that $\hat{f}(\lambda)$ is even, continuously extended on $\mathsf{F}(\rho)$, holomorphic in the interior, and

$$|\hat{f}(\lambda)| \leq \|f\|_1, \quad \lambda \in \mathsf{F}(\rho).$$

For $f \in C_c^\infty(G//K)$ the Paley-Wiener theorem (cf. [**2**, Theorem 4]) implies that $\hat{f}(\lambda)$ is holomorphic on F_c of exponential type. Furthermore, it satisfies the inversion formula

$$f(x) = \int_{-\infty}^\infty \hat{f}(\lambda)\varphi_\lambda(x)|C(\lambda)|^{-2}d\lambda$$

and the Plancherel formula

$$\int_0^\infty |f(x)|^2\Delta(x)dx = \int_0^\infty |\hat{f}(\lambda)|^2|C(\lambda)|^{-2}d\lambda.$$

Therefore, the spherical Fourier transform $f \mapsto \hat{f}$ of $C_c^\infty(G//K)$ is uniquely extended to an isometry between $L^2(G//K) = L^2(\mathbf{R}_+, \Delta(x)dx)$ and $L^2(\mathbf{R}_+, |C(\lambda)|^{-2}d\lambda)$ (cf. [**2**, Proposition 3], [**12**, Theorem 9.2.2.13]).

For $f \in C_c^\infty(G//K)$ we define the Abel transform F_f^s, $s \in \mathbf{R}$, of f as

$$F_f^s(x) = e^{\rho(1+s)x}\int_N f(a_x n)dn. \tag{3}$$

Then the Euclidean Fourier transform $(F_f^s)^\sim(\lambda)$ is holomorphic on F_c of exponential type, because $F_f^s(f) \in C_c^\infty(\mathbf{R})$, and it coincides with the spherical Fourier transform of f:

$$\hat{f}(\lambda + is\rho) = (F_f^s)^\sim(\lambda), \quad \lambda \in \mathsf{F}_c. \tag{4}$$

(cf. [**9**, §3]). Especially, F_f^0 is even on $\mathbf{R}$. The integral over N in (3) can be explicitly rewritten by using a generalized Weyl type fractional integral operator W_μ^σ: For $\sigma > 0$ and $\mu \in \mathbf{C}$, we define $W_\mu^\sigma(f)(y)$, $y > 0$, as

$$W_\mu^\sigma(f)(y) = \frac{(-1)^n}{\Gamma(\mu+n)}\int_y^\infty \frac{d^n f(x)}{d(\mathrm{ch}\sigma x)^n}(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu+n-1}d(\mathrm{ch}\sigma x), \tag{5}$$

where $n = 0$ if $\Re\mu > 0$ and $-n < \Re\mu \leq -n+1$, $n = 0, 1, 2, \cdots$, if $\Re\mu \leq 0$ (see [**9**, (3.11)]). Koornwinder obtains that for $x > 0$,

$$F_f^0(x) = W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(f)(x).$$

(see [**9**, (2.18), (2.19), (3.5)]). In the following, for simplicity, we denote $W_+(f)(x) = F_f^1(|x|)$, that is,

$$W_+(f)(x) = e^{\rho x} W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(f)(|x|),\ x \in \mathbf{R} \tag{6}$$

and for a function F on $\mathbf{R}_+$,

$$W_-(F)(x) = W^2_{-(\beta+1/2)} \circ W^1_{-(\alpha-\beta)}(e^{-\rho x}F),\ x \in \mathbf{R}_+. \tag{7}$$

Then $W_- \circ W_+(f) = f$ and $W_+ \circ W_-(F) = F$.

For $f \in L^1(G//K)$, $W_+(f)$ belongs to $L^1(\mathbf{R})$, because $W_+(f)(x) = e^{2\rho x}\displaystyle\int_N f(a_x n)dn$, $x > 0$ (see (3)) and thus, the integral formula for the Iwasawa decomposition of G yields that

$$\|W_+(f)\|_{L^1(\mathbf{R})} \leq \|f\|_1 \tag{8}$$

(cf. [**9**, (3.5), (2.20)]). Hence $W_+(f)^\sim(\lambda)$, $\lambda \in \mathsf{F}$, is well-defined and by (4)

$$\hat{f}(\lambda + i\rho) = W_+(f)^\sim(\lambda), \quad \lambda \in \mathsf{F}. \tag{9}$$

For $f, g \in L^1(G//K)$, since $f * g \in L^1(G//K)$ and $(f*g)^\wedge(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$ (cf. [**2**, Theorem 5], [**3**, §5]), it follows that

$$W_+(f*g) = W_+(f) * W_+(g), \tag{10}$$

We say that a function F on $\mathbf{R}$ is W_+-smooth if $W_-(F)$ is well-defined and continuous. Then, for W_+-smooth functions F, G on $\mathbf{R}$ with compact support such that $e^{-\rho x}F$ and $e^{-\rho x}G$ are even, it follows that

$$W_-(F*G) = W_-(F) * W_-(G).$$

3 Radial maximal functions

As in the Euclidean case, to define a radial maximal function we need to define a dilation ϕ_t, $t > 0$, of a function ϕ on G. Let ϕ be a positive compactly supported C^∞, K-bi-invariant function on G such that

$$\int_G \phi(g)dg = \int_0^\infty \phi(x)\Delta(x)dx = 1. \tag{11}$$

and furthermore, there exists $M \in \mathbf{N}$ such that

$$\phi(x) = O(x^{2M}). \tag{12}$$

We define the dilation ϕ_t of ϕ as

$$\phi_t(x) = \frac{1}{t}\frac{1}{\Delta(x)}\Delta\left(\frac{x}{t}\right)\phi\left(\frac{x}{t}\right).$$

Clearly, ϕ_t has the same L^1-norm of ϕ: $\|\phi_t\|_1 = \|\phi\|_1$ and, for $1 \leq p \leq \infty$, it gives an approximate identity in $L^p(G//K)$ (see [**2**, Lemma 16]). We here introduce the radial maximal function $M_\phi f$ on G as follows.

Definition 3.1. *For $f \in L^1_{\text{loc}}(G//K)$,*

$$(M_\phi f)(g) = \sup_{0<t<\infty} |(f * \phi_t)(g)|, \quad g \in G.$$

As shown in [**5**, Theorem 3.4 and Theorem 3.5], M_ϕ satisfies the maximal theorem and, for $1 \leq p \leq \infty$, $\|f\|_p \leq \|M_\phi f\|_p$ if the both sides exist. By using $W_+(\phi_t)$, we shall define a maximal function on $\mathbf{R}$ as follows.

Definition 3.2. *For $F \in L^1_{\text{loc}}(\mathbf{R})$,*

$$(M^{\mathbf{R}}_\phi F)(x) = \sup_{0<t<\infty} |(F * W_+(\phi_t)(x)|, \quad x \in \mathbf{R}.$$

Since $W_+(f * \phi_t) = W_+(f) * W_+(\phi_t)$ (see (10)) and W_+ is an integral operator with a positive kernel (see (6)), it follows that

$$\sup_{0<t<\infty} |W_+(f) * W_+(\phi_t)(x)| \leq W_+\left(\sup_{0<t<\infty} |f * \phi_t|\right)(x).$$

Therefore, from (8) we have a relation between M_ϕ and $M^{\mathbf{R}}_\phi$:

Proposition 3.3. *For $f \in L^1_{\text{loc}}(G//K)$,*

$$(M^{\mathbf{R}}_\phi W_+(f))(x) \leq W_+(M_\phi f)(x), \quad x \in \mathbf{R}.$$

In particular,

$$\|M^{\mathbf{R}}_\phi W_+(f)\|_{L^1(\mathbf{R})} \leq c\|M_\phi f\|_1$$

if the both sides exist.

We note that $W_+(\phi_t)^\sim(\lambda) = \hat{\phi}_t(\lambda + i\rho)$ (see (9)) has similar properties of the Euclidean Fourier transform of a Euclidean dilation: Let M be the same as (12).

(1) *There exists c such that for all $t > 0$, $\lambda \in \mathbf{R}$ and $0 \leq k \leq M$,*

$$\left|\left(\frac{d}{d\lambda}\right)^n \hat{\phi}_t(\lambda + i\rho)\right| \leq ct^n(1+t)^k(1+|t\lambda|)^{-2k}.$$

(2) *There exists c such that for all $t > 1$ and $\lambda \in \mathbf{R}$,*

$$\left|\left(\frac{d}{d\lambda}\right)^n \hat{\phi}_t(\lambda + i\rho)\right| \leq ct^n(1 + |t\lambda|)^{-(2M+\alpha+1/2)}.$$

(3) $\hat{\phi}_t(\lambda + i\rho) \to 1 \quad \text{as } |t\lambda| \to 0.$

(4) $|\hat{\phi}_t(\lambda + i\rho)| \geq 1/2 \quad \text{if } 0 \leq |t\lambda| \leq 2.$

These properties mean that $W_+(\phi)$ behaves like a Euclidean dilation on $\mathbf{R}$. Hence, we can deduce that the maximal operator $M_\phi^{\mathbf{R}}$ can characterize $H^1(\mathbf{R})$, that is, $F \in H^1(\mathbf{R})$ if and only if $M_\phi^{\mathbf{R}}(F) \in L^1(\mathbf{R})$:

Theorem 3.4. *Let ϕ be as above and suppose that $M \geq 2$. Then $F \in H^1(\mathbf{R})$ if and only if $M_\phi^{\mathbf{R}} F \in L^1(\mathbf{R})$:*

$$\|F\|_{H^1(\mathbf{R})} \approx \|M_\phi^{\mathbf{R}} F\|_{L^1(\mathbf{R})}.$$

4 Real Hardy spaces

Let ϕ be the same as in the previous section (see (11), (12)) and $M_\phi, M_\phi^{\mathbf{R}}$ the corresponding radial maximal operators on G and $\mathbf{R}$ respectively (see Definitions 3.1 and 3.2). In this section we shall define two real Hardy spaces $H_\phi^1(G//K)$ and $W_-(H^1(\mathbf{R}))$ on G and give a relation between them.

Definition 4.1. *We define*

$$H_\phi^1(G//K) = \{f \in L^1_{\mathrm{loc}}(G//K) \ ; \ M_\phi f \in L^1(G//K)\}$$

and $\|f\|_{H_\phi^1(G)} = \|M_\phi f\|_1$.

Clearly, since $\|f\|_1 \leq \|M_\phi f\|_1$, it follows that

$$H_\phi^1(G//K) \subset L^1(G//K).$$

Next we shall introduce a pull-back of the real Hardy space $H^1(\mathbf{R})$ on $\mathbf{R}$ to G via W_+ (see (6)). Let M_s, $s \geq 0$, denote the Euclidean Fourier multiplier given by

$$\mathsf{M}_s(F)^\sim(\lambda) = (\lambda + i\rho)^s F^\sim(\lambda).$$

Definition 4.2. *For $s \geq 0$, we define*

$$W_-(\mathsf{M}_{-s}(H^1(\mathbf{R}))) = \{f \in L^1_{\mathrm{loc}}(G//K) \ ; \ \mathsf{M}_s \circ W_+(f) \in H^1(\mathbf{R})\}$$

and give the norm by $\|\mathsf{M}_s \circ W_+(f)\|_{H^1(\mathbf{R})}$. We denote $W_-(\mathsf{M}_0(H^1(\mathbf{R})))$ by $W_-(H^1(\mathbf{R}))$ for simplicity.

Obviously, Proposition 3.3 and Theorem 3.4 yield the following.

Corollary 4.3. *Let $M \geq 2$. There exists a positive constant c such that $\|W_+(f)\|_{H^1(\mathbf{R})} \leq c\|f\|_{H^1_\phi(G)}$ for all $f \in H^1_\phi(G//K)$ and thus,*

$$H^1_\phi(G//K) \subset W_-(H^1(\mathbf{R})).$$

Let $s_\alpha = \alpha + 1/2$. Then we see that

$$W_-(\mathsf{M}_{-s_\alpha}(H^1(\mathbf{R}))) \subset W_-(H^1(\mathbf{R})). \tag{13}$$

Actually, if we put $F = W_+(f)$ for $f \in W_-(\mathsf{M}_{-s_\alpha}(H^1(\mathbf{R})))$, then $\mathsf{M}_{s_\alpha}(F)$ belongs to $H^1(\mathbf{R})$. Since the Fourier multiplier M_{-s_α} satisfies the Hörmander condition (cf. [**11**, §5 in Chap.11]), it is bounded on $H^1(\mathbf{R})$ (cf. [**11**, Theorem 4.4 in Chap.14]). Thereby, $F \in H^1(\mathbf{R})$ and the desired inclusion follows. Similarly, since the Fourier multipliers $\mathsf{M}_{s_\alpha}^{-1} \circ W^{\mathbf{R}}_{-\gamma}$, $0 \leq \gamma \leq s_\alpha$, which correspond to $(i\lambda)^\gamma/(\lambda + i\rho)^{s_\alpha}$, satisfy the Hörmander condition, they are bounded on $H^1(\mathbf{R})$. Hence, each $W^{\mathbf{R}}_{-\gamma}(F)$ also belongs to $H^1(\mathbf{R})$: For $0 \leq \gamma \leq s_\alpha$,

$$\|W^{\mathbf{R}}_{-\gamma}(F)\|_{H^1(\mathbf{R})} = \|M^{\mathbf{R}}_\phi(W^{\mathbf{R}}_{-\gamma}(F))\|_{L^1(\mathbf{R})} \leq c\|\mathsf{M}_{s_\alpha}(F)\|_{H^1(\mathbf{R})}.$$

Now we shall characterize the H^1_ϕ-norm of $f \in H^1_\phi(G//K)$ and show that $H^1_\phi(G//K)$ locates between $W_-(\mathsf{M}_{-s_\alpha}(H^1(\mathbf{R})))$ and $W_-(H^1(\mathbf{R}))$ (see (13)). We recall that

$$f * \phi_t = W_-(W_+(f) * W_+(\phi_t)) = W_-(F * W_+(\phi_t)).$$

Therefore, roughly speaking, the H^1_ϕ-norm of f, that is, the L^1-norm of $M_\phi f$ on G (see Definition 4.1) can be characterized in terms of the L^1-norm of $M^{\mathbf{R}}_\phi(W_-(F))$ (see Definition 3.1). Let $\delta = (\alpha-\beta)-[\alpha-\beta]$ and $\delta' = (\beta - 1/2) - [\beta - 1/2]$, where $[\,\cdot\,]$ is the Gauss symbol, and put $\underline{n} = [s_\alpha]$, $\underline{\delta} = \delta + \delta'$ and $\underline{D} = \{\delta, \delta', \delta + \delta'\}$. Then the local and global forms of the operator W_- in (7) can be rewritten by using the Weyl type fractional operator $W^{\mathbf{R}}_\mu$ on $\mathbf{R}$:

$$W^{\mathbf{R}}_\mu(F)(y) = \frac{(-1)^n}{\Gamma(\mu+n)} \int_y^\infty \frac{d^n F(x)}{dx^n}(x-y)^{\mu+n-1}dx,$$

which corresponds to the case of $f(x) = F(\mathrm{ch}x)$ and $\alpha = \beta = 1/2$ in (5).

(1) *If F is W_+-smooth and supported on $0 < x \leq 1$, then*

$$\begin{aligned} &|W_-(F)(x)| \\ \leq\ & c\sum_{m,\xi}\left(x^{-2s_\alpha+\underline{\delta}+m}W^{\mathbf{R}}_{-(m+\xi)}(F)(x) + \int_x^\infty W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^1_m(x,s)ds\right), \end{aligned}$$

where the sum is taken over $0 \le m \le \underline{n}$ *and* $\xi \in \underline{D}$, *and* $A^1_m(x,s)$ *satisfies*

$$0 \le A^1_m(x,s) \le x^{-2s_\alpha+\underline{\delta}+m-1} \quad \text{for all } 0 < x \le s. \qquad (14)$$

(2) *If* F *is* W_+*-smooth and supported on* $x \ge 1$, *then*

$$\begin{aligned} & |W_-(F)(x)| \\ \le\ & c\sum_{m,\xi}\Bigg(\Big(x^{-2s_\alpha+\underline{\delta}+m}W^{\mathbf{R}}_{-(m+\xi)}(F)(x)+\int_x^\infty W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^2_m(x,s)ds \\ & +x^{-2\alpha+\underline{\delta}+m}\int_x^\infty W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^3_m(x,s)ds\Big)\chi_{[0,1]}(x) \\ & +e^{-2\rho x}\Big(W^{\mathbf{R}}_{-(m+\xi)}(F)(x)+\int_x^\infty W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^4_m(x,s)ds\Big)\chi_{[1,\infty)}(x)\Bigg), \end{aligned}$$

where $A^2_m(x,s)$ *satisfies* (14) *and for* $j=3,4$, $A^j_m(x,s)\ge 0$ *and there exists a positive constant* c *such that*

$$\int_0^s A^j_m(x,s)dx \le c \quad \text{for all } s > 0.$$

This means that the L^1-norm of $M_\phi f$ on G can be characterized in terms of $M^{\mathbf{R}}_\phi(W^{\mathbf{R}}_{-\gamma}(F))$ on $\mathbf{R}$. Finally, we have the following.

Theorem 4.4 *Let* $M \ge 2$ *and* $F = W_+(f)$ *for* $f \in W_-(\mathsf{M}_{-s_\alpha}(H^1(\mathbf{R})))$. *Then there exist* c_1, c_2 *such that for all* $0 \le \gamma \le s_\alpha$,

$$\begin{aligned} c_1\|M^{\mathbf{R}}_\phi \circ W^{\mathbf{R}}_{-\gamma}(F)(x)(\mathrm{th}x)^\gamma\|_{L^1(\mathbf{R})} &\le \|f\|_{H^1_\phi(G)} \\ \le\ c_2\sum_{m=0}^{\underline{n}}\sum_{\xi\in\underline{D}}\|M^{\mathbf{R}}_\phi \circ W^{\mathbf{R}}_{-(m+\xi)}(F)(x)(\mathrm{th}x)^{m+\underline{\delta}}\|_{L^1(\mathbf{R})}. & \end{aligned}$$

Especially,

$$\begin{aligned} \|f\|_{H^1_\phi(G)} &\approx \sum_{m=0}^{\underline{n}}\sum_{\xi\in\underline{D}}\|M^{\mathbf{R}}_\phi \circ W^{\mathbf{R}}_{-(m+\xi)}(F)(x)(\mathrm{th}x)^{m+\xi}\|_{L^1(\mathbf{R})} \\ &\le c\sum_{m=0}^{\underline{n}}\sum_{\xi\in\underline{D}}\|W^{\mathbf{R}}_{-(m+\xi)}(F)\|_{H^1(\mathbf{R})} \\ &\le c\|\mathsf{M}_{s_\alpha}(F)\|_{H^1(\mathbf{R})} \end{aligned}$$

and thus,

$$W_-(\mathsf{M}_{-s_\alpha}(H^1(\mathbf{R}))) \subset H^1_\phi(G//K) \subset W_-(H^1(\mathbf{R})).$$

Remark 4.5. Let $C(\lambda)$ be Harish-Chandra's C-function (see (2)) and M_{C_ρ} the Euclidean Fourier multiplier corresponding to $C_\rho(\lambda) = C(\lambda + i\rho)$:

$$\mathsf{M}_{C_\rho}(F)^\sim(\lambda) = C(\lambda + i\rho)F^\sim(\lambda).$$

If we define

$$W_-(\mathsf{M}_{C_\rho}(H^1(\mathbf{R}))) = \{f \in L^1_{\mathrm{loc}}(G//K) \ ; \ \mathsf{M}_{C_\rho}^{-1} \circ W_+(f) \in H^1(\mathbf{R})\},$$

then it easily follows from Theorem 4.4 that

$$W_-(\mathsf{M}_{C_\rho}(H^1(\mathbf{R}))) \subset H^1_\phi(G//K) \subset W_-(H^1(\mathbf{R})).$$

This is one of main results in [**6**]. However, the proof was a little bit complicated, because to obtain the first inclusion we used the Harish-Chandra expansion of the zonal spherical function φ_λ and also the Gangolli expansion of Φ_λ (see (2), [**2**, §3]). Thereby, to sum up the estimates of each expanded terms we required a sharp estimate and a deep theory of H^1 Fourier multipliers on $\mathbf{R}$. We here obtain the desired inclusion as an easy consequence of Theorem 4.4.

5 Atomic Hardy spaces

We introduce *atomic* Hardy spaces on G. In the Euclidean space the atomic Hardy space $H^1_{\infty,0}(\mathbf{R})$ coincides with $H^1(\mathbf{R})$ (cf. [**4**, Theorem 3.30], [**10**, §2 in Chap.3]). However, it may be not true in our setting, because the Lebesgue measure dx is replaced by the weighted measure $\Delta(x)dx$ (see (1)). We denote the interval $[x_0 - r, x_0 + r]$ by $R(x_0, r)$ and set the volume by

$$|R(x_0, r)| = \int_{x_0 - r}^{x_0 + r} \Delta(x)dx.$$

We say that a K-bi-invariant function a on G is a $(1, \infty, 0)$-atom on G provided that there exist $x_0 \geq 0$ and $r > 0$ such that

$$\begin{aligned} &(i) \quad \mathrm{supp}(a) \subset R(x_0, r), \\ &(ii) \quad \|a\|_\infty \leq |R(x_0, r)|^{-1}, \\ &(iii) \int_0^\infty a(x)\Delta(x)dx = 0. \end{aligned} \tag{15}$$

Here a is identified with a function on $\mathbf{R}_+$. Then the $(1, \infty, 0)$-atomic Hardy space $H^1_{\infty,0}(G//K)$ is defined as follows.

Definition 5.1. *Let notations be as above. We define*

$$H^1_{\infty,0}(G//K) = \{f = \sum_i \lambda_i a_i \ ; \ a_i \text{ is } (1,\infty,0)\text{-atom on } G \text{ and } \textstyle\sum_i |\lambda_i| < \infty\}$$

and denote the norm by

$$\|f\|_{H^1_{\infty,0}(G)} = \inf \sum_i |\lambda_i|,$$

where the infimum is taken over all such representations $f = \sum_i \lambda_i a_i$. *We also define* $H^{1,\epsilon}_{\infty,0}(G//K)$ $(\epsilon \geq 0)$ *and* $H^{1,+}_{\infty,0}(G//K)$ *by replacing* (ii) *and* (iii) *of the above definition of* $(1,\infty,0)$*-atom* a *on* G*, respectively, with*

$$(ii)_\epsilon \quad \|a\|_\infty \leq |R(x_0,r)|^{-1}(1+r)^{-\epsilon} \tag{16}$$

and

$$(iii)_+ \quad \int_0^\infty a(x)\Delta(x)dx = 0 \quad \text{if } r \leq 1.$$

Moreover, we define the small Hardy space $h^1_{\infty,0}(G//K)$ *on* G *by restricting* $(1,\infty,0)$*-atoms in the definition of* $H^1_{\infty,0}(G//K)$ *to ones with radius* ≤ 1.

Clearly, for $\epsilon \geq 0$,

$$h^1_{\infty,0}(G//K) \subset H^{1,\epsilon}_{\infty,0}(G//K) \subset H^1_{\infty,0}(G//K) \subset H^{1,+}_{\infty,0}(G//K).$$

Let χ_1 denote the characteristic function of $B(1) = R(0,1)$ and set

$$\theta(g) = |B(1)|^{-1}\chi_1(g), \quad g \in G.$$

Moreover, for each (not necessary K-bi-invariant) function f on G, we define a K-bi-invariant function $f^\flat_x$, $x \in G$, as

$$f^\flat_x(g) = \int_K \int_K f(x^{-1}kgk')dkdk', \quad g \in G.$$

Then the difference between $h^1_{\infty,0}(G//K)$ and $H^{1,+}_{\infty,0}(G//K)$ is given as follows.

Proposition 5.2. *For* $f \in H^{1,+}_{\infty,0}(G//K)$ *there exist* $f_0 \in h^1_{\infty,0}(G//K)$ *and* $x_i \in G$, $\lambda_i \in \mathbf{R}$ *such that*

$$f = f_0 + \sum_i \lambda_i \theta^\flat_{x_i},$$

where $\|f_0\|_{H^{1,\mathrm{loc}}_{\infty,0}(G)}$ *and* $\sum_i |\lambda_i|$ *are respectively bounded by* $\|f\|_{H^{1,+}_{\infty,0}(G)}$.

As in the Euclidean case, the truncated maximal operator $M_\phi^{\rm loc}$ on G defined by

$$(M_\phi^{\rm loc} f)(g) = \sup_{0<t<1} |(f * \phi_t)(g)|, \quad g \in G,$$

is bounded from $H^{1,+}_{\infty,0}(G//K)$ to $L^1(G//K)$ (see [**7**]). As for M_ϕ, we see from [**7**, Theorem 5.3] that M_ϕ is bounded from $H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ to $L^1(G//K)$:

Proposition 5.3. *Let $M \geq 2$. M_ϕ is bounded from $H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ to $L^1(G//K)$, that is, there exists a constant $c > 0$ such that*

$$\|M_\phi f\|_1 \leq c \left(\|f\|_{H^{1,+}_{\infty,0}(G)} + \|W_+(f)\|_{H^1(\mathbf{R})} \right)$$

for all $f \in H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ and thus,

$$H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R})) \subset H^1_\phi(G//K).$$

Let a be a $(1,\infty,0,1)$-atom on G supported on $R(x_0, r)$. (15) and (16) imply that $\|a\|_\infty \leq |R(x_0,r)|^{-1}(1+r)^{-1}$ and $\int_G a(g)dg = 0$. Then we can deduce that $A = W_+(a)$ is supported on $R(x_0, r)$ and

$$\int_{-\infty}^{\infty} A(x)dx = A^{\sim}(0) = \hat{a}(i\rho) = \int_G a(g)dg = 0.$$

Moreover, we see that $|A(x)| \leq ce^{2\rho x}{\rm th}(x_0+r)^{2s_\alpha}\|a\|_\infty$ (see (6) and cf. [**8**, Lemma 3.4]).

Case I: $x_0 - r \geq 1$. Since A is supported on $R(x_0, r)$ and

$$|R(x_0,r)| \sim \int_{x_0-r}^{x_0+r} e^{2\rho x}dx \sim e^{2\rho x_0}{\rm sh}r,$$

it follows that $|A(x)| \leq ce^{2\rho(x_0+r)}(e^{2\rho x_0}{\rm sh}r)^{-1}(1+r)^{-1} \leq cr^{-1}$.

Case II: $x_0 - r < 1$ and $r \geq 1$. Since $x_0 + r \geq 1$,

$$|R(x_0,r)| \geq c\int_1^{x_0+r} e^{2\rho x}dx \sim e^{2\rho(x_0+r)}.$$

Therefore, as in Case I, we have $\|A\|_\infty \leq cr^{-1}$.

Case III: $x_0 - r < 1$, $r < 1$ and $x_0 > 2r$. Since $x_0 > 2r$, it follows that $x_0 + r \geq 3$ and thus

$$|R(x_0,r)| \sim \int_{x_0-r}^{x_0+r} x^{2s_\alpha}dx \leq c(x_0-r)^{s_\alpha}r.$$

Since $(x_0+r)/(x_0-r) \leq 3$, we have $|A(x)| \leq c\mathrm{th}(x_0+r)^{s_\alpha}((x_0-r)^{s_\alpha}r)^{-1} \leq cr^{-1}$.

Case IV: $x_0 - r < 1$, $r < 1$ and $x_0 \leq 2r$. Since $x_0 + r \leq 3r < 3$ and $|R(x_0,r)| \geq |B(r)| \sim |B(3r)|$, we may suppose that a is a centered atom supported on $B(3r)$. Then $|A(x)| \leq c(\mathrm{th}3r)^{s_\alpha}|B(3r)|^{-1} \leq cr^{-1}$.

These four cases imply that cA is a $(1,\infty,0)$-atom on $\mathbf{R}$ and c is independent of a. Therefore, we obtain the following.

Theorem 5.4. *Let $M \geq 2$. Then*

$$H^{1,1}_{\infty,0}(G//K) \subset H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R})).$$

Especially, M_ϕ is bounded from $H^{1,1}_{\infty,0}(G//K)$ to $L^1(G//K)$, that is, there exists a constant $c > 0$ such that

$$\|M_\phi f\|_1 \leq c\|f\|_{H^{1,1}_{\infty,0}(G)}$$

for all $f \in H^{1,1}_{\infty,0}(G//K)$.

6 Characterization of $H^1_\phi(G//K)$

We shall prove that the inclusion in Proposition 5.3 is the equality. First we shall introduce a subspace of $H^1(\mathbf{R})$. Let s_α be as above,

$$d_\alpha(x_0,r) = \int_{\max\{0,|x_0|-r\}}^{|x_0|+r} (\mathrm{th}x)^{s_\alpha}\,dx,$$

and we define $H^{1,+}_{\infty,0}(\mathbf{R})_\alpha$ as the space of all $F = \sum_i \lambda_i A_i$ such that $\sum_i |\lambda_i| < \infty$ and each A_i satisfies

$$\begin{array}{ll} (i) & \mathrm{supp}(A_i) \subset R(x_i,r_i) \\ (ii) & \|W^{\mathbf{R}}_{-s_\alpha}(A_i)\|_\infty \leq d_\alpha(x_i,r_i)^{-1} \\ (iii) & \int_{-\infty}^{\infty} A_i(x)dx = 0 \text{ if } r_i < 1. \end{array} \tag{17}$$

Definition 6.1. *We define*

$$W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha) = \{f \in L^1_{\mathrm{loc}}(G//K)\ ;\ W_+(f) \in H^{1,+}_{\infty,0}(\mathbf{R})_\alpha\}.$$

For $f \in W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$ we can give a $(1,\infty,+)$-atomic decomposition f on G. Let $F = W_+(f) = \sum_i \lambda_i A_i$ be the decomposition of F given by (17). Here we may suppose that $r_i \leq 1$. Actually, when $r_i > 1$, we decompose the support of A_i by using a smooth decomposition of 1, where each piece is supported in the interval with radius

≤ 1 and thus, we have $A_i = \sum_j A_{ij}$ and each A_{ij} satisfies (17) with radius ≤ 1. Moreover, we may suppose that $x_i = 0$ with $r_i \leq 3$ or $|x_i| > 2r_i$. Hence, we can rearrange the decomposition of F as

$$F = \sum_i \lambda_i A_i + \sum_j \mu_j B_j + \sum_k \gamma_k E_k,$$

where each A_i satisfies (i), (ii) with $x_i = 0$, $r_i \leq 3$, $\int A_i(x)dx = 0$; each B_j satisfies (i) to (iii) with $|x_j| \geq 2r_j$, $r_j < 1$; each E_k satisfies (i), (ii) with $|x_k| \geq 2r_k$, $r_k \geq 1$, and moreover, $\sum_i |\lambda_i| + \sum_j |\mu_j| + \sum_k |\gamma_k| < \infty$. Since F is W_+-smooth, finally, we have

$$f = \sum_i \lambda_i a_i + \sum_j \mu_j b_j + \sum_k \gamma e_k, \tag{18}$$

where $a_i = W_-(A_i)$, $b_j = W_-(B_j)$ and $e_k = W_-(E_k)$. Here it is easy to see that each a_i, b_j, e_k have the same supports of A_i, B_j, E_k respectively.

Now we apply fractional calculus in [**8**] to estimate each a_i, b_j, e_k. For simplicity, we abbreviate the suffices i, j, k and denote the supports of a, b, e by $R(x_0, r)$. Without loss of generality, we may suppose that $x_0 \geq 0$.

As for e, since e is supported on $R(x_0, 1)$ and $x_0 \geq 2$, it follows that $x_0 - 1 \geq 1$ and thus, $d_\alpha(x_0, 1) \sim 1$. Thereby, (ii) and [**8**, Lemma 3.3] imply that on the support of e

$$|e(x)| \leq c(\text{th}x)^{-(\alpha+1/2)} e^{-2\rho x} \leq ce^{-2\rho x} \leq c|R(x,1)|^{-1}.$$

This means that $c^{-1}e$ is a $(1, \infty, +)$-atom on G.

As for b, we recall that $x_0 = 0$ or $x_0 > 2r$.

Case I. $x_0 - r \geq 1$: Since $x_0 - r \geq 1$, $d_\alpha(x_0, r) \sim r$. Thereby, (ii) and [**8**, Lemma 3.3] imply that on the support of b

$$|b(x)| \leq c(\text{th}x)^{-(\alpha+1/2)} e^{-2\rho x} r^{-1} \leq ce^{-2\rho x} r^{-1} \leq c|R(x,r)|^{-1}.$$

This means that $c^{-1}b$ is a $(1, \infty, 0)$-atom on G.

Case II. $x_0 - r < 1$: Since $r < 1$ and $x_0 > 2r$, it follows that $x_0 < r + 1 < 2$, $x_0 - r > x_0/2$, and $x_0 + r < 3x_0/2 < 3$. Therefore, $d_\alpha(x_0, r) \leq c(x_0 - r)^{s_\alpha} r$ and thus, on the support of b

$$|b(x)| \leq c(\text{th}x)^{-(\alpha+1/2)} e^{-2\rho x} r^{-1} (x_0 - r)^{-s_\alpha} \leq c(x_0 - r)^{-(2\alpha+1)} r^{-1}.$$

Since $(x_0 + r)/(x_0 - r) \leq 3$, it follows that

$$|R(x_0, r)| \leq c(x_0 + r)^{2\alpha+1} r \leq c(x_0 - r)^{2\alpha+1} r.$$

Therefore, $|b(x)| \le c|R(x_0,r)|^{-1}$ on the support. This means that $c^{-1}b$ is a $(1,\infty,0)$-atom on G.

As for a, since $x_0 = 0$ and $r < 1$, it follows that $d_\alpha(0,r) \sim r^{s_\alpha+1}$ and

$$|a(x)| \le c(\mathrm{th}x)^{-(\alpha+1/2)} e^{-2\rho x} r^{-1} r^{-(s_\alpha+1)} \le c\Delta(x)^{-1} r^{-1}. \qquad (19)$$

We put

$$a_+(x) = c\Delta(x)^{-1} r^{-1} \chi_{[0,r]}(x), \ x > 0.$$

Clearly, $|a(x)| \le a_+(x)$ and a_+ is a non-increasing function on $\mathbf{R}_+$ with finite L^1-norm:

$$\|a_+\|_{L^1(\Delta)} = \int_0^\infty a_+(x)\Delta(x)dx = c_0.$$

Since a is supported on $B(r)$ and $\int_G a(g)dg = \int_{-\infty}^\infty A(x)dx = 0$, it follows that $|B(s)|^{-1} \int_s^\infty a(x)\Delta(x)dx$ is also supported on $B(r)$ and

$$\frac{1}{|B(s)|}\int_s^\infty a(x)\Delta(x)dx = \frac{1}{|B(s)|}\int_0^s a(x)\Delta(x)dx \le c\Delta(s)^{-1}r^{-1}.$$

Here we used (19) and $|B(s)| \sim \Delta(s)s$ if $s \le r \le 1$ (see (1)). Hence,

$$\frac{1}{|B(s)|}\int_s^\infty a(x)\Delta(x)dx \le a_+(s). \qquad (20)$$

This means that ca_+ is an L^1 non-increasing denominator of a satisfying (20). Then [**5**, Theorem 4.5] yields that a has a centered $(1,\infty,0)$-atomic decomposition $a = \sum_j \gamma_j a_j$ on G such that $\sum_j |\gamma_j| \le c\|a_+\|_{L^1(\Delta)} \le cc_0$. Especially, $a \in H^1_{\infty,0}(G//K)$ and $\|a\|_{H^1_{\infty,0}(G)} \le cc_0$.

These three cases imply that all a_i, b_j, e_k in (18), and thus f belongs to $H^{1,+}_{\infty,0}(G//K)$:

Proposition 6.2. *Functions in $W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$ have $(1,\infty,+)$-atomic decompositions, that is, $W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha) \subset H^{1,+}_{\infty,0}(G//K)$.*

We shall prove that $H^1(G//K) \subset W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$ in the case of $s_\alpha = \alpha+1/2$ is integer. Let $f \in H^1(G//K)$ and put $F = W_+(f)$. Then it follows from Theorem 4.4 that $\|M^{\mathbf{R}}_\phi \circ W^{\mathbf{R}}_{-s_\alpha}(F)(x)(\mathrm{th}x)^{s_\alpha}\|_{L^1(\mathbf{R})} < \infty$. We note that $(\mathrm{th}x)^{s_\alpha}$ is an A_1-weight. Therefore, $W^{\mathbf{R}}_{-s_\alpha}(F)$ has a $(1,\infty,s_\alpha)$-atomic decomposition with respect to this weight:

$$W^{\mathbf{R}}_{-s_\alpha}(F) = \sum_i \lambda_i B_i,$$

where B_i is supported on $R(x_i, r_i)$, $\int_{-\infty}^{\infty} B_i(x)x^k dx = 0, \leq k \leq s_\alpha$, $\|B_i\|_\infty \leq d_\alpha(x_i, r_i)^{-1}$, and $\sum_i |\lambda_i| < \infty$. We set

$$F = \sum_i \lambda_i W^{\mathbf{R}}_{s_\alpha}(B_i) = \sum_i \lambda_i A_i.$$

Since s_α is integer and each B_i satisfies the s_α-th moment condition, it follows that A_i is supported on $R(x_i, r_i)$ and $\int_{-\infty}^{\infty} A_i(x)dx = 0$. Moreover, $\|W^{\mathbf{R}}_{-s_\alpha}(A_i)\|_\infty = \|B_i\|_\infty \leq d_\alpha(x_i, r_i)^{-1}$. Therefore, A_i satisfies (17) and thus, $F \in H^{1,+}_{\infty,0}(\mathbf{R})_\alpha$.

Furthermore, we can drop the assumption that s_α is integer and we have $H^1(G//K) \subset W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$ in general. This means that $H^1_\phi(G//K) \subset H^{1,+}_{\infty,0}(G//K)$ by Proposition 6.2. Finally, as a refinement of proposition 5.3, we have the following main theorem.

Theorem 6.3. *Let notations be as above. Then*

$$H^1_\phi(G//K) = H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R})).$$

As an easy consequence of the previous arguments, we have

Theorem 6.4. *Let $\epsilon \geq 0$. Then $H^{1,\epsilon}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ is dense in $W_-(H^1(\mathbf{R}))$. Especially, $H^1_\phi(G//K)$ is dense in $W_-(H^1(\mathbf{R}))$.*

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Generalized Hardy's theorem for the Jacobi transform

R. Daher and T. Kawazoe

Abstract

The classical Hardy theorem on **R** was generalized by Miyachi [**?**] and Bonami, Demange, and Jaming [**?**]. In this paper we show that Miyachi's theorem and Bonami-Demange-Jaming' one can be reformulated for the Jacobi transform in terms of the heat kernel.

1. Introduction. For $f \in L^1(\mathbf{R})$ we define the Fourier transform $\tilde{f}(\lambda)$, $\lambda \in \mathbf{R}$, of f by

$$\tilde{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx.$$

Let us take two positive numbers a, b which satisfy the relation $ab = 1/4$. Miyachi's theorem in [**?**] states that if $f \in L^1(\mathbf{R})$ satisfies

$$e^{ax^2}f(x) \in L^1(\mathbf{R}) + L^\infty(\mathbf{R})$$

and

$$\int_{-\infty}^{+\infty} \log^+ \frac{|\tilde{f}(\lambda)e^{b\lambda^2}|}{C} d\lambda < \infty$$

for some $C > 0$, then f is a constant multiple of e^{-ax^2}, where $L^1(\mathbf{R}) + L^\infty(\mathbf{R})$ is the set of functions of the form $f = f_1 + f_2$, $f_1 \in L^1(\mathbf{R})$, $f_2 \in L^\infty(\mathbf{R})$, and $\log^+ x = \log x$ if $x > 1$ and $\log^+ x = 0$ if $x \leq 1$. On the other hand, one dimendional case of Bonami-Demange-Jaming's theorem in [**?**] states that $f \in L^2(\mathbf{R})$ satisfies

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(x)||\tilde{f}(y)|}{(1+|x|+|y|)^N} e^{|xy|} dxdy < \infty$$

for some $N \geq 0$ if and only if f is written as $f(x) = P(x)e^{-ax^2}$, where P is a polynomial of degree $< (N-1)/2$. Both theorems are generalizations of the classical Hardy theorem and the Cowling-Price theorem which is an L^p version of the classical Hardy one (see [?] and [?]).

Recently, Hardy's theorem on Lie groups has been investigated by various people. As remarked by V.S. Varadarajan some years ago, Hardy's theorem can be written in terms of the heat kernel of the Laplacian on the groups. Then, considerable attention has been paid to discover a connection between the heat kernel and analogues of Hardy's theorem and Cowling-Price's theorem on Lie groups. For this subject we refer to [?], [?], [?], and [?]. Moreover, N.B. Andersen [?] and the second auther of this article and J. Liu [?] obtained independently an analogue of Hardy's theorem and its L^p version for the Jacobi transform. The aim of this article is to show that the above two theorems can be restated for the Jacobi transform in terms of the heat kernel.

2. Notations. We collect relevant material from the harmonic analysis associated with the Jacobi transform. General references for this section are [?], [?] and [?]. For $\alpha, \beta, \lambda \in \mathbf{C}$ and $x \in \mathbf{R}_+ = [0, \infty)$, the Jacobi function $\phi_\lambda(x)$ of order (α, β), $\alpha \neq -1, -2, \cdots$, is the unique solution on $\mathbf{R}_+$ of the differential equation:

$$L_{\alpha,\beta} u = -(\lambda^2 + \rho^2) u, \quad u(0) = 1, \text{ and } u'(0) = 0,$$

where $\rho = \alpha + \beta + 1$ and

$$L_{\alpha,\beta} = \frac{d^2}{dx^2} + ((2\alpha + 1)\coth x + (2\beta + 1)\tanh x)\frac{d}{dx}.$$

In the following we suppose that $\alpha \geq \beta \geq -1/2$. Then $\phi_\lambda(x)$ is estimated as

$$|\phi_\lambda(x)| \leq \begin{cases} 1 & \text{if } |\Im\lambda| \leq \rho, \\ e^{(|\Im\lambda| - \rho)x} & \text{if } |\Im\lambda| > \rho, \\ \phi_{i\Im\lambda}(x) & \end{cases} \tag{1}$$

for all $x \in \mathbf{R}_+$ (see [?, Lemma 11]). For a compactly supported C^∞ function f on $\mathbf{R}_+$ the Jacobi transform $\hat{f}(\lambda)$, $\lambda \in \mathbf{C}$, of f is given by

$$\hat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x)\Delta_{\alpha,\beta}(x)dx, \tag{2}$$

where $\Delta_{\alpha,\beta}(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$. We recall that for all $\lambda \in \mathbf{C}$,

$$(L_{\alpha,\beta} f)^\wedge(\lambda) = -(\lambda^2 + \rho^2)\hat{f}(\lambda). \tag{3}$$

The Abel transform $F_f(x)$, $x \in \mathbf{R}_+$, of f is given as

$$F_f(x) = \int_x^\infty f(s)A(x,s)ds, \quad x \geq 0, \tag{4}$$

where $A(x,s)$ is positive, even with respect to x and moreover, it satisfies

$$\Delta_{\alpha,\beta}(s)\phi_\lambda(s) = c\int_0^s \cos(\lambda x)A(x,s)dx, \quad s \geq 0. \tag{5}$$

We refer to [**?**, (2.16), (3.5)] for the explicit form of $A(x,s)$. We recall that

$$\hat{f}(\lambda) = \tilde{F}_f(\lambda), \quad \lambda \in \mathbf{C}, \tag{6}$$

where f and F_f are regarded as even functions on $\mathbf{R}$ and the right hand side $\tilde{F}_f$ denotes the Euclidean Fourier transform of F_f. We note that the Jacobi transform is extended to functions for which the right hand side of (2) is well-defined. For example, if $f \in L^1(\mathbf{R}_+, \Delta_{\alpha,\beta}(x)dx)$, then $\hat{f}(\lambda)$, $\lambda \in \mathbf{R}$, is well-defined and it has a holomorphic extension on the tube domain $|\Im\lambda| < \rho$ (see (1)). Also the relations (3) and (6) hold for $|\Im\lambda| < \rho$. Moreover, the map $f \to \hat{f}$ extends to an isometry between $L^2(\mathbf{R}_+, \Delta_{\alpha,\beta}(x)dx)$ and $L^2(\mathbf{R}_+, |C_{\alpha,\beta}(\lambda)|^{-2}d\lambda)$, where $C_{\alpha,\beta}(\lambda)$ denotes the Harish-Chandra C-function (cf. [**?**, (2.6)]).

For $t > 0$ let $h_t(x)$, $x \in \mathbf{R}$, denote the heat kernel associated to $L_{\alpha,\beta}$, that is, the even C^∞ function on $\mathbf{R}$ such that

$$\hat{h}_t(\lambda) = e^{-t(\lambda^2+\rho^2)}, \quad \lambda \in \mathbf{R}. \tag{7}$$

We recall that

$$h_t(x) \sim t^{-\alpha-1}e^{\rho^2 t}e^{-\rho x - x^2/4t}(1+t+x)^{\alpha-1/2}(1+x), \quad x \geq 0, \tag{8}$$

where "$\sim$" means that the ratio of the left side and the right side is bounded below and above by positive constants (see [**?**, Corollary 1], cf. [**?**, Theorem 3.1]). Hence (8) and (1) imply that $\hat{h}_t(\lambda)$ is entire and (7) holds for $\lambda \in \mathbf{C}$.

3. Miyachi's theorem. We shall obtain an extension of Miyachi's theorem for the Jacobi transform. We put

$$d_\alpha x = (\tanh x)^{2\alpha+1}(1+x)^{\alpha+1/2}dx \text{ on } \mathbf{R}_+$$

and

$$L^\infty(\mathbf{R}_+) + L^1(\mathbf{R}_+, d_\alpha x) = \{f_1 + f_2\ ;\ f_1 \in L^\infty(\mathbf{R}_+), f_2 \in L^1(\mathbf{R}_+, d_\alpha x)\}.$$

Theorem 3.1. *Let us take positive constants a, b which satisfy $ab = 1/4$. Suppose f is a measurable function on $\mathbf{R}_+$ satisfying*

$$(A) \quad : \quad f(x)h_{1/4a}^{-1}(x) \in L^\infty(\mathbf{R}_+) + L^1(\mathbf{R}_+, d_\alpha x)$$

$$(B) \quad : \quad \int_{-\infty}^{\infty} \log^+ \frac{|\hat{f}(\lambda)e^{b\lambda^2}|}{C} d\lambda < \infty \text{ for some } 0 < C < \infty.$$

Then f is a constant multiple of $h_{1/4a}$.

Proof. The first condition (A) implies that $fh_{1/4a}^{-1} = u+v$, where $u \in L^\infty(\mathbf{R}_+)$ and $v \in L^1(\mathbf{R}_+, d_\alpha x)$ and hence, $f = h_{1/4a}u + h_{1/4a}v$. As for the first term, it follows from (1) that for all $\lambda = \xi + i\eta \in \mathbf{C}$,

$$\begin{aligned} |(h_{1/4a}u)^\wedge(\lambda)| &\leq \|u\|_\infty \int_0^\infty h_{1/4a}(x)\phi_{i\eta}(x)\Delta_{\alpha,\beta}(x)dx \\ &= c\hat{h}_{1/4a}(i\eta) = ce^{b\eta^2}. \end{aligned}$$

As for the second term, it follows from (1) and (7) that, if $|\eta| > \rho$, then

$$\begin{aligned} &|(h_{1/4a}v)^\wedge(\lambda)| \\ \leq\ & c\int_0^\infty |v(x)|e^{-\rho x - ax^2}(1+x)^{\alpha-1/2}(1+x)e^{(|\eta|-\rho)x}\Delta_{\alpha,\beta}(x)dx \\ \leq\ & c\int_0^\infty |v(x)|(\tanh x)^{2\alpha+1}(1+x)^{\alpha+1/2}e^{-a(x-|\eta|/2a)^2}dx \cdot e^{\eta^2/4a} \\ \leq\ & c\|v\|_{L^1(\mathbf{R}_+, d_\alpha x)}e^{b\eta^2} \end{aligned}$$

and, if $|\eta| \leq \rho$, since $e^{-ax^2} \leq ce^{-\rho x}$ for $x \geq 0$, it follows that

$$|(h_{1/4a}v)^\wedge(\lambda)| \leq c\|v\|_{L^1(\mathbf{R}_+, d_\alpha x)} \leq ce^{b\eta^2}.$$

Hence, $\hat{f}(\lambda)$ is entire and it satisfies $|\hat{f}(\lambda)| \leq ce^{b\eta^2}$ for all $\lambda \in \mathbf{C}$ and (B). We here recall the lemma which is used in the proof of Miyachi's theorem (see [**?**, Lemma 4]):

Lemma 3.2. *Suppose $F(\lambda)$ is an entire function and there exist constant $A, B > 0$ such that*

$$|F(\lambda)| \leq Ae^{B(\Re\lambda)^2} \quad \text{and} \quad \int_{-\infty}^{\infty} \log^+ |F(\lambda)|d\lambda < \infty.$$

Then F is a constant function.

Therefore, applying this lemma to $\hat{f}(\lambda)e^{-b\lambda^2}/C$, we see that $\hat{f}(\lambda) = ce^{-b\lambda^2}$ and thus, $f(x) = ch_{1/4a}(x)$. ■

4. Bonami-Demange-Jaming's theorem. We shall obtain an extension of Bonami-Demange-Jaming's theorem for the Jacobi transform.

Theorem 4.1 *Let us take a function* $f \in L^2(\mathbf{R}_+, \Delta_{\alpha,\beta}(x)dx)$ *and a non-negative integer* N. *Then the inequality*

$$(A) \quad \int_0^\infty \int_0^\infty \frac{|f(x)||\hat{f}(\lambda)|}{(1+\lambda)^N}\phi_{i\lambda}(x)\Delta_{\alpha,\beta}(x)dxd\lambda < \infty$$

holds if and only if f *can be written as*

$$(B) \quad f(x) = P(L_{\alpha,\beta})h_a(x),$$

where $a > 0$ *and* P *is a polynomial of* $\deg P < (N-1)/4$.

Proof. First we shall prove that (A) implies (B) by reducing the case to the original Bonami-Demange-Jaming theorem on $\mathbf{R}$. Since $\hat{f}(\lambda) = \tilde{F}_f(\lambda)$ (see (6)), it follows from (4), (5) and (A) that

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{|F_f(x)||\tilde{F}_f(\lambda)|}{(1+x+\lambda)^N}e^{x\lambda}dxd\lambda \\
\le\ & \int_0^\infty \int_0^\infty |f(s)||\hat{f}(\lambda)|\Big(\int_0^s \frac{A(x,s)e^{x\lambda}}{(1+x+\lambda)^N}dx\Big)dsd\lambda \\
\le\ & 2\int_0^\infty \int_0^\infty \frac{|f(s)||\hat{f}(\lambda)|}{(1+\lambda)^N}\Big(\int_0^s A(x,s)\cos(ix\lambda)dx\Big)dsd\lambda \\
=\ & 2c\int_0^\infty \int_0^\infty \frac{|f(s)||\hat{f}(\lambda)|}{(1+\lambda)^N}\phi_{i\lambda}(s)\Delta_{\alpha,\beta}(s)dsd\lambda < \infty.
\end{aligned}$$

As in the first step of the proof of Proposition 2.2 in [**?**], F_f belongs to $L^1(\mathbf{R}_+)$. Hence $\hat{f} = \tilde{F}_f$ is bounded on $\mathbf{R}$. Since $\tilde{F}_f \in L^2(\mathbf{R}_+, |C_{\alpha,\beta}(\lambda)|^{-2}d\lambda)$ and $|C_{\alpha,\beta}(\lambda)|^{-2}$ is polynomial growth of order $\alpha + 1/2$, it easily follows that $\tilde{F}_f \in L^2(\mathbf{R})$ and thus, $F_f \in L^2(\mathbf{R})$ as an even function on $\mathbf{R}$. Then F_f satisfies the condition of Theorem 1.1 in [**?**], which yields that

$$\tilde{F}_f(\lambda) = Q(\lambda)e^{-a\lambda^2},$$

where $a > 0$ and Q is an even polynomial of degree $< (N-1)/2$. Since Q is even, this relation can be rewritten as

$$\tilde{F}_f(\lambda) = P(-(\lambda^2+\rho^2))e^{-a(\lambda^2+\rho^2)},$$

where P is a polynomial of $\deg P < (N-1)/4$. Since the map $f \to \tilde{F}_f$ is bijectiove on $L^2(\mathbf{R}_+, \Delta_{\alpha,\beta}(x)dx)$, it easily follows from (3) that $f(x) = P(L_{\alpha,\beta})h_a(x)$.

Next we suppose that $f(x) = P(L_{\alpha,\beta})h_a(x)$, where $a > 0$ and P is a polynomial of $\deg P < (N-1)/4$. Then, $\hat{f}(\lambda) = \tilde{F}_f(\lambda)$ is of the form $Q(\lambda)e^{-a\lambda^2}$, where Q is an even polynomial of degree $< (N-1)/2$. We note that, if $f \geq 0$, then

$$\int_0^\infty \int_0^\infty \frac{|f(x)||\hat{f}(\lambda)|}{(1+\lambda)^N}\phi_{i\lambda}(x)\Delta_{\alpha,\beta}(x)dxd\lambda$$
$$= \int_0^\infty \frac{\hat{f}(i\lambda)|\hat{f}(\lambda)|}{(1+\lambda)^N}d\lambda = \int_0^\infty \frac{Q(i\lambda)|Q(\lambda)|}{(1+\lambda)^N}d\lambda < \infty.$$

We recall that for $x \geq 0$, $f(x) = P(L_{\alpha,\beta})h_a(x) \sim U(x)h_a$, where $U(x)$ is a polynomial of degree $d = 2\deg P$, because $h_a = h_a^{\alpha,\beta}$ is defined by $h_a^{\alpha,\beta} \sim W^2_{-\beta-1/2}W^1_{-\alpha+\beta}(e^{-x^2/4a})$ as a function of x (cf. [?, §3]) and thus, $dh_a^{\alpha,\beta}/dx = \sinh(2x)W^2_{-1}(h_a^{\alpha,\beta}) \sim \sinh(2x)h_a^{\beta+1,\alpha+1} \sim xh_a^{\alpha,\beta}$ (see (8)). Here we may suppose that the coefficient of x^d is positive. Since there exists a positive constant c such that $h_a(x) \geq c(1+x)^{\alpha+1/2}e^{-x^2/4a-\rho x} \geq ce^{-x^2/4a-\rho x}$ for $x \geq 0$ (see (8)), there exists a positive constant A such that $f(x)+Ah_a(x) \geq 0$ for $x \geq 0$. Hence, $|f(x)| = |f(x)+Ah_a(x)-Ah_a(x)| \leq f(x)+2Ah_a(x)$. Then, replacing $|f(x)|$ with $f(x)+2Ah_a(x) \geq 0$, that is, $Q(i\lambda)$ with $Q(i\lambda)+2Ae^{-a\rho^2}$ in the above calculation, we have the desired result. ■

As an easy consequence of Theorem 4.1, we can deduce the Beurling theorem for the Jacobi transform.

Theorem 4.2. *Suppose that* $f \in L^1(\mathbf{R}_+, \Delta_{\alpha,\beta}(x)dx)$ *satisifes*

$$\int_0^\infty \int_0^\infty |f(x)||\hat{f}(\lambda)|\phi_{i\lambda}(x)\Delta_{\alpha,\beta}(x)dxd\lambda < \infty.$$

Then $f = 0$.

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Added in proof. After we have accomplished this paper, we were informed that R. P. Sarkar and J. Sengupta also investigated a generalization of Beurling's theorem in the paper titled Beurling's theorem for Riemaniann symmetric spaces of noncompact type.

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Uncertainty principle for the Fourier-Jacobi transform

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Abstract

We obtain a uncertainty principle for the Fourier-Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$. When $|\beta| \leq \alpha + 1$, as in the Euclidean case, an analogues of the uncertainty principle holds, because there is no discrete part in the Parseval formula. Moreover, we can obtain a new type of a uncertainty inequality: the L^2-norm of $\hat{f}_{\alpha,\beta}(\lambda)\lambda$ is estimated below by the L^2-norm of $(\alpha+\beta+1)f(x)(\cosh x)^{-1}$. Otherwise, the discrete part of f appears in the Parseval formula and it influences the uncertainty principle.

1. Notation. Let $\alpha, \beta \in \mathbb{C}$, $\Re\alpha > -1$ and $\rho = \alpha + \beta + 1$. For $\lambda \in \mathbb{C}$, let $\phi_\lambda(x)$ denote the Jacobi function of the first kind, that is, the unique solution of $(L + \lambda^2 + \rho^2)f = 0$ satisfying $f(0) = 1$ and $f'(0) = 0$, where

$$L = \Delta(x)^{-1}\frac{d}{dx}\Big(\Delta(x)\frac{d}{dx}\Big) \tag{1}$$

and $\Delta(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$. For $\lambda \neq -i, -2i, -3i, \ldots$, let $\Phi_\lambda(x)$ denote the Jacobi function of the second kind which satisfies

$$2\pi^{1/2}\Gamma(\alpha+1)^{-1}\phi_\lambda(x) = C(\lambda)\Phi_\lambda(x) + C(-\lambda)\Phi_{-\lambda}(x), \tag{2}$$

where $C(\lambda)$ is Harish-Chandra's C-function. Then the following estimates are well-known (cf. [2, 3]): For $x \geq 0$ and $\lambda \in \mathbb{C}$ with $|\Im\lambda| \leq \rho$

$$|\phi_\lambda(x)| \leq 1 \tag{3}$$

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and for $\delta > 0$ there exist a positive constant K_δ such that for $x \geq \delta$, $\lambda \in \mathbb{R}$

$$|\Phi_\lambda(x)| \leq K_\delta e^{-\rho x}, \tag{4}$$

and there exists a positive constant K such that for $\lambda \in \mathbb{R}$

$$|C(-\lambda)|^{-1} \leq K(1+|\lambda|)^{\alpha+1/2}. \tag{5}$$

Let $f \in C^\infty_{c,e}(\mathbb{R})$, the space of all even C^∞ functions on $\mathbb{R}$ with compact support. Then the Fourier-Jacobi transform $\hat{f}(\lambda)$ is defined as

$$\hat{f}(\lambda) = \frac{\pi^{1/2}}{\Gamma(\alpha+1)} \int_0^\infty f(x)\phi_\lambda(x)\Delta(x)dx. \tag{6}$$

This transform $f \rightarrow \hat{f}$ satisfies analogous properties of the classical cosine Fourier transform; the inversion formula, the Paley-Wiener theorem, and the Plancherel formula are obtained in [2, 3]. For convenience we suppose that $\alpha, \beta \in \mathbb{R}$ in the following. We define

$$D_{\alpha,\beta} = \{i(|\beta| - \alpha - 1 - 2m); m = 0, 1, 2, \cdots, |\beta| - \alpha - 1 - 2m > 0\}.$$

Then the inversion formula is given as follows. For $f \in C^\infty_{c,e}(\mathbb{R})$,

$$f(x) = \frac{1}{2\pi} \int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda + \sum_{\mu \in D_{\alpha,\beta}} \hat{f}(\mu)\phi_\mu(x)d(\mu),$$

where $d(\mu) = -i\mathrm{Res}_{\lambda=\mu}(C(\lambda)C(-\lambda)^{-1})$. We denote this decomposition as

$$f = f_P + {}^\circ f \tag{7}$$

and we call f_P and ${}^\circ f$ the principal and discrete part of f respectively. We here recall that for each $\mu \in D_{\alpha,\beta}$, there exists a positive constant $K(\mu)$ such that

$$|\phi_\mu(x)| \leq K(\mu)e^{-(\rho+|\mu|)x}. \tag{8}$$

We denote by $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\mu\})$ a function on $\mathbb{R}_+ \cup D_{\alpha,\beta}$ defined by

$$\boldsymbol{F}(\nu) = \begin{cases} F(\lambda) & \text{if } \nu = \lambda \in \mathbb{R}_+ \\ a_\mu & \text{if } \nu = \mu \in D_{\alpha,\beta}. \end{cases}$$

We put $\overline{\boldsymbol{F}}(\nu) = (\overline{F(\lambda)}, \{\overline{a_\mu}\})$ and define the product of $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\mu\})$ and $\boldsymbol{G}(\nu) = (G(\lambda), \{b_\mu\})$ as

$$(\boldsymbol{F}\boldsymbol{G})(\nu) = (F(\lambda)G(\lambda), \{a_\mu b_\mu\}).$$

For a function $h(\lambda)$ on $\mathbb{C}$, we define $\boldsymbol{F}(\nu)h(\nu)$ by regarding $h(\nu)$ as a function on $D_{\alpha,\beta}$. Let $d\nu$ denote the measure on $\mathbb{R}_+ \cup D_{\alpha,\beta}$ defined by

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} \boldsymbol{F}(\nu)d\nu = \frac{1}{2\pi}\int_0^\infty F(\lambda)|C(\lambda)|^{-2}d\lambda + \sum_{\mu \in D_{\alpha,\beta}} a_\mu d(\mu).$$

For $f \in C_{c,e}^\infty(\mathbb{R})$, we put

$$\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{\hat{f}(\mu)\}).$$

Then the Parseval formula for the Fourier-Jacobi transform on $C_{c,e}^\infty(\mathbb{R})$ can be stated as follows (see [3, Theorem 2.4]):

$$\int_0^\infty f(x)\overline{g(x)}\Delta(x)dx = \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} \hat{\boldsymbol{f}}(\nu)\overline{\hat{\boldsymbol{g}}(\nu)}d\nu \tag{9}$$

for $f, g \in C_{c,e}^\infty(\mathbb{R})$. This map $f \to \hat{\boldsymbol{f}}$, $f \in C_{c,e}^\infty(\mathbb{R})$, is extended to an isometry between $L^2(\Delta) = L^2(\mathbb{R}_+, \Delta(x)dx)$ and $L^2(\nu) = L^2(\mathbb{R}_+ \cup D_{\alpha,\beta}, d\nu)$. Actually, each function f in $L^2(\Delta)$ is of the form $f = f_P + {}^\circ f$ (see (7)) and their L^2-norms are given as

$$\int_0^\infty |f_P(x)|^2\Delta(x) = \frac{1}{2\pi}\int_0^\infty |\hat{f}(\lambda)|^2|C(\lambda)|^{-2}d\lambda,$$

$$\int_0^\infty |{}^\circ f(x)|^2\Delta(x)dx = \sum_{\mu \in D_{\alpha,\beta}} |a_\mu|^2 d(\mu) \;\text{ if } {}^\circ f(x) = \textstyle\sum_{\mu \in D_{\alpha,\beta}} a_\mu \phi_\mu(x)d(\mu).$$

Therefore, if we put $\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{a_\mu\})$, then $\|f\|_{L^2(\Delta)} = \|\hat{\boldsymbol{f}}\|_{L^2(\nu)}$ and

$$f(x) = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda + \sum_{\mu \in D_{\alpha,\beta}} a_\mu \phi_\mu(x)d(\mu). \tag{10}$$

We define

$$B(x) = \int_0^x \Delta(t)dt, \; x \geq 0 \tag{11}$$

and put

$$\theta(x)=\frac{B(x)}{\Delta(x)} \quad \text{and} \quad \Theta(\lambda)=(\lambda^2+\rho^2)^{1/2}.$$

2. Main theorem. We keep the notation in §1 and prove the following.

Theorem 2.1. *Let $\alpha>-1$, $\beta\in\mathbb{R}$. For $f\in L^2(\Delta)$, we suppose that $f\theta\in L^2(\Delta)$ and $\hat{\boldsymbol{f}}\Theta\in L^2(\nu)$. Then*

$$\|f\theta\|^2_{L^2(\Delta)}\int_{\mathbb{R}_+\cup D_{\alpha,\beta}}|\hat{\boldsymbol{f}}(\nu)|^2\Theta(\nu)^2d\nu\geq\frac{1}{4}\|f\|^4_{L^2(\Delta)}, \tag{12}$$

where the equality holds if and only if f is of the form

$$f(x)=ce^{\gamma\int_0^x\theta(t)dt}$$

for some $c,\gamma\in\mathbb{C}$.

Proof. Without loss of generality we may supppose that $f\in C^\infty_{c,e}(\mathbb{R})$ and f is real valued. Since

$$\hat{\boldsymbol{f}}(\nu)=(\hat{f}(\lambda),\{\hat{f}(\mu)\}).$$

and $(-Lf)^\wedge(\lambda)=\hat{f}(\lambda)(\lambda^2+\rho^2)=\hat{f}(\lambda)\Theta(\lambda)^2$, the Parseval formula (9) yields that

$$\begin{aligned}\int_{\mathbb{R}_+\cup D_{\alpha,\beta}}|\hat{\boldsymbol{f}}(\nu)|^2\Theta(\nu)^2d\nu &= \int_0^\infty f(x)(-Lf)(x)\Delta(x)dx\\ &= \int_0^\infty (f'(x))^2\Delta(x)dx.\end{aligned}$$

Hence it follows that

$$\begin{aligned}&\int_0^\infty f(x)^2\theta(x)^2\Delta(x)dx\int_{\mathbb{R}_+\cup D_{\alpha,\beta}}|\hat{\boldsymbol{f}}(\nu)|^2\Theta(\nu)^2d\nu\\ =&\ \int_0^\infty f(x)^2\theta(x)^2\Delta(x)dx\int_0^\infty(f'(x))^2\Delta(x)dx\\ \geq&\ \Big(\int_0^\infty f(x)f'(x)\theta(x)\Delta(x)dx\Big)^2\\ =&\ \frac{1}{4}\Big(\int_0^\infty (f(x)^2)'B(x)dx\Big)^2=\frac{1}{4}\Big(\int_0^\infty f(x)^2\Delta(x)dx\Big)^2.\end{aligned}$$

Here we used the fact that $B' = \Delta$ (see (11)). Clearly, the equality holds if and only if $f\theta = cf'$ for some $c \in \mathbb{R}$, that is, $f'/f = c^{-1}\theta$. This means that $\log(f) = c^{-1}\int_0^x \theta(t)dt + C$ and thus, the desired result follows. ■

We recall that $\Theta^2(\lambda) = \lambda^2 + \rho^2$. Then (12) and the Parseval formula imply the following.

Corollary 2.2. *Let f be the same as in Theorem 2.1.*

$$\|f\theta\|^2_{L^2(\Delta)}\int_{\mathbb{R}_+\cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2\nu^2 d\nu \geq \frac{1}{4}\|f\|^2_{L^2(\Delta)}\int_0^\infty |f(x)|^2(1-4\rho^2\theta^2)\Delta(x)dx. \quad (13)$$

We shall estimate θ and $1-4\rho^2\theta^2$. Since $\alpha > -1$, it follows that

$$\begin{aligned}
B(x) &= \int_0^x (2\sinh s)^{2\alpha+1}(2\cosh s)^{2\beta+1}ds \\
&= 2^{2\rho}\int_0^{\sinh x} t^{2\alpha+1}(1+t^2)^\beta dt \\
&= 2^{2\rho}(\sinh x)^{2\alpha+2}\int_0^1 t^{2\alpha+1}(1+(\sinh x)^2t^2)^\beta dt \\
&= 2^{2\rho-1}(\sinh x)^{2\alpha+2}(\cosh x)^{2\beta}\int_0^1 (1-s)^\alpha(1-(\tanh x)^2s)^\beta ds \\
&= 2^{2\rho-1}(\sinh x)^{2\alpha+2}(\cosh x)^{2\beta}\frac{1}{\alpha+1}F(1,-\beta,2+\alpha;(\tanh x)^2)
\end{aligned}$$

and thus,

$$\theta(x) = \frac{1}{2(\alpha+1)}F(1,-\beta,2+\alpha;(\tanh x)^2)\tanh x. \quad (14)$$

Lemma 2.3. *Let $k = 0,1,2,\cdots$ and $0 \leq x \leq 1$. We suppose that $k-(\alpha+1) < \beta \leq k+(2k+1)\alpha$. Then $x^{2k+1}F(k+1,k-\beta,k+2+\alpha,x^2)$ is increasing and*

$$0 \leq x^{2k+1}F(k+1,k-\beta,k+2+\alpha;x^2) \leq \frac{\Gamma(k+2+\alpha)\Gamma(\rho-k)}{\Gamma(1+\alpha)\Gamma(\rho+1)}.$$

Proof. When $k-(\alpha+1) < \beta \le k$, it follows that $F(k+1, k-\beta, k+2+\alpha; x)$ is increasing on $0 \le x \le 1$. Hence $H(x) = H_k(\alpha, \beta, x) = x^{2k+1}F(k+1, k-\beta, k+2+\alpha; x^2)$ is dominated by $H(1) = \Gamma(k+2+\alpha)\Gamma(\rho-k)/\Gamma(1+\alpha)\Gamma(\rho+1)$. Let $k < \beta \le k + (2k+1)\alpha$. We shall prove that $H(x)$ is increasing and $H(x) \le H(1)$ as before. In order to prove that $H(x)$ is increasing, we shall show that its derivative is positive. We note that

$$\begin{aligned} H'(x) &= (1+2k)x^{-1}H_k(\alpha,\beta,x) + \frac{2(1+k)(k-\beta)x^{-1}}{2+k+\alpha}H_{k+1}(\alpha,\beta,x) \quad (15) \\ &= (1+2k)x^{-1}H_k(\alpha,\beta,x) \\ &\quad +2(1+k+\alpha)x^{-1}\Big(H_k(\alpha-1,\beta,x) - H_k(\alpha,\beta,x)\Big) \\ &= x^{2k}K(x^2), \end{aligned}$$

where $K(x) = (1+2k)F(1+k, k-\beta, 2+k+\alpha; x) + 2(1+k+\alpha)(F(1+k, k-\beta, 1+k+\alpha; x) - F(1+k, k-\beta, 2+k+\alpha, x))$. Then

$$\begin{aligned} K'(x) &= (1+k)(k-\beta)x^{-(2k+3)}\Big(\frac{(1+2k)}{2+k+\alpha}H_{k+1}(\alpha,\beta,x) \\ &\quad +2(1+k+\alpha)\Big(\frac{H_{k+1}(\alpha-1,\beta,x)}{1+k+\alpha} - \frac{H_{k+1}(\alpha,\beta,x)}{2+k+\alpha}\Big)\Big). \end{aligned}$$

Since $\beta > k$, $x^{-(2k+3)}H_{k+1}(\alpha,\beta,x) = F(2+k, 1+k-\beta, 3+k+\alpha; x) \le F(2+k, 1+k-\beta, 2+k+\alpha; x) = x^{-(2k+3)}H_{k+1}(\alpha-1,\beta,x)$ and $1/(1+k+\alpha) - 1/(2+k+\alpha) > 0$, it follows that $K'(x) < 0$ and thus, $K(x)$ is decreasing. Therefore, $H'(x)$ is decreasing and

$$H'(x) \ge H'(1) = (k+(2k+1)\alpha-\beta)\frac{\Gamma(2+k+\alpha)\Gamma(\alpha+\beta-k)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)} \ge 0$$

under the assumption on β. Hence $H(x)$ is increasing. ■

Lemma 2.4. *Let notation be as above and $\rho > 0$. If $-(\alpha+1) < \beta \le \alpha$, then*

$$0 \le \theta(x) \le 1/2\rho$$

and if $\alpha < \beta$ and k is an integer such that $k-1+(2k-1)\alpha < \beta \le k+(2k+1)\alpha$, then

$$0 \le \theta(x) \le \frac{1}{2\rho}\frac{(2k)!!}{(2k-1)!!}\frac{\Gamma(\beta+1)\Gamma(\rho-k)}{\Gamma(\beta-k+1)\Gamma(\rho)} = O(\sqrt{\beta}).$$

Proof. Clearly, $\theta(x) \geq 0$ from Euler's integral expression of hypergeometric functions. The first assertion follows from (9) and Lemma 2.3 with $k = 0$. We suppose that $\alpha < \beta \leq 1 + 3\alpha$, that is, the case of $k = 1$ in Lemma 2.3. Since β is out of the range when $k = 0$, we couldn't conclude that $H(x) = xF(1, -\beta, 2 + \alpha; x^2)$ is increasing on $0 \leq x \leq 1$. Let $x = x_0$ be the maximaum point of $H(x)$. Since

$$H'(x) = \Big(F(1, -\beta, \alpha + 2, x^2) - \frac{2\beta}{2 + \alpha} x^2 F(2, 1 - \beta, \alpha + 3, x^2)\Big)$$

and $H'(x_0) = 0$, it follows that

$$H(x_0) = \frac{2\beta}{2 + \alpha} x_0^3 F(2, 1 - \beta, \alpha + 3, x_0^2).$$

Since $\alpha < \beta \leq 1 + 3\alpha$, applying Lemma 2.3 with $k = 1$, we see that

$$\begin{aligned} \theta(x) \leq \frac{H(x_0)}{2(\alpha + 1)} &\leq \frac{1}{2(\alpha + 1)} \frac{2\beta}{2 + \alpha} \frac{\Gamma(3 + \alpha)\Gamma(\rho - 1)}{\Gamma(1 + \alpha)\Gamma(\rho + 1)} \\ &= \frac{1}{2\rho} \frac{2!!}{1!!} \frac{\Gamma(1 + \beta)\Gamma(\rho - 1)}{\Gamma(\beta)\Gamma(\rho)}. \end{aligned}$$

When $1 + 3\alpha < \beta \leq 2 + 5\alpha$, we couldn't apply Lemma 2.3 in the above argument to conclude that $x^3 F(2, 1 - \beta, \alpha + 3, x^2)$ is increasing on $0 \leq x \leq 1$. Hence, we shall consider its derivative and the maximum point again. Then we can apply Lemma 2.3 with $k = 2$ to the derivative. Generally, when $(k - 1) + (2k - 1)\alpha < \beta \leq k + (2k + 1)\alpha$, $H_l(\alpha, \beta, x) = x^{2l+1} F(l + 1, l - \beta, l + 2 + \alpha; x^2)$, $0 \leq l \leq k - 1$, are not increasing and $H_k(\alpha, \beta, x)$ is increasing. Then it follows from (9) that

$$H_{l-1}(\alpha, \beta, x_{l-1}) = \frac{2l(\beta - l + 1)}{(2l - 1)(1 + l + \alpha)} H_l(\alpha, \beta, x_{l-1}),$$

where x_{l-1} is the maximum point of $H_{l-1}(\alpha, \beta, x)$ and thus,

$$\begin{aligned} \theta(x) &\leq \frac{1}{2(\alpha + 1)} \prod_{l=1}^{k} \frac{2l(\beta - l + 1)}{(2l - 1)(1 + l + \alpha)} \frac{\Gamma(k + 2 + \alpha)\Gamma(\rho - k)}{\Gamma(1 + \alpha)\Gamma(\rho + 1)} \\ &= \frac{1}{2\rho} \frac{(2k)!!}{(2k - 1)!!} \frac{\Gamma(\beta + 1)\Gamma(\rho - k)}{\Gamma(\beta - k + 1)\Gamma(\rho)}. \end{aligned}$$

The asymptotic behavior of $\theta(x)$ follows from Wallis' formula. ■

Lemma 2.5. *Let $\Upsilon(x) = 1 - 4\rho^2\theta(x)^2$. If $-(\alpha+1) < \beta \leq 0$, then $\Upsilon(x) \geq (\cosh x)^{-2}$. Generally,*

$$\Upsilon(x) = \begin{cases} O((\cosh x)^{-2}) & \text{if } x \to \infty, \\ O(1) & \text{if } x \to 0 \end{cases}$$

and if $\beta \leq \alpha$, then $\Upsilon(x) \geq 0$.

Proof. Since $F(1, -\beta, 2+\alpha; 0) = 1$ and $F(1, -\beta, 2+\alpha; 1) = (\alpha+1)/\rho$, the asymptotic behaviour easily follows. If $-(\alpha+1) < \beta \leq 0$, then $F(1, -\beta, 2+\alpha; x)$ is increasing with respect to x. Hence $\theta(x) \leq F(1, -\beta, 2+\alpha; 1) \tanh x /2(\alpha+1) \leq (1/2\rho) \tanh x$ and thus, $\Upsilon(x) \geq (\cosh x)^{-2}$. If $0 < \beta \leq \alpha$, then $\Upsilon(x) \geq 0$ from Lemma 2.4. ■

We put

$$\varepsilon_k = \varepsilon_k(\alpha, \beta) = \frac{(2k-1)!!}{(2k)!!} \frac{\Gamma(\beta - k + 1)\Gamma(\rho)}{\Gamma(\beta+1)\Gamma(\rho - k)}. \tag{16}$$

Lemma 2.4 implies that, if $k - 1 + (2k-1)\alpha < \beta \leq k + (2k+1)\alpha$, then

$$\theta(x) \leq (2\rho\epsilon_k)^{-1}. \tag{17}$$

The following assertion follows from Theorem 2.1, Corollary 2.2, Lemma 2.4 and Lemma 2.5.

Corollary 2.6. *Let $\rho > 0$ and f be the same as in Theorem 2.1. If $-(\alpha+1) < \beta \leq \alpha$, then $f = f_P$,*

$$\int_0^\infty |\hat{f}(\lambda)|^2 \Theta(\lambda)^2 |C(\lambda)|^{-2} d\lambda \geq \rho^2 \|f\|^2_{L^2(\Delta)}$$

and

$$\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda \geq \rho^2 \int_0^\infty |f(x)|^2 \Upsilon(x) \Delta(x) dx,$$

where $\Upsilon(x) = 1 - 4\rho^2\theta(x)^2 \geq 0$, and if $k - 1 + (2k-1)\alpha < \beta \leq k + (2k+1)\alpha$, then

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \Theta(\nu)^2 d\nu \geq \rho^2 \varepsilon_k^2 \|f\|^2_{L^2(\Delta)},$$

and

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \nu^2 d\nu \geq \rho^2 \varepsilon_k^2 \int_0^\infty |f(x)|^2 \Upsilon(x) \Delta(x) dx. \qquad (18)$$

The shapes of $\theta(t)$ with $1/2\rho$ and $\Upsilon(t)$, $t = \tanh x$, are respectively given as follows.

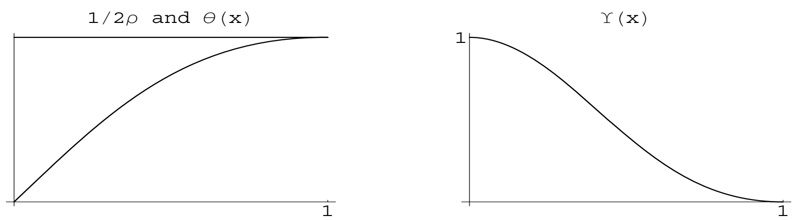

Figure 1: The case of $\beta \leq \alpha$.

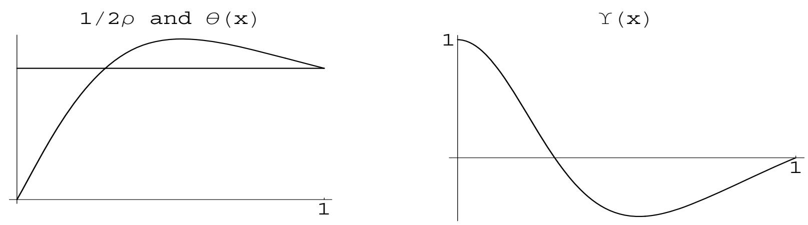

Figure 2: The case of $\beta > \alpha$.

3. Uncertainty. We shall apply the inequalities obtained in the previous section to deduce some information on the concentration of f and $\hat{f}$. Let f be a non-zero function in $L^2(\Delta)$ satisfying $f\theta \in L^2(\Delta)$ and $\hat{\boldsymbol{f}}\Theta \in L^2(\nu)$. We recall that

$$f = f_P + {}^\circ f, \quad {}^\circ f(x) = \sum_{\mu \in D_{\alpha,\beta}} a_\mu \phi_\mu(x) d(\mu)$$

and $\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{a_\mu\})$.

Definition 3.1. *Let* $0 < \epsilon < 1/4\rho^2$ *and* $M > 0$.

(1) We say that a function $f(x)$ on $\mathbb{R}_+$ is (θ,ϵ)-concentrated at $x=0$ if

$$\|f\theta\|_{L^2(\Delta)} \leq \epsilon \|f\|_{L^2(\Delta)}$$

and is (θ,ϵ)-nonconcentrated at $x=0$ if the reverse holds.

(2) We say that a function $\hat{f}(\lambda)$ on $\mathbb{R}_+$ is (λ,ϵ)-concentrated at $\lambda=0$ if

$$\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda \leq \epsilon \|f\|^2_{L^2(\Delta)}$$

and is (λ,ϵ)-nonconcentrated at $\lambda=0$ if the reverse holds.

(3) We say that a function $f(x)$ on $\mathbb{R}_+$ is (μ,ϵ)-concentrated at $x=0$ if

$$\sum_{\mu\in D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) \leq \epsilon \|f\|^2_{L^2(\Delta)}.$$

(4) We say that a function $f(x)$ on $\mathbb{R}_+$ is (Υ,ϵ)-nonconcentrated at $x=0$ if

$$\Big| \int_0^\infty |f(x)|^2 \Upsilon(x) \Delta(x) dx \Big| \leq \epsilon \|f\|^2_{L^2(\Delta)}.$$

(5) We say that a function $f(x)$ on $\mathbb{R}_+$ is (x_0,ϵ)-bounded if

$$|f(x)| \leq \epsilon \|f\|_{L^2(\Delta)} \text{ if } x \geq x_0.$$

Now we suppose that $f(x)$ is (θ,ϵ)-concentrated at $x=0$. Since

$$\begin{aligned} & \int_{\mathbb{R}_+\cup D_{\alpha,\beta}} |\boldsymbol{\hat{f}}(\nu)|^2 \Theta(\nu)^2 d\nu \\ = & \int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda - \sum_{D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) + \rho^2 \|f\|^2_{L^2(\Delta)}, \end{aligned}$$

it follows from (12) that

$$\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda - \sum_{D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) \geq (1/4\epsilon - \rho^2) \|f\|^2_{L^2(\Delta)}.$$

Therefore, $\hat{f}(\nu)$ is $(\lambda, 1/4\epsilon - \rho^2)$-nonconcentrated at $\lambda = 0$.

Conversely, we suppose that $\hat{f}(\nu)$ is (ν,ϵ)-concentrated at $\lambda = 0$. Then it follows from (17) that, if $k-1+(2k-1)\alpha < \beta \le k+(2k+1)\alpha$, then

$$\int_0^\infty |f_P(x)|^2 \Upsilon(x)\Delta(x)dx \ge (1-\varepsilon_k^{-2})\|f_P\|^2_{L^2(\Delta)}.$$

Here we recall that $1-\varepsilon_k^{-2} \le 0$. Moreover, letting $A = \int_0^\infty |f_P(x)|^2\Upsilon(x)|\Delta(x)dx$ and $B = \|f_P\|^2_{L^2(\Delta)}$, we see from (13) for $f = f_P$ that

$$(B-A)\epsilon B \ge \rho^2 AB$$

and thus,

$$A \le \frac{\epsilon B}{\rho^2+\epsilon} \le \frac{\epsilon}{\rho^2}B. \tag{19}$$

Therefore, $f_P(x)$ is (Υ,δ)-nonconcentrated at $x=0$, where

$$\delta = \max\{\varepsilon_k^{-2}-1, \rho^{-2}\epsilon\}.$$

Moreover, let $x_0 = 1$. Then it follows from (2), (3), and (4) that for $x \ge 1$,

$$\begin{aligned}
|f_P(x)| &\le \Big|\int_0^\infty \hat{f}(\lambda)\Phi_\lambda(x)C(\lambda)^{-1}d\lambda\Big| \\
&\le e^{-\rho x}K_{x_0}\Big(\int_0^{\sqrt{\epsilon}} |\hat{f}(\lambda)||C(-\lambda)|^{-1}d\lambda + \int_{\sqrt{\epsilon}}^\infty |\hat{f}(\lambda)||C(-\lambda)|^{-1}d\lambda\Big) \\
&\le e^{-\rho x}K_{x_0}\Big(\epsilon^{1/4}\|f_P\|_{L^2(\Delta)} \\
&\qquad + \Big(\int_{\sqrt{\epsilon}}^\infty |\hat{f}(\lambda)|^2\lambda^2|C(\lambda)|^{-2}d\lambda\Big)^{1/2}\Big(\int_{\sqrt{\epsilon}}^\infty \lambda^{-2}d\lambda\Big)^{1/2}\Big) \\
&\le 2e^{-\rho x}K_{x_0}\epsilon^{1/4}\|f_P\|_{L^2(\Delta)}.
\end{aligned}$$

Hence we have the following.

Theorem 3.2 *Let $\rho > 0$ and $f \in L^2(\Delta)$ satisfy $f\theta \in L^2(\Delta)$ and $\hat{\boldsymbol{f}}\Theta \in L^2(\nu)$. Let k be an integer such that $k-1+(2k-1)\alpha < \beta \le k+(2k+1)\alpha$, where $k=0$ if $\beta \le \alpha$. We define ε_k by (16) where $\epsilon_0 = 1$. If $f(x)$ is (θ,ϵ)-concentrated at $x=0$, then $\hat{f}(\lambda)$ is $(\lambda, 1/4\epsilon-\rho^2)$-nonconcentrated at $\lambda=0$. Conversely, if $\hat{f}(\lambda)$ is (λ,ϵ)-concentrated at $\lambda=0$, then $f_P(x)$ is (Υ,δ)-nonconcentrated* *at $x=0$, where $\delta = \max\{\epsilon_k^{-2} - 1, \rho^{-2}\epsilon\}$, and there exists a positive constant c such that $f_P(x)$ is $(1, c\epsilon^{1/4})$-bounded.*

When $\beta \leq \alpha$, we recall that $f = f_P$ and $\varepsilon_0 = 1$ $(k = 0)$. Hence, the above theorem implies that $f(x)$ is $(\Upsilon, \rho^{-2}\epsilon)$-nonconcentrated at $x = 0$ and $(1, c\epsilon^{1/4})$-bounded. Therefore, $f(x)$ is spread if ϵ goes to 0. However, when $\beta > \alpha$, then $\varepsilon_k < 1$ and it is not clear that $f(x)$ is spread if ϵ goes to 0. This implies that the discrete part of f influences the uncertaintity.

We now suppose that $\beta > \alpha$, $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$ and $f(x)$ is (μ, ϵ_d)-concentrated at $x = 0$. We shall prove that $f(x)$ is spread if ϵ and ϵ_d go to 0. As in the previous argument, let $A = \int_0^\infty |f(x)|^2 \Upsilon(x)| \Delta(x)dx$ and $B = \|f\|^2_{L^2(\Delta)}$. Then it follows from (13) that

$$(B - A)(\epsilon + \epsilon_d)B \geq \rho^2 AB$$

and thus, $A \leq \rho^{-2}(\epsilon + \epsilon_d)B$. Let $x_0 > 0$ the point such that $\Upsilon(x_0) = 0$ and $x \geq x_0$. As before, it follows that

$$|f_P(x)| \leq e^{-\rho x} K_{x_0} \epsilon^{1/4} \|f_P\|_{L^2(\Delta)}.$$

On the other hand, it follows from (8) that

$$\begin{aligned} |{}^\circ f(x)| &\leq \sum_{\mu \in D_{\alpha,\beta}} |a_\mu||\phi_\mu(x)|d(\mu) \\ &\leq e^{-\rho x}\Big(\sum_{\mu \in D_{\alpha,\beta}} e^{-2|\mu|x_0}|\mu|^{-2}d(\mu)\Big)^{1/2} \epsilon_d^{1/2}\|{}^\circ f\|_{L^2(\Delta)}. \end{aligned}$$

Hence, for $x \geq x_0$, we see that there exist a positive constant c such that

$$|f(x)e^{\rho x}| \;\leq\; c(\epsilon^{1/4} + \epsilon_d^{1/2})\|f\|_{L^2(\Delta)}.$$

Therefore, it follows that

$$\begin{aligned} \int_0^\infty |f(x)|^2 \Upsilon(x)\Delta(x)dx &\geq c\int_{x_0}^\infty |f(x)e^{\rho x}|^2 \Upsilon(x)dx \\ &\geq c^2(\epsilon^{1/4} + \epsilon_d^{1/2})^2 \|f\|^2_{L^2(\Delta)} \int_{x_0}^\infty \Upsilon(x)dx \\ &= c_\Upsilon(\epsilon^{1/4} + \epsilon_d^{1/2})^2\|f\|^2_{L^2(\Delta)}. \end{aligned}$$

Here $c_\Upsilon < 0$, because $\displaystyle\int_{x_0}^{\infty} \Upsilon(x)dx < 0$.

Theorem 3.3 *Let $\beta > \alpha$ and $\alpha > -1$. Let $f \in L^2(\Delta)$ satisfy $f\theta \in L^2(\Delta)$ and $\hat{\boldsymbol{f}}\Theta \in L^2(\nu)$. We suppose that $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$ and $f(x)$ is (μ, ϵ_d)-concentrated at $x = 0$. Then there exists constants $c_\Upsilon < 0, c > 0$ such that $f(x)$ is (Υ, δ)-nonconcentrated at $x = 0$, where $\delta = \max\{-c_\Upsilon(\epsilon^{1/4} + \epsilon_d^{1/2}), \rho^{-2}(\epsilon + \epsilon_d)\}$, and is $(x_0, c(\epsilon^{1/4} + \epsilon_d^{1/2}))$-bounded.*

We suppose that f is supported on $[R, \infty)$. Then there exists a constant $0 < \varepsilon(R) \le 1$ such that

$$0 \le \theta(x) \le \frac{1}{2\rho\varepsilon(R)}$$

and $\varepsilon(R) \to 1$ if $R \to \infty$. Then it follows from (13) that

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \nu^2 d\nu \ge \rho^2 \int_0^\infty |f(x)|^2 (\varepsilon(R)^2 - 1)\Delta(x)dx.$$

Then we obtain the following.

Proposition 3.4. *Let $\rho > 0$ and suppose that $f \in L^2(\Delta)$ satisfies $f\theta \in L^2(\Delta)$ and $\hat{\boldsymbol{f}}\Theta \in L^2(\nu)$. We suppose that f is supported on $[R, \infty)$. Then*

$$\sum_{\mu \in D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) \le \int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda + \rho^2 (1 - \varepsilon(R)^2) \|f\|^2_{L^2(\Delta)}.$$

Remark 3.5. When $\beta = 0$, we can calculate more precisely. In this case $\theta = (2\rho)^{-1}\tanh x$ and $1 - 4\rho^2\theta^2 = (\cosh x)^{-2}$. Therefore, (12) and (13) became

$$\|f(x)\tanh x\|^2_{L^2(\Delta)} \|\hat{f}(\lambda)(\lambda^2 + \rho^2)^{1/2}\|^2_{L^2(|C|^{-2})} \ge \rho^2 \|f\|^4_{L^2(\Delta)},$$

where the equality holds if and only if f is of the form $c(\cosh x)^\gamma$, and

$$\|f(x)\tanh x\|^2_{L^2(\Delta)} \|\hat{f}(\lambda)\lambda\|^2_{L^2(|C|^{-2})} \ge \rho^2 \|f\|^2_{L^2(\Delta)} \|f(x)(\cosh x)^{-1}\|^2_{L^2(\Delta)}.$$

Since the Jacobi transform of $(\cosh \lambda)^\gamma$ is explicitly calcurated in [1], we can directly check the equality condition for these inequalities.

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Hardy's theorem on $SU(1,1)$

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1. Introduction. The classical Hardy's theorem [6] asserts f and its Fourier transform $\tilde{f}$ can not both be "very rapidly decreasing". More precisely, suppose a measurable function f on $\mathbb{R}$ and its Fourier transform $\tilde{f}$ on $\mathbb{R}$ satisfing

$$|f(x)| \leq Ae^{-ax^2} \quad \text{and} \quad |\tilde{f}(\lambda)| \leq Be^{-b\lambda^2} \tag{1}$$

for some positive constants A, B, a and b. If $ab > 1/4$, then $f = 0$, and if $ab = 1/4$, then f is a constant multiple of e^{-ax^2}. Recently, an analogue of Hardy's theorem was established for some Lie Groups; for the case of $ab > 1/4$ see [2], [3], [4], [5], [7], [11], [13], [14], [15], [19] and for the case of $ab = 1/4$ see [10], [17], [20], [21], [22]. In the last case the heat kernel on Lie groups plays an essential role to control the decay of f. Moreover, Hardy's theorem is generalized for the Fourier-Jacobi transform (see [1] and [8]) and for the Heckman-Opdam transform (see [16]). However, as pointed in [1], Hardy's theorem on $SU(1,1)$ does not hold if no assumption on the K-types of f is imposed: Let $G = SU(1,1)$, and for $g \in G$ let $g = k_\phi a_x k_\psi$, $0 \leq x$, $0 \leq \phi, \psi \leq 4\pi$, denote the Cartan decomposition of g. Let h_t denote the heat kernel on G and for integrabl functions f on G let $\tilde{f}_{n,m}$, $n, m \in \mathbb{Z}$, the spherical Fourier transform of f corresponding to the K-type (n,m) (see (6)). We suppose that a measurable function f on G and its spherical Fourier transform $\tilde{f}_{n,m}$ on $\mathbb{R}$ satisfing

$$|f(g)| \leq Ah_{1/4a}(g) \quad \text{and} \quad |\tilde{f}_{n,m}(\lambda)| \leq Be^{-b\lambda^2} \text{ for all } n, m \in \mathbb{Z} \tag{2}$$

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for some positive constants A, B, a and b. Then, $f=0$ if $ab>1/4$, however, there are infinitely many linearly independent functions on G satisfying the above condition if $ab=1/4$ (see Theorem 3.2).

In this paper, we will show that the condition (2) under $ab=1/4$ determines a function on G uniquely in the following sense: In the classical case the condition (1) under $ab=1/4$ guarantees the limit

$$\lim_{x\to\infty} e^{ax^2} f(x) = c$$

and then f is uniquely determined as $f(x)=ce^{-ax^2}$. On $SU(1,1)$, similary, the condition (2) under $ab=1/4$ guarantees the limit

$$\lim_{x\to\infty} (h_{1/4a}(x))^{-1} f(k_\phi a_x) = F(\phi)$$

and then f is uniquely determined by using the Fourier coefficient of F (see Theorem 5.1).

2. Notation. Let $G=SU(1,1)$ and A, K the subgroups of G of the matrices

$$a_x = \begin{pmatrix} \mathrm{ch}x/2 & \mathrm{sh}x/2 \\ \mathrm{sh}x/2 & \mathrm{ch}x/2 \end{pmatrix},\ x\in\mathbb{R} \quad \text{and} \quad k_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix},\ 0\le\phi\le 4\pi$$

respectively. According to the Cartan decomposition of G, each $g\in G$ can be written uniquely as $g=k_\phi a_x k_\psi$ where $0\le x$, $0\le\phi,\psi\le 4\pi$. Let $\pi_{j,\lambda}$ ($j=0,1/2,\lambda\in\mathbf{R}$) denote the principal series representation of G. In the following, we shall consider functions f on G satisfying

$$f(a_x)=f(a_{-x}),\quad x\in\mathbb{R}$$

and we shall identify f with an even function on $\mathbb{R}$, which is denoted by the same letter f. Under this restriction the (vector-valued) spherical Fourier transform $\int_G f(f)\pi_{j,\lambda}(\lambda)dg$, dg a Haar measureon G, is supported on $j=0$ and $\lambda>0$ (cf. [18, §8]).

Before introducing the explicit form of the spherical Fourier transform of f on G, we shall define the Jacobi transform of f on $\mathbb{R}$. Let $\alpha,\beta,\lambda\in\mathbf{C}$ and $x\in\mathbb{R}$. We shall consider the differential equation

$$(L_{\alpha,\beta}+\lambda^2+\rho^2)f(x)=0, \tag{3}$$

where $\rho = \alpha + \beta + 1$ and

$$L_{\alpha,\beta} = \frac{d^2}{dx^2} + ((2\alpha+1)\mathrm{cth}x + (2\beta+1)\mathrm{th}x)\frac{d}{dx}.$$

Then, for $\alpha \notin -\mathbf{N}$, the Jacobi function of the first kind with order (α, β)

$$\phi_\lambda^{\alpha,\beta}(x) = F\left(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -\mathrm{sh}^2 x\right) \tag{4}$$

is a unique solution of (3) satisfying $\phi_\lambda^{\alpha,\beta}(0) = 1$ and $d\phi_\lambda^{\alpha,\beta}/dx(0) = 0$. Then for an even function f on $\mathbb{R}$ the Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$ is given by

$$\hat{f}_{\alpha,\beta}(\lambda) = \int_0^\infty f(x)\phi_\lambda^{\alpha,\beta}(x)\Delta_{\alpha,\beta}(x)dx, \tag{5}$$

where $\Delta_{\alpha,\beta}(x) = (2\mathrm{sh}x)^{2\alpha+1}(2\mathrm{ch}x)^{2\beta+1}$.

Let $n, m \in \mathbf{Z}$ and $\psi_\lambda^{n,m}(g)$ $(\lambda \in \mathbb{R},\ g \in G)$ the matrix coefficient of $\pi_{0,\lambda}(g)$ with the K-type (m, n). Then the (scalar-valued) spherical Fourier transform $\tilde{f}_{n,m}$ of type (n, m) is defined as

$$\tilde{f}_{n,m}(\lambda) = \int_0^\infty f(x)\psi_\lambda^{(n,m)}(x)\Delta_{0,0}(x)dx. \tag{6}$$

We recall the explicit form of $\psi_\lambda^{n,m}(g)$

$$\psi_\lambda^{n,m}(g) = (\mathrm{ch}x)^{2n}(\mathrm{th}x)^{|n-m|}Q_{n,m}(\lambda)\phi_\lambda^{|n-m|,n+m}(x)e^{in\phi}e^{im\psi},$$

where $g = k_\phi a_x k_\psi$ and

$$Q_{n,m}(\lambda) = \begin{pmatrix} -1/2 - i\lambda \mp m \\ |n-m| \end{pmatrix}. \tag{7}$$

Here $\mp m$ is equal to $-m$ if $m \geq n$ and m if $m \leq n$ (see [12, (3.4.10)]). Hence from (5) and (6) we see that

$$\tilde{f}_{n,m}(\lambda) = Q_{n,m}(\lambda)(f(x)(\mathrm{sh}x)^{-|n-m|}(\mathrm{ch}x)^{-(n+m)})^\wedge_{|n-m|,n+m}(\lambda). \tag{8}$$

Let $h_t^{\alpha,\beta}$ denote the heat kernel for the Jacobi transform, that is, an even function on $\mathbb{R}$ satisfying

$$(h_t^{\alpha,\beta})^\wedge_{\alpha,\beta}(\lambda) = e^{-t(\lambda^2+\rho^2)},\ t, \lambda \in \mathbb{R}.$$

Then, it follows from (8) that

$$\left(h_t^{|n-m|,n+m}(x)(\mathrm{sh}x)^{|n-m|}(\mathrm{ch}x)^{n+m}\right)^{\sim}_{n,m}(\lambda) = Q_{n,m}(\lambda)e^{-t(2n+1)^2}e^{-t\lambda^2}. \qquad (9)$$

3. Hardy's theorem. We keep the notations in the previous section. As an application of the Hardy's theorem for the Jacobi transform (see [1], [8]), we have the following.

Theorem 3.1. *Let f be a mesurable function on G of the K-type (n, m) and satisfy*

$$(i)\quad f(x) = O\left(h_{1/4a}^{|n-m|,n+m}(x)(\mathrm{sh}x)^{|n-m|}(\mathrm{ch}x)^{n+m}\right)$$
$$(ii)\quad \tilde{f}_{n,m}(\lambda) = O\left(Q_{n,m}(\lambda)e^{-b\lambda^2}\right).$$

If $ab > 1/4$, then $f = 0$, and if $ab = 1/4$, then f is a constant multiple of $h_b^{|n-m|,n+m}(x)(\mathrm{sh}x)^{|n-m|}(\mathrm{ch}x)^{n+m}$.

Proof. Let $g(x) = f(x)(\mathrm{sh}x)^{-|n-m|}(\mathrm{ch}x)^{-(n+m)}$. Then

$$g(x) = O\left(h_{1/4a}^{|n-m|,n+m}(x)\right)$$

and

$$\hat{g}_{n-m,n+m}(\lambda) = \tilde{f}_{n,m}(\lambda)Q_{n,m}^{-1}(\lambda) = O\left(e^{-b\lambda^2}\right).$$

By Hardy's theorem for the Jacobi transform (see [1], [8]), it follows that, if $ab > 1/4$, then $g = 0$ almost everywhere, and if $ab = 1/4$, then g is a constant multiple of $h_b^{|n-m|,n+m}(x)$ and thus, f is the desired form. ■

We here recall the asymptotic behaviour of the heat kernel:

$$h_t^{\alpha,\beta}(x) \sim t^{-\alpha-1}e^{-\rho^2 t}e^{-\rho x}e^{-x^2/4t}(1+t+x)^{\alpha-1/2}(1+x) \qquad (10)$$

(cf. [8, Theorem 3.1]). Then, as functions of x,

$$h_t^{\alpha,\beta}(x)(\mathrm{sh}x)^{\alpha}(\mathrm{ch}x)^{\beta} \sim h_t^{0,0}(x)(1+x)^{\alpha},\ x \to \infty. \qquad (11)$$

Theorem 3.2. *Let f be a mesurable function on G and satisfy*

$$(i)\quad f(x) = O\left(h_{1/4a}^{0,0}(x)\right)$$
$$(ii)\quad \tilde{f}_{n,m}(\lambda) = O\left(e^{-b\lambda^2}\right) \text{ for all } n, m \in \mathbb{Z}.$$

If $ab > 1/4$, then $f = 0$, and if $ab = 1/4$, then f is of the form

$$f(g) = \sum_{n\in\mathbb{Z}} a_n h_b^{0,2n}(x)(\mathrm{ch}x)^{2n} e^{in(\phi+\psi)},$$

where $g = k_\phi a_x k_\psi$ and $a_n \in \mathbb{C}$.

Proof. Let $f = \sum_{n,m\in\mathbb{Z}} f_{n,m}$ be the K-type decomposition of f. Clearly $f_{n,m}(x) = O(h_{1/4a}^{0,0}(x)) = O(h_{1/4a}^{|n-m|,n+m}(x)\ (\mathrm{sh}x)^{|n-m|}(\mathrm{ch}x)^{n+m}(1+x)^{-|n-m|})$ (see (11)) and $\tilde{f}_{n,m}(\lambda) = O(e^{-b\lambda^2}) = O(Q_{n,m}(\lambda)e^{-b\lambda^2})$ (see (7)). Hence Theorem 3.1 implies that, if $ab > 1/4$, then $f_{n,m} = 0$, for all $n, m \in \mathbb{Z}$, and thus $f = 0$. If $ab = 1/4$, then $f_{n,m}$ is a constant multiple of $h_b^{|n-m|,n+m}(x)(\mathrm{sh}x)^{|n-m|}(\mathrm{ch}x)^{n+m}$. Since $f_{n,m}(x) = O(h_{1/4a}^{0,0}(x))$, it follows that $|n-m| = 0$. Therefore, f must be of the desired form. ■

4. Asymptotic behavior. We fix $t > 0$ and we shall consider an asymptotic behavior of $h_t^{0,2n}(x)$ when $x \to \infty$. For an even function f on $\mathbb{R}$ let $W_\mu^\sigma(f)$, $\mu \in \mathbf{C}$, $\sigma > 0$, denote the Weyl type fractional integral of f, which is defined by

$$W_\mu^\sigma(f)(y) = \Gamma(\mu)^{-1}\int_y^\infty f(x)(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu-1} d(\mathrm{ch}\sigma x) \tag{12}$$

for $\Re\mu > 0$ and is extended to an entire function in μ (see [9, (3,10), (3.11)]). We recall

$$(h_t^{\alpha,\beta})^\wedge_{\alpha,\beta}(\lambda) = \mathcal{F}\left(2^{3\alpha+3/2} W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(h_t^{\alpha,\beta})\right) = e^{-t\rho^2} e^{-t\lambda^2},$$

where $\mathcal{F}$ denotes the Euclidean Fourier transform (see [9, (3.12)]). Therefore, letting $\alpha = 0$ and $\beta = 2n$, $e^{-t((2n+1)^2} W^1_{-2n} \circ W^2_{2n+1/2}(h_t^{0,2n})$ does not depend on n. Hence, it follows that

$$h_t^{(0,2n)} = e^{-t((2n+1)^2-1)} W^2_{-1/2} \circ W^2_{-2n} \circ W^1_{2n} \circ W^2_{1/2}(h_t^{0,0}). \tag{13}$$

Lemma 4.1. *For $n = 0, 1, 2, \cdots$,*

$$W^2_{-n} \circ W^1_n(f)(x) = \sum_{l=0}^{n-1} c_l^n (\mathrm{ch}x)^{-(n+l)} W^1_l(f)(x)$$

and $c_0^n = 2^{-2n}$, $|c_l^n| \le |c_{n-1}^n| \le 2^{-2n}(2n-3)!!$ $(0 < l \le n-1)$.

Proof. Since

$$W_{-1}^2 = \frac{1}{2\text{sh}2x}\frac{d}{dx} = \frac{1}{4\text{ch}x}W_{-1}^1,$$

it follows from the induction on n that

$$W_{-n}^2 \circ W_n^1(f)(x) = \sum_{l=0}^{n-1} c_l^n (\text{ch}x)^{-(n+l)} W_l^1(f)(x)$$

and $c_l^n = \frac{1}{4}(c_l^{n-1} - (n+l-2)c_{l-1}^{n-1})$. Since $c_0^n = \frac{1}{4}c_l^{n-1}$ and $|c_l^n| \le |c_{n-1}^n| \le \frac{(2n-3)}{4}|c_{n-2}^{n-1}|$, the desired estimates on c_l^n follow. ■

Lemma 4.2. *Let* $a, b, c, \sigma, l > 0$ *and* $f(x) = O\left(e^{-ax-bx^2}(1+x)^c\right)$*. Then for* $\alpha = n + \mu \ge 0$ *where* $n \in \mathbb{N}$ *and* $0 < \mu \le 1$

$$(i)\ \ W_\alpha^\sigma(f)(x) = O\left(e^{\alpha\sigma x - ax - bx^2}(1+x)^{c-(n+1)}\right)$$

$$(ii)\ W_\alpha^\sigma\left((\text{ch}x)^{-l}f(x)\right) = (\text{ch}x)^{-l}W_\alpha^\sigma(f)(x) + O\left(e^{(\alpha\sigma-l)x-ax-bx^2}(1+x)^{c-(n+1)}\right).$$

Proof. We may suppose that x is sufficiently large and $n = 0$. As for (i), we need to estimate the integral of the form

$$\int_x^\infty e^{(\alpha-a)s-bs^2}s^c ds$$

(see (12)). Since $e^{(\alpha-a)s-bs^2}s^c \sim e^{-b(s-(\alpha-a)/2b)^2}(s-(\alpha-a)/2b)s^{c-1}$, the above integral behaves as $e^{(\alpha-a)x-bx^2}x^{c-1}$. Then the desired estimate follows from integration by parts. As for (ii), it is clear from (i). ■

Now we shall estimate $I(x) = W_{-1/2}^2 \circ W_{-2n}^2 \circ W_{2n}^1 \circ W_{1/2}^2(h_t^{0,0})(x)$ (see (13)). Applying Lemma 4.1, we see that

$$I(x) \quad = \quad W_{-1/2}^2 \circ \left(\sum_{l=0}^{2n-1} c_l^{2n}(\text{ch}x)^{-(2n+l)}W_l^1\right) \circ W_{1/2}^2(h_t^{0,0})(x)$$

$$
\begin{aligned}
&= W^2_{1/2} \circ W^2_{-1} \circ \left(\sum_{l=0}^{2n-1} c_l^{2n} (\mathrm{ch}x)^{-(2n+l)} W_l^1 \right) \circ W^2_{1/2}(h_t^{0,0})(x) \\
&= \sum_{l=0}^{2n-1} c_l^{2n} \Big(-\frac{2n+l}{4} W^2_{1/2} \left((\mathrm{ch}x)^{-(2n+l+2)} W_l^1 \circ W^2_{1/2}(h_t^{0,0}) \right)(x) \\
&+ W^2_{1/2} \left((\mathrm{ch}x)^{-(2n+l)} W^2_{-1} \circ W_l^1 \circ W^2_{1/2}(h_t^{0,0}) \right)(x) \Big) \\
&= \sum_{l=0}^{2n-1} c_l^{2n} \Big(-\frac{2n+l}{4} W^2_{1/2} \left((\mathrm{ch}x)^{-(2n+l+2)} W_l^1 \circ W^2_{1/2}(h_t^{0,0}) \right)(x) \Big) \\
&+ \sum_{l=1}^{2n-1} c_l^{2n} \Big(W^2_{1/2} \left((\mathrm{ch}x)^{-(2n+l)} W^2_{-1} \circ W_l^1 \circ W^2_{1/2}(h_t^{0,0}) \right)(x) \Big) \\
&+ c_0^{2n} W^2_{1/2} \left((\mathrm{ch}x)^{-2n} W^2_{-1/2}(h_t^{0,0}) \right)(x). \qquad (14)
\end{aligned}
$$

We recall that $h_t^{0,0}(x) \sim e^{x-x^2/4t}(1+x)^{1/2}$ (see (10)). Then Lemma 4.2 (i) implies that each term in the first sum in (14) behaves as $c_l^{2n} e^{-2n+x-x^2/4t}(1+x)^{1/2-\gamma} \sim c_l^{2n} h_t^{0,0}(x)(\mathrm{ch}x)^{-2n}(1+x)^{-\gamma}$, $\gamma \geq 1$. Similarly, since $W^2_{-1} \circ W^1_1 = (2\mathrm{ch}x)^{-1} W^2_1$, each term in the second sum in (14) has the same behaviour. On the other hand, it follows from Lemma 4.2 (ii) that the last term in (14) behaves as $c_0^{2n} h_t^{0,0}(x)(\mathrm{ch}x)^{-2n}(1+O((1+x)^{-\gamma}))$, $\gamma \geq 1$. Therefore, we see that

$$
I(x) = c_0^{2n} h_t^{0,0}(x)(\mathrm{ch}x)^{-2n} \left(1 + O\left((1 + 4n \frac{|c_{2n-1}^{2n}|}{c_0^{2n}})(1+x)^{-1} \right) \right).
$$

Since $e^{-t((2n+1)^2-1)} n(4n-3)!! \leq C$, it follows that

$$
h_t^{0,2n}(x) = h_t^{0,0}(x)(\mathrm{ch}x)^{-2n} \left(2^{-2n} e^{-t((2n+1)^2-1)} + O\left((1+x)^{-1} \right) \right). \qquad (15)
$$

5. Main theorem. Let $ab = 1/4$ and f be an L^2 function on G satisfying the assumptions in Theorem 3.2. Then Theorem 3.2 and (15) imply that f is of the form

$$
\begin{aligned}
f(g) &= \sum_{n \in \mathbb{Z}} a_n h_b^{0,2n}(x)(\mathrm{ch}x)^{2n} e^{in(\phi+\psi)} \\
&= \sum_{n \in \mathbb{Z}} a_n h_b^{0,0}(x) \left(2^{-2n} e^{-b((2n+1)^2-1)} + O((1+x)^{-1}) \right) e^{in(\phi+\psi)},
\end{aligned}
$$

where $g = k_\phi a_x k_\psi$ and $a_n \in \mathbb{C}$. Since $f \in L^2(G)$, the Plancherel formula on G and (9) imply that

$$\sum_{n\in\mathbb{Z}} |a_n|^2 e^{-2b((2n+1)^2} < \infty.$$

Therefore, it easily follows that, as an L^2 function on K,

$$\lim_{x\to\infty} (h_b^{0,0}(x))^{-1} f(k_\phi a_x) = \sum_{n\in\mathbb{Z}} a_n 2^{-2n} e^{-b((2n+1)^2-1)} e^{in\phi}. \quad (16)$$

Finally, we have the following.

Theorem 5.1 *Let $ab = 1/4$ and let f be an L^2 function on $SU(1,1)$ satisying*

$$(i) \quad f(x) = O\left(h_{1/4a}^{0,0}(x)\right)$$
$$(ii) \quad \tilde{f}_{n,m}(\lambda) = O\left(e^{-b\lambda^2}\right) \text{ for all } n, m \in \mathbb{Z}.$$

Then, as an L^2 function on K, $\lim_{x\to\infty}(h_b^{0,0}(x))^{-1} f(k_\phi a_x)$ exists;

$$\lim_{x\to\infty} (h_b^{0,0}(x))^{-1} f(k_\phi a_x) = F(\phi)$$

and moreover, f is uniquely determined as

$$f(g) = \sum_{n\in\mathbb{Z}} c_n 2^{2n} e^{b((2n+1)^2-1)} h_b^{0,2n}(x) (\mathrm{ch} x)^{2n} e^{in(\phi+\psi)},$$

where $g = k_\phi a_x k_\psi$ and $\{c_n\}$ the Fourier coefficients of F.

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An uncertainty principle on Sturm-Liouville hypergroups

R. Daher and T. Kawazoe

Abstract

As an analogue of the classical uncertainty inequality on the Euclidean space, we shall obtain a generalization on the Sturm-Liouville hypergroups $(\mathbb{R}_+, *(A))$. Especially, we shall obtain a condition on A under which the discrete part of the Plancherel formula vanishes.

1. Sturm-Liouville hypergroups. Sturm-Liouville hypergroups are a class of one-dimensional hypergroups on $\mathbb{R}_+ = [0, \infty)$ with the convolution structure related to the second order differential operators

$$L = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx},$$

where A satisfies the following conditions (see [**1**], [**2**]):

(1) $A > 0$ on $R_+^* = (0, \infty)$, and is in $C^2(\mathbb{R}_+^*)$,

(2) on a neighborhood of 0, $\dfrac{A'(x)}{A(x)} = \dfrac{2\alpha+1}{x} + B(x)$, $\alpha \geq -\frac{1}{2}$ and

  (a) if $\alpha > 0$, B and B' are integrable,

  (b) if $\alpha = 0$, $\log xB$ and $x \log xB'$ are integrable,

  (c) if $-\frac{1}{2} < \alpha < 0$, $x^{2\alpha}B$ and $x^{2\alpha+1}B'$ are integrable,

  (d) if $\alpha = -\frac{1}{2}$, B' is integrable,

(3) $\dfrac{A'}{A} \geq 0$ on $\mathbb{R}_+^*$ and $\lim_{x\to\infty} \dfrac{A'(x)}{A(x)} = 2\rho$,

(4) $\dfrac{1}{2}\Big(\dfrac{A'}{A}\Big)' + \dfrac{1}{4}\Big(\dfrac{A'}{A}\Big)^2 - \rho^2$ is integrable at ∞.

Since $A'/A = (\log A)'$, (3) implies that A is increasing, and thus, $A(0) < \infty$. Under the conditions (1) to (3), the second order differential equation: $Lu + (\lambda^2 + \rho^2) = 0$, $\lambda \in \mathbb{C}$, has a unique solution satisfying $u(0) = 1$, $u'(0) = 0$, which we denote by ϕ_λ. Furthermore, under (4), if $\Im\lambda \geq 0$, then there exists another solution $\psi_\lambda(x)$, which behaves as $\sqrt{\pi/2}\sqrt{\lambda x}H_\alpha^{(1)}$ at ∞, where $H_\alpha^{(1)}$ is the Hankel function. Similarly, we have $\psi_\lambda^-(x)$ for $\Im\lambda \leq 0$, and for $\lambda \in \mathbb{R}_+^*$, there exists $C(\lambda) \in \mathbb{C}$ such that $\phi_\lambda(x) = C(\lambda)\psi_\lambda(x) + \overline{C(\lambda)}\psi_\lambda^-(x)$.

Let $C_{c,e}^\infty(\mathbb{R})$ denote the set of C^∞ even functions f on $\mathbb{R}$. For $f \in C_{c,e}^\infty(\mathbb{R})$ the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x)A(x)dx.$$

Then the inverse transform is given as

$$f(x) = \sum_{\Lambda\in D} \pi_\Lambda \hat{f}(\Lambda)\phi_\Lambda(x) + \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)\frac{d\lambda}{|C(\lambda)|^2},$$

where D is a finite set in the interval $i(0,\rho)$ and $\pi_\lambda = \|\phi_\Lambda\|^{-2}_{L^2(\mathbb{R}_+,Adx)}$. We denote this decomposition as

$$f = {}^\circ f + f_P$$

and we call f_P and ${}^\circ f$ the principal part and the discrete part of f respectively. We denote by $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\Lambda\})$ a function on $\mathbb{R}_+ \cup D$ defined by

$$\boldsymbol{F}(\nu) = \begin{cases} F(\lambda) & \text{if } \nu = \lambda \in \mathbb{R}_+ \\ a_\Lambda & \text{if } \nu = \Lambda \in D. \end{cases}$$

We put $\overline{\boldsymbol{F}}(\nu) = (\overline{F(\lambda)}, \{\overline{a_\Lambda}\})$ and define the product of $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\Lambda\})$ and $\boldsymbol{G}(\nu) = (G(\lambda), \{b_\Lambda\})$ as

$$(\boldsymbol{F}\boldsymbol{G})(\nu) = (F(\lambda)G(\lambda), \{a_\Lambda b_\Lambda\}).$$

Let $d\nu$ denote the measure on $\mathbb{R}_+ \cup D$ defined by

$$\int_{\mathbb{R}_+\cup D} \boldsymbol{F}(\nu)d\nu = \sum_{\Lambda\in D} \pi_\Lambda a_\Lambda + \frac{1}{2\pi}\int_0^\infty F(\lambda)|C(\lambda)|^{-2}d\lambda.$$

For $f \in C_{c,e}^\infty(\mathbb{R})$, we put

$$\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{\hat{f}(\Lambda)\}).$$

Then the Parseval formula on $C_{c,e}^\infty(\mathbb{R})$ can be stated as follows: For $f, g \in C_{c,e}^\infty(\mathbb{R})$

$$\int_0^\infty f(x)\overline{g(x)}A(x)dx = \int_{\mathbb{R}_+\cup D} \hat{\boldsymbol{f}}(\nu)\overline{\hat{\boldsymbol{g}}(\nu)}d\nu. \tag{5}$$

The map $f \to \hat{\boldsymbol{f}}$, $f \in C^\infty_{c,e}(\mathbb{R})$, is extended to an isometry between $L^2(A) = L^2(\mathbb{R}_+, A(x)dx)$ and $L^2(\nu) = L^2(\mathbb{R}_+ \cup D, d\nu)$. Actually, each function f in $L^2(A)$ is of the form

$$\begin{aligned} f(x) &= \sum_{\Lambda \in D} \pi_\Lambda \hat{f}_\Lambda \phi_\Lambda(x) + \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda \\ &= {}^\circ f + f_P \end{aligned}$$

and their L^2-norms are given as

$$\int_0^\infty |{}^\circ f(x)|^2 A(x)dx = \sum_{\Lambda \in D} \pi_\Lambda |\hat{f}_\Lambda|^2,$$

$$\int_0^\infty |f_P(x)|^2 A(x) = \frac{1}{2\pi}\int_0^\infty |\hat{f}_P(\lambda)|^2 |C(\lambda)|^{-2} d\lambda.$$

Therefore, if we define $\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{\hat{f}_\Lambda\})$, then $\|f\|_{L^2(A)} = \|\hat{\boldsymbol{f}}\|_{L^2(\nu)}$ holds. In particular, if $f \in C^\infty_{c,e}(\mathbb{R})$, then $\hat{f}_\Lambda = \hat{f}(\Lambda)$ for all $\Lambda \in D$.

2. Uncertainty inequality. We retain the notations in the previous sections. We put for $x \in \mathbb{R}_+$,

$$a(x) = \int_0^x A(t)dt \quad \text{and} \quad v(x) = \frac{a(x)}{A(x)} \tag{6}$$

and for $\lambda \in \mathbb{C}$,

$$w(\lambda) = (\lambda^2 + \rho^2)^{1/2}.$$

Theorem 2.1. *For all $f \in L^1(A) \cap L^2(A)$,*

$$\|fv\|^2_{L^2(A)} \int_{\mathbb{R}_+ \cup D} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \geq \frac{1}{4}\|f\|^4_{L^2(A)}, \tag{7}$$

where the equality holds if and only if f is of the form

$$f(x) = ce^{\gamma \int_0^x v(t)dt}$$

for some $c, \gamma \in \mathbb{C}$ and $\Re\gamma < 0$.

Proof. Without loss of generality we may suppose that $f \in C^\infty_{c,e}(\mathbb{R})$. Since $(-Lf)^\wedge(\lambda) = \hat{f}(\lambda)(\lambda^2+\rho^2) = \hat{f}(\lambda)w(\lambda)^2$ and $w(\lambda)$ is positive on $\mathbb{R}_+ \cup D$, the Parseval formula (5) yields that

$$\begin{aligned}\int_{\mathbb{R}_+\cup D} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu &= \int_0^\infty (-Lf)(x)\overline{f(x)}A(x)dx \\ &= \int_0^\infty |f'(x)|^2 A(x)dx.\end{aligned}$$

Hence it follows that

$$\begin{aligned}&\int_0^\infty |f(x)|^2 v(x)^2 A(x)dx \int_{\mathbb{R}_+\cup D} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \\ =& \int_0^\infty |f(x)|^2 v(x)^2 A(x)dx \int_0^\infty |f'(x)|^2 A(x)dx \\ \geq& \Big(\int_0^\infty \Re(f(x)f'(x))v(x)A(x)dx\Big)^2 \\ =& \frac{1}{4}\Big(\int_0^\infty (|f(x)|^2)'a(x)dx\Big)^2 = \frac{1}{4}\Big(\int_0^\infty |f(x)|^2 A(x)dx\Big)^2.\end{aligned}$$

Here we used the fact that $a' = A$ (see (6)). Clearly, the equality holds if and only if $fv = cf'$ for some $c \in \mathbb{C}$, that is, $f'/f = c^{-1}v$. This means that $\log(f) = c^{-1}\int_0^x v(t)dt + C$ and thus, the desired result follows. ■

Remark 2.2. When $(\mathbb{R}_+, *(A))$ is the Bessel-Kingman hypergroup, the equality holds for $e^{\gamma x^2}$, $\Re\gamma < 0$. However, when it is the Jacobi hypergroup, each function satisfying the equality has an exponential decay $e^{\gamma x}$.

Since $w^2(\lambda) = \lambda^2 + \rho^2$, (7) can be rewritten as follows.

Corollary 2.3. *Let f be the same as in Theorem 2.1.*

$$\|fv\|^2_{L^2(A)}\int_{\mathbb{R}_+\cup D}|\hat{\boldsymbol{f}}(\nu)|^2\nu^2 d\nu \geq \frac{1}{4}\|f\|^2_{L^2(A)}\int_0^\infty |f(x)|^2(1-4\rho^2 v(x)^2)A(x)dx. \quad (8)$$

3. Vanishing condition of the discrete part. We shall prove that under the assumption:

$$0 \leq v(x) \leq \frac{1}{2\rho}, \quad (9)$$

it follows that $D = \emptyset$. We suppose that $D \neq \emptyset$ and we take $f = \pi_\Lambda \phi_\Lambda$, $\Lambda \in D$. Then, since $\hat{\boldsymbol{f}}(\nu) = 1$ if $\nu = \Lambda$ and 0 otherwise, it follows from (8) that

$$\|fv\|_{L^2(A)}^2 \pi_\Lambda \Lambda^2 \geq \frac{1}{4} \|f\|_{L^2(A)}^2 \int_0^\infty |f(x)|^2 (1 - 4\rho^2 v(x)^2) A(x) dx.$$

Here we recall that $\Lambda^2 < 0$, because $D \subset i(0, \rho)$ and $1 - 4\rho^2 v(x)^2 \geq 0$ by (9). This is contradiction. Therefore, we obtain the following

Theorem 3.1. *If* $0 \leq v \leq \dfrac{1}{2\rho}$, *then* $D = \emptyset$.

For example, if A satisfies the inequality:

$$a(x)A'(x) = \int_0^x A(x)dx \cdot A'(x) \leq A^2(x), \tag{10}$$

then A satisfies (9). Actually, (10) implies

$$v'(x) = \frac{A^2(x) - a(x)A'(x)}{A^2(x)} \geq 0.$$

Hence v is increasing on $\mathbb{R}_+$ and $v(x) = a(x)/A(x) \leq A(x)/A'(x)$ because $A/A' > 0$ by (3). Then it follows from (3) that A satisfies (9).

Corollary 3.2. *If A satisfies the inequality* (10), *then* $D = \emptyset$.

Remark 3.3. It is well-known that $D = \emptyset$ for Chébli-Triméche hypergroups where A'/A is decreasing and (4) is not required (cf. [**1**]). This fact easily follows from our argument. Since A/A' is increasing and $0 \leq A/A' \leq 1/2\rho$ by (3), we see that $a \leq A/2\rho$ by integration and thus, (9) holds. Hnece $D = \emptyset$ by Theomre 3.1.

4. Uncertainty principle. We suppose that $D = \emptyset$. Then (8) is of the form:

$$\|fv\|_{L^2(A)}^2 \frac{1}{2\pi} \int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 \frac{d\lambda}{|C(\lambda)|^2} \geq \frac{1}{4} \|f\|_{L^2(A)}^2 \int_0^\infty |f(x)|^2 (1 - 4\rho^2 v(x)^2) A(x) dx.$$

Since v is increasing, $v(0) = 0$, and $1 - 4\rho^2 v(x)^2 \geq 0$ by (9), it follows that f and $\hat{f}$ both cannot be concentrated around the origin.

In general, if $D \neq \emptyset$, then we must pay attention to the discrete part of f to consider uncertainty principles. We refer to [**3**] for the Jacobi hypergroups.

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H^1-estimates of the Littlewood-Paley and Lusin functions on real rank 1 semisimple Lie groups

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1. Introduction.

Let G be a connected semisimple Lie group with finite center. On G the Littlewood-Paley g-function and the Lusin area function $S(f)$ are introduced as

$$g(f)(x) = \Big(\int_0^\infty \Big| t \frac{\partial}{\partial t} f * p_t(x) \Big|^2 \frac{dt}{t} \Big)^{1/2},$$

$$S(f)(x) = \Big(\int_0^\infty \chi_t * \Big| t \frac{\partial}{\partial t} p_t * f(x) \Big|^2 \frac{dt}{t} \Big)^{1/2}.$$

Here $*$ denotes the convolution on G, p_t the Poisson kernel on G, and $\chi_t(x) = |B(t)|^{-1}\chi_{B(t)}(x)$, where $\chi_{B(t)}$ is the characteristic function of the ball $B(t)$ on G with radius t centered at the origin and $|B(t)|$ the volume of $B(t)$. As shown in [1], [5], [6], these operators satisfy the maximal theorem: Let $L^p(G/K)$ denote the space of all p-th integrable right K-invariant functions on G. Then g and S are bounded from $L^p(G/K)$ to $L^p(G/K)$ for $1 < p < \infty$ and satisfy a weak type L^1 estimate on G. As well-known, in the Euclidean space $\mathbb{R}$ these operators are bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$. Our aim is to introduce H^1 space on G and obtain an analogous result on G when $p = 1$.

In the following, we restrict our attention to real rank one semisimple Lie groups G and we treat K-bi-invariant functions on G. Since G has the Cartan decomposition $G = K\overline{A^+}K$, we can identify K-bi-invariant functions on G with even functions on $\mathbb{R}$. Therefore, under the restriction, harmonic analysis for K-bi-invariant functions on (G, dg), dg a Haar measure on G, is reduced to one on $(\mathbb{R}_+, \Delta(x)dx)$ where $\Delta(x)$ is a weight function on $\mathbb{R}_+$. Here the space $(\mathbb{R}_+, \Delta(x)dx)$ is not of homogeneous type, because Δ has a exponential growth order (see (1)).

In §3 we introduce H^1-real Hardy spaces on G as

$$H^1_\sharp(G//K) \subset H^1_\flat(G//K) \subset H^1(G//K) \subset L^1(G//K)$$

and give their characterizations (see (7)). In §4 we shall consider the H^1-estimate of the Littlewood-Paley g-function on G and show that g is bounded from $H^1(G//K)$ to $L^1(G//K)$ (see Theorem 4.5). In §5 we treat a modified Lusin area function $S_+(f)$ on G:

$$S_+(f)(x) = \Big(\int\int \Theta(x,y)\chi_t(xy^{-1}) \Big| t\frac{\partial}{\partial t} f * p_t(y) \Big|^2 dy \frac{dt}{t} \Big)^{1/2},$$

where $\Theta(x,y) \geq 1$ if $\sigma(y) \geq \sigma(x)$ and $\Theta(x,y) < 1$ otherwise (see (14)). We obtain that S_+ is bounded from $H^1_\flat(G//K)$ to $L^1(G//K)$ (see Theorem 5.1).

2. Notations.

We suppose that G is of real rank one, that is, $\dim A = 1$ and A is identified with $\mathbb{R}$. Let $\mathfrak{a}$ be the Lie algebra of A and $\mathsf{F} = \mathfrak{a}^*$ the dual space of $\mathfrak{a}$. Let γ be a positive simple root of (G, A) and H the unique element in $\mathfrak{a}$ satisfying $\gamma(H) = 1$. Then we can parameterize each element in A, $\mathfrak{a}$, and F as $a_x = \exp(xH)$, xH, and $x\gamma$ $(x \in \mathbb{R})$

respectively. Let m_1 and m_2 denote the multiplicities of γ and 2γ respectively and let

$$\alpha = \frac{m_1 + m_2 - 1}{2},\ \beta = \frac{m_2 - 1}{2},\ \rho = \alpha + \beta + 1,\ \gamma_\alpha = \alpha + 1/2.$$

According to the classification of G, these numbers belong to $\mathbb{Z}/2$. All K-bi-invariant functions f on G are identified with even functions on $\mathbb{R}$, which we denoted by the same symbol: $f(g) = f(a_{\sigma(g)}) = f(\sigma(g))$, where $\sigma : G \to \mathbb{R}_+$ is the distance function on G (cf. [**9**, 8.1.2]). Conversely, for a function F on $\mathbb{R}$, we define $F_+(g) = F(\sigma(g))$, $g \in G$, that is, $F_+(x) = F(|x|)$, $x \in \mathbb{R}$. When F is even, we abbreviate F_+ as F.

Let $dg = \Delta(x)dkdxdk'$ denote the decomposition of a Haar measure dg on G according to the Cartan decomposition of G, where dk, dx denote Haar measures on K, A respectively. We normalize $\Delta(x)$, $x \geq 0$, as

$$\Delta(x) = 2^{2\rho}(\mathrm{sh}x)^{2\alpha+1}(\mathrm{ch}x)^{2\beta+1} \tag{1}$$

and extend it as an even function on $\mathbb{R}$. We note $\Delta(x) \sim e^{2\rho x}$ as x goes to ∞. Let $L^p(G//K)$ denote the space of K-bi-invariant functions on G with finite L^p-norm: $\|f\|_{L^p(G)} = \left(\int_0^\infty |f(x)|^p \Delta(x)dx\right)^{1/p}$. We denote by $L^1_{\mathrm{loc}}(G//K)$ the space of locally integrable, K-bi-invariant functions on G and by $C_c^\infty(G//K)$ the space of compactly supported C^∞, K-bi-invariant functions on G.

The spherical Fourier transform $\hat{f}$ for $f \in C_c^\infty(G//K)$ is defined by

$$\hat{f}(\lambda) = \int_G f(g)\phi_\lambda(g)dg = \int_0^\infty f(x)\phi_\lambda(x)\Delta(x)dx,\ \lambda \in \mathbb{R}, \tag{2}$$

where $\phi_\lambda(g)$ denotes the zonal spherical function on G(cf. [10,Chap.9]). We refer to [4] for some basic properties of $\hat{f}$. The map $f \to \hat{f}$ is a bijection of $C_c^\infty(G//K)$ onto the space of entire holomorphic functions of exponential type, and the inverse transform is given as

$$f(x) = \int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda,\ x \in \mathbb{R},$$

where $C(\lambda)$ is Harish-Chandra's C-function (cf. [4, (2.6)]). Furthermore, the map $f \to \hat{f}$ extends to an isometry of $L^2(G//K)$ onto $L^2(\mathbb{R}_+, |C(\lambda)|^{-2}d\lambda)$.

For $f \in C_c^\infty(G//K)$ we define the Abel transform F_f^s, $s \in \mathbb{R}$, as $F_f^s(x) = e^{\rho(1+s)x}\int_N f(a_x n)dn,\ x \in \mathbb{R}$, where N is a maximal nilpotent subgroup of G. We put $W_+^s(f) = F_f^s$ and W_-^s the inverse operator of W_+^s. These operators are explicitly given by using generalized Weyl type fractional operators on G (see [4, §6]):

$$W_+^s(f) = e^{s\rho x}W^1_{\alpha-\beta}\circ W^2_{\beta+1/2}(f),\ W_-^s(f) = W^2_{-(\beta+1/2)}\circ W^1_{-(\alpha-\beta)}(e^{-s\rho x}f),$$

where $W_\mu^\sigma(f)$, $\mu, \sigma > 0$, is the fractional integral on G defined by

$$W_\mu^\sigma(f)(y) = \Gamma(\mu)^{-1}\int_y^\infty f(x)(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu-1}d(\mathrm{ch}\sigma x)$$

and $W_\mu^\sigma(f)$, $\mu < 0$, the fractional derivative on G, is defined by the analytic continuation on μ. Clearly, $F_f^s \in C_c^\infty(\mathbb{R})$ and the Euclidean Fourier transform $(F_f^s)^\sim$ is related with $\hat{f}$ as follows (cf. [**4**, (3.7)]):

$$\hat{f}(\lambda + is\rho) = (F_f^s)^\sim(\lambda), \quad \lambda \in \mathbb{C}. \tag{3}$$

We suppose that $f \in L^p(G//K)$, $1 \leq p < 2$. We recall (cf. [**2**, Lemma 13]) that there exists a positive constant c such that for all $\lambda \in \mathbb{C}$ and $x \geq 0$, $|\phi_\lambda(x)| \leq c(1+x)e^{(\Im\lambda-\rho)x}$. Hence $|\phi_\lambda(x)|^{p'}\Delta(x)$ is dominated by $(1+x)^{p'}\ e^{(p|\Im\lambda|+(p-2)\rho)x/(p-1)}$ (see (1)). Then, applying

Hörder's inequality to (2), we see that $\hat{f}(\lambda)$ has a bounded holomorphic extension on the tube domain:

$$\mathsf{F}_p = \{\lambda \in \mathbb{C}\ ;\ |\Im\lambda| < (2/p-1)\rho\}$$

and (3) holds for $0 \le s < 2/p - 1$.

3. Real Hardy spaces.

We shall introduce real Hardy spaces on G. As shown in [3, Proposition 4.5, Lemma 4.4] the Weyl type fractional derivative W_-^1 on $(\mathbb{R}_+, \Delta(x)dx)$ is related to the classical Weyl type derivative $W_{-\gamma}^{\mathbb{R}}$ on $(\mathbb{R}_+, dx)$ as follows. For a smooth function F on $\mathbb{R}_+$,

$$\begin{aligned} |W_-^1(F)(x)| \ \le\ & c\Delta(x)^{-1}\sum_{\gamma\in\Gamma}\Big((\mathrm{th}x)^{\gamma}|W_{-\gamma}^{\mathbb{R}}(F)(x)| \\ & +(\mathrm{th}x)^{\gamma-1}\int_x^\infty |W_{-\gamma}^{\mathbb{R}}(F)(s)|A_\gamma(x,s)ds\Big), \qquad (4)\end{aligned}$$

where Γ is a finite set of real numbers γ such that $0 \le \gamma \le \gamma_\alpha$, $A_\gamma(x,s) \equiv 0$ if $\gamma = 0$, $A_\gamma(x,s) \ge 0$, and there exists a constant c such that

$$\int_0^s (\mathrm{th}x)^{\gamma-1}A_\gamma(x,s)dx \le c(\mathrm{th}s)^\gamma \text{ for all } s \ge 0. \qquad (5)$$

More precisely, if $\alpha - \beta$ and $\beta + 1/2$ are both integer, then γ_α and all $\gamma \in \Gamma$ are integers and no integral terms appear in (4). In other cases, γ_α and all $\gamma \in \Gamma$ are half-integer, and $A_\gamma(x,s)$ is dominated by $\chi_{[0,\infty)}(s-x)\chi_{[0,1]}(s)$ or $B_\gamma(s-x)$, where $B_\gamma(x)$ is a bounded integrable function on $\mathbb{R}_+$. We note that, if γ_α is half-integer, then $1/2 \le \gamma \le \gamma_\alpha$ and each integral term in (4) can be rewritten as

$$\begin{aligned} & (\mathrm{th}x)^{\gamma-1}\int_x^\infty |W_{-\gamma}^{\mathbb{R}}(F)(s)|A_\gamma(x,s)ds \\ = \ & (\mathrm{th}x)^{\gamma-1/2}\int_x^\infty |W_{-\gamma}^{\mathbb{R}}(F)(s)|\frac{A_\gamma^+(x,s)}{\sqrt{s}}ds, \qquad (6)\end{aligned}$$

where $A_\gamma^+(x,s)$ satisfy the same property of $A(x,s)$ (see (5)). We suppose that $f \in L^1(G//K)$ and put $F = W_+^1(f)$. Since $f = W_-^1 \circ W_+^1(f) = W_-^1(F)$, the argument used in the proof of [3, Theorem 4.6] with (4) and (5) yields that

$$\|f\|_{L^1(G)} \sim \sum_{\gamma\in\Gamma} \|W_{-\gamma}^{\mathbb{R}}(F)\|_{L^1_{w_\gamma}(\mathbb{R})},$$

where $W_{-\gamma}^{\mathbb{R}}(F)$ is regarded as an even function on $\mathbb{R}$, $w_\gamma(x) = (\mathrm{th}_+ x)^\gamma$ and $L^1_{w_\gamma}(\mathbb{R})$ is the w_γ-weighted L^1-space on $\mathbb{R}$.

Let ϕ be a smooth K-bi-invariant function on G satisfying $\int_G \phi(g)\, dg \neq 0$ and M_ϕ the corresponding radial maximal operator on G (see [3, §3]). As in the Euclidean space, for $p \geq 1$ we put

$$H^p(G//K) = \{f \in L^1_{\mathrm{loc}}(G//K)\ ;\ M_\phi(f) \in L^p(G//K)\}$$

and $\|f\|_{H^1(G)} = \|M_\phi(f)\|_{L^p(G)}$. Then it follows from [3, §4] that

$$H^1(G//K) \subset L^1(G//K), \quad H^p(G//K) = L^p(G//K) \text{ for } 1 < p < \infty.$$

Moreover, [3, Theorem 4.6] yields that

$$\|f\|_{H^1(G)} \sim \sum_{\gamma\in\Gamma} \|W_{-\gamma}^{\mathbb{R}}(F)\|_{H^1_{w_\gamma}(\mathbb{R})}, \tag{7}$$

where $F = W_+^1(f)$ and $H^1_{w_\gamma}(\mathbb{R})$ is the w_γ-weighted H^1-space on $\mathbb{R}$. We recall that, if γ_α is a half-integer, then (6) holds. Hence we introduce

$$\|f\|_{H^1_\flat(G)} = \sum_{\gamma\in\Gamma} \|W_{-\gamma}^{\mathbb{R}}(F)\|_{H^1_{w_{\gamma-1/2}}(\mathbb{R})}$$

and $H^1_\flat(G//K)$ as the space of all $f \in L^1_{\mathrm{loc}}(G//K)$ with finite $H^1_\flat(G)$-norm. Clearly (7) implies that $H^1_\flat(G//K) \subset H^1(G//K)$.

Let $s = 2/p - 1$ for $1 \leq p \leq 2$ and M_{C_s} the Euclidean Fourier multiplier corresponding to a C-function $C_s(\lambda) = C(-(\lambda + is\rho))$. We define

$$H^p_\sharp(G//K) = \{f \in L^1_{\rm loc}(G//K) \; ; \; \mathsf{M}^{-1}_{C_s} \circ W^s_+(f) \in H^p(\mathbb{R})\}$$

and $\|f\|_{H^p_\sharp(G)} = \|\mathsf{M}^{-1}_{C_s} \circ W^s_+(f)\|_{H^p(\mathbb{R})}$. Here, when $1 \leq p < 2$, f is in $H^p_\sharp(G//K)$ implies that $\hat{f}(\lambda)$ has a holomorphic extension on F_p, $(W^s_+(f))^\sim(\lambda) = \hat{f}(\lambda + is\rho)$ is well-defined as a locally integrable function on the boundary of F_p and

$$\begin{aligned} f_{\sharp,p}(x) &= \mathsf{M}^{-1}_{C_s} \circ W^s_+(f)(x) \\ &= \int_{-\infty}^{\infty} \hat{f}(\lambda + is\rho) C(-(\lambda + is\rho))^{-1} e^{i\lambda x} d\lambda, \; x \in \mathbb{R} \end{aligned}$$

is well-defined as an H^p-function on $\mathbb{R}$. When $p = 2$, the Plancherel formula on G implies that $H^2_\sharp(G//K) = L^2(G//K)$. We recall that $C(-(\lambda + i\rho)) \sim (1 + |\lambda|)^{-\gamma_\alpha}$ (cf. [2, Theorem 2]). Since $(i\lambda)^\gamma(\lambda + i\rho)^{-\gamma_\alpha}$, $0 \leq \gamma \leq \gamma_\alpha$, satisfies the Hörmander condition (cf. [9, p.318]), it follows from [9, p.363] that, if $H \in H^1(\mathbb{R})$, then $W^{\mathbb{R}}_{-\gamma}(\mathsf{M}_{C_1}(H))$ belongs to $H^1(\mathbb{R})$ and $\|W^{\mathbb{R}}_{-\gamma}(\mathsf{M}_{C_1}(H))\|_{H^1(\mathbb{R})} \leq c\|H\|_{H^1(\mathbb{R})}$. In particular, if $f \in H^1_\sharp(G//K)$, then $\mathsf{M}^{-1}_{C_\rho}(F)$, $F = W^1_+(f)$, belongs to $H^1(\mathbb{R})$ and $\|\mathsf{M}^{-1}_{C_1}(F)\|_{H^1(\mathbb{R})} = \|f\|_{H^1_\sharp(G)}$ by the definition and $\|W^{\mathbb{R}}_{-\gamma}(F)\|_{H^1_{w_{\gamma'}}(\mathbb{R})} \leq \|W^{\mathbb{R}}_{-\gamma}(\mathsf{M}_{C_1}(\mathsf{M}^{-1}_{C_1}(F))\|_{H^1(\mathbb{R})} \leq c\|f\|_{H^1_\sharp(G)}$ for $0 \leq \gamma, \gamma' \leq \gamma_\alpha$. Finally, it follows that

$$H^1_\sharp(G//K) \subset H^1_\flat(G//K) \subset H^1(G//K) \subset L^1(G//K).$$

4. Littlewood-Paley g-function.

As an application of the real Hardy space $H^1(G//K)$, we shall consider an (H^1, L^1) bound on G for the Littelwood-Paley g-function $g(f)$ on G (see §1). For simplicity, we put $K_t = t(\partial/\partial t)p_t$. Since $t(\partial/\partial t)f * p_t = W_-^1(W_+^1(f) \circledast W_+^1(K_t)) = W_-^1(F \circledast W_+(K_t))$, where $F = W_+^1(f)$ and $\circledast$ is the convolution on $\mathbb{R}$, $g(f)$ can be rewritten as

$$g(f)(x) = \Big(\int_0^\infty \Big| W_-^1(F \circledast W_+^1(K_t))(x) \Big|^2 \frac{dt}{t} \Big)^{1/2}. \tag{8}$$

Proposition 4.1. *We define an operator $g_{\mathbb{R}}$ on $\mathbb{R}_+$ by*

$$g_{\mathbb{R}}(H)(x) = \Big(\int_0^\infty |H \circledast W_+(K_t)(x)|^2 \frac{dt}{t} \Big)^{1/2}.$$

Then

$$\|g(f)\|_{L^1(G)} \leq c \sum_{\gamma \in \Gamma} \|g_{\mathbb{R}}(W_{-\gamma}^{\mathbb{R}}(F))\|_{L^1_{w_\gamma}(\mathbb{R})}.$$

Proof. We substitute (4) to (8) and take the integration over $\mathbb{R}_+$ with respect to $\Delta(x)dx$. Since $W_{-\gamma}^{\mathbb{R}}(F \otimes W_+^1(K_t)) = W_{-\gamma}^{\mathbb{R}}(F) \otimes W_+^1(K_t)$, the $\Delta(x)dx$-integration of the first term in the right hand side of (4) is clearly dominated by $\|g_{\mathbb{R}}(W_{-\gamma}^{\mathbb{R}}(F))\|_{L^1_{w_\gamma}(\mathbb{R})}$. As for the integral term in (4), we denote $H(s,t) = W_{-\gamma}^{\mathbb{R}}(F) \circledast W_+(K_t)(s)$ and $A(x,s) = (\text{th}x)^{\gamma-1}A_\gamma(x,s)$ for simplicity. Then by (5) the $\Delta(x)dx$-integration is estimated as

$$\begin{aligned}
&\int_0^\infty \Big(\int_0^\infty \Big| \Delta(x)^{-1} \int_x^\infty H(s,t)A(x,s)ds \Big|^2 \frac{dt}{t} \Big)^{1/2} \Delta(x)dx \\
\leq\ & \int_0^\infty \int_x^\infty \Big(\int_0^\infty |H(s,t)|^2 \frac{dt}{t} \Big)^{1/2} A(x,s)dsdx \\
=\ & \int_0^\infty \Big(\int_0^\infty |H(s,t)|^2 \frac{dt}{t} \Big)^{1/2} \Big(\int_0^s A(x,s)dx \Big) ds \\
\leq\ & c \int_0^\infty \Big(\int_0^\infty |H(s,t)|^2 \frac{dt}{t} \Big)^{1/2} (\text{th}s)^\gamma ds = \|g_{\mathbb{R}}(W_{-\gamma}^{\mathbb{R}}(F))\|_{L^1_{w_\gamma}(\mathbb{R})}. \quad \blacksquare
\end{aligned}$$

Now we suppose that $f \in H^1(G//K)$, that is, each $W^{\mathbb{R}}_{-\gamma}(F)$ belongs to $H^1_{w_\gamma}(\mathbb{R})$ (see (7)). Therefore, Proposition 4.1 implies that the (H^1, L^1) bound on G for g is reduced to an $(H^1_{w_\gamma}, L^1_{w_\gamma})$ bound on $\mathbb{R}$ for $g_{\mathbb{R}}$. Actually, if $g_{\mathbb{R}}$ is bounded from $H^1_{w_\gamma}(\mathbb{R})$ to $L^1_{w_\gamma}(\mathbb{R})$ for $\gamma \in \Gamma$, it follows that $\|g(f)\|_{L^1(G)} \leq c\sum_{\gamma\in\Gamma} \|g_{\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))\|_{L^1_{w_\gamma}(\mathbb{R})} \leq c\sum_{\gamma\in\Gamma} \|W^{\mathbb{R}}_{-\gamma}(F)\|_{H^1_{w_\gamma}(\mathbb{R})} = \|f\|_{H^1(G)}$.

We shall prove that $g_{\mathbb{R}}$ is bounded form $H^1_{w_\gamma}(\mathbb{R})$ to $L^1_{w_\gamma}(\mathbb{R})$. Let $H \in H^1_{w_\gamma}$ and $H = \sum_i \lambda_{\gamma,i} A_{\gamma,i}$ denote a $(1,\infty,1)$-atomic decomposition of H, where $\lambda_{\gamma,i} \geq 0$, $A_{\gamma,i}$ is a $(1,\infty,1)$-atom on $\mathbb{R}$ supported on $B_{\gamma,i} = B(x_{\gamma,i}, r_{\gamma,i})$ and

$$\| \sum_i \lambda_{\gamma,i} r^{-1}_{\gamma,i} \chi_{B_{\gamma,i}} \|_{L^1_{w_\gamma}(\mathbb{R})} \leq \|H\|_{H^1_{w_\gamma}(\mathbb{R})} \tag{9}$$

(see [8, Chap. 8]). In what follows we shall determine a shape of $g_{\mathbb{R}}(A)(x)$ for each $(1,\infty,1)$-atom A on $\mathbb{R}$. We may suppose that A is centered, that is, A is supported on $[-r,r]$, $\|A\|_\infty \leq (2r)^{-1}$ and $\int_{-\infty}^{\infty} A(x)x^k dx = 0$ for $k = 0, 1$.

Proposition 4.2. *$g_{\mathbb{R}}$ is L^2 bounded on $\mathbb{R}$.*

Proof. We recall that $\hat{p}_t(\lambda) = e^{-t\sqrt{\lambda^2+\rho^2}}$ and $W_+(K_t)^\sim = t(\partial/\partial t)\hat{p}_t$. Hence

$$\begin{aligned}
& \|g_{\mathbb{R}}(H)\|^2_{L^2(\mathbb{R})} \qquad (10)\\
= & \int_0^\infty \|H \circledast W_+(K_t)\|^2_{L^2(\mathbb{R})} \frac{dt}{t} \\
= & \int_0^\infty \|\tilde{H} \cdot W_+(K_t)^\sim\|^2_{L^2(\mathbb{R})} \frac{dt}{t} \\
= & \int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 \left(\int_0^\infty t|\lambda(\lambda+2i\rho)| e^{-2t\Re\sqrt{\lambda(\lambda+2i\rho)}} dt \right) d\lambda \\
= & \int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 \left(\int_0^\infty t r e^{-2t\sqrt{r}\cos(\theta/2)} dt \right) d\lambda \leq c\|H\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where we set $\lambda(\lambda+2i\rho)=re^{i\theta}$ and we used the fact that $\cos\theta\geq 0$ and $\cos(\theta/2)=\sqrt{(\cos\theta+1)/2}\geq 1/\sqrt{2}$. ■

In particular, we have

$$\int_0^\infty g_{\mathbb{R}}(A)^2(x)dx \leq c\|A\|^2_{L^2(\mathbb{R})} \leq cr^{-1}. \tag{11}$$

Next we suppose that $|x|\geq 2r$. We recall the asymptotic behavior of $W_+(K_t)$ (cf. [1], p. 289):

$$\begin{aligned} W_+(K_t)(x) &= te^{\rho x}(\partial/\partial t)W_+(p_t)(x) = te^{\rho x}(\partial/\partial t)F^0_{p_t}(x) \\ &= Cte^{\rho x}(\partial/\partial t)\Big(t(t^2+x^2)^{-1/2}\mathsf{K}_1(\rho(t^2+x^2)^{1/2}) \\ &\qquad \times e^{\rho(t^2+x^2)^{1/2}}\cdot e^{-\rho(t^2+x^2)^{1/2}}\Big), \end{aligned}$$

where K_ν is the modified Bessel function, which satisfies $(d/dx)^k\mathsf{K}_\nu(x) = O(x^{-1/2-k}e^{-x})$ if $x\to\infty$, and $O(x^{-\nu-k})$ if $x\to 0$. Moreover, as a function $t\in\mathbb{R}_+$, $t^{2l}e^{-\rho(t^2+x^2)^{1/2}}$, $l\in\mathbb{R}$, has the maximum $O(|x|^l e^{-\rho x})$ at $t\sim|x|^{1/2}$. Hence we can deduce the following estimates.

Lemma 4.3. *Let notation be as above.*

(g_1): $W_+(K_t)(x)\leq ct(t^2+x^2)^{-3/4}$ *if* $t^2+x^2\geq 1$,
(l_1): $W_+(K_t)(x)\leq ct(t^2+x^2)^{-1}$ *if* $t^2+x^2\leq 1$,
(g_2): $(d/dx)^2(W_+(K_t))(x)\leq ct(t^2+x^2)^{-7/4}$ *if* $t^2+x^2\geq 1$,
(l_2): $(d/dx)^2(W_+(K_t))(x)\leq ct(t^2+x^2)^{-2}$ *if* $t^2+x^2\leq 1$,
(g_3): $(d/dx)^2(W_+(K_t))(x)\leq ct^{-2}(t^2+x^2)^{-1}$ *if* $t^2+x^2\geq 1$.

Applying this lemma, we can estimate $|A\circledast W_+(K_t)(x)|$ as follows.

Lemma 4.4. *Suppose $|x| \geq 2r$. Then $|A \circledast W_+(K_t)(x)|$ is dominated by*

$$G_r(t,x) = c\begin{cases} t(t+|x|)^{-3/2} & \text{if } t+|x| \geq 1 \qquad (G_1) \\ t(t+|x|)^{-2} & \text{if } t+|x| \leq 1 \qquad (L_1) \\ r^2t^{-2}(t+|x|)^{-2} & \text{if } t+|x| \geq 1 \qquad (G_3) \\ r^2t^{-1}(t+|x|)^{-2} & \text{if } t+|x| \leq 1. \qquad (L_2) \end{cases}$$

Proof. Let $|y| \leq r$. Since $|x| \geq 2r$, $|x-y| \leq |x|+r \leq 3|x|/2$ and $|x-y| \geq |x|-r \geq |x|/2$, that is, $|x-y| \sim |x|$ and $t+|x-y| \sim t+|x|$. Therefore, since $A \circledast W_+(K_t)(x) = \int_{-\infty}^{\infty} A(y)W_+(K_t)(x-y)dy$ and $\|A\|_{L^1(\mathbb{R})} = 1$, (G_1) and (L_1) follow from (g_1) and (l_1) in Lemma 4.3 respectively. Since A satisfies the moment conditions, $B(x) = \int_{-\infty}^{x}\int_{-\infty}^{u} A(v)dvdu$ is supported on $[-r,r]$, $\|B\|_\infty \leq 2r$ and thereby $\|B\|_{L^1(\mathbb{R})} \leq 4r^2$. Integration by parts implies that $A \circledast W_+(K_t)(x) = \int_{-\infty}^{\infty} B(y)(d/dy)^2(K_t(x-y))dy$. Then (G_3) and (L_2) follow from (g_3) and (l_2) in Lemma 4.3 respectively. ■

We return to the estimate of $g_{\mathbb{R}}(A)(x)$ for $|x| \geq 2r$. Since

$$g_{\mathbb{R}}(A)(x) \leq \Big(\int_0^\infty G_r(t,x)^2 \frac{dt}{t}\Big)^{1/2},$$

it follows that: Case I: $r \geq 1$. Since $|x| \geq 2$, (G_1) and (G_3) in Lemma 4.4 imply that $g_{\mathbb{R}}(A)^2(x) \leq c|x|^{-3}\int_0^{\sqrt{r}} tdt + cr^4|x|^{-4}\int_{\sqrt{r}}^{\infty} t^{-5}dt \leq cr|x|^{-3} + cr^2|x|^{-4} \leq cr|x|^{-3}$. Case II: $r < 1$. When $|x| \geq 2$, the same argument in Case I yields $g_{\mathbb{R}}(A)^2(x) \leq cr|x|^{-3}$. When $|x| \leq 2$, we can use (L_1) and (L_2) if $t \leq 1$, and (G_3) if $t \geq 1$. Hence, $g_{\mathbb{R}}(A)^2(x)$ is dominated by $c|x|^{-4}\int_0^r tdt + cr^4|x|^{-4}\int_r^1 t^{-3}dt + cr^4|x|^{-4}\int_1^\infty t^{-5}dt \leq cr^2|x|^{-4} \leq cr|x|^{-3}$. Therefore, in both cases we can deduce that

$$g_{\mathbb{R}}(A)(x) \leq cr^{1/2}|x|^{-3/2} \quad \text{if } |x| \geq 2r. \tag{12}$$

Finally, combining (11) and (12), we see that

$$\begin{aligned} g_{\mathbb{R}}(A)(x) &\le g_{\mathbb{R}}(A)(x)\chi_{B(0,2r)}(x) + cr^{1/2}|x|^{-3/2}\chi_{B(0,2r)^c}(x) \\ &\le ca(x) + c\sum_{k=2}^{\infty} r^{-1}2^{-3k/2}\chi_{B(0,2^k r)}(x), \end{aligned} \tag{13}$$

where $a \ge 0$, a is supported on $B(0,2r)$ and $\|a\|_{L^2(\mathbb{R})} \le r^{-1/2}$.

Applying (13) to $g_{\mathbb{R}}(H)$, $H = \sum_i \lambda_{\gamma,i}A_{\gamma,i}$, we see that

$$g_{\mathbb{R}}(H)(x) \le c\sum_i \lambda_{\gamma,i}\Big(a_{\gamma,i}(x) + \sum_{k=2}^{\infty} r_{\gamma,i}^{-1}2^{-3k/2}\chi_{B(x_{\gamma,i},2^k r_{\gamma,i})}(x)\Big),$$

where $a_{\gamma,i} \ge 0$ is supported on $B(x_{\gamma,i}, 2r_{\gamma,i})$ and $\|a_{\gamma,i}\|_{L^2(\mathbb{R})} \le r_{\gamma,i}^{-1/2}$. Therefore, it follows from [8, Lemmas 4 and 5 in Chap. 8] and (9) that

$$\begin{aligned} \|g_{\mathbb{R}}(H)\|_{L^1_{w_\gamma}(\mathbb{R})} &\le \|\sum_i\sum_{k=1}^{\infty}\lambda_{\gamma,i}r_{\gamma,i}^{-1}2^{-3k/2}\chi_{B(x_{\gamma,i},2^k r_{\gamma,i})}\|_{L^1_{w_\gamma}(\mathbb{R})} \\ &\le c\|\sum_i\sum_{k=1}^{\infty}\lambda_{\gamma,i}r_{\gamma,i}^{-1}2^{-k/2}\chi_{B(x_{\gamma,i},r_{\gamma,i})}\|_{L^1_{w_\gamma}(\mathbb{R})} \\ &\le c\|\sum_i\lambda_{\gamma,i}r_{\gamma,i}^{-1}\chi_{B(x_{\gamma,i},r_{\gamma,i})}\|_{L^1_{w_\gamma}(\mathbb{R})} \le c\|H\|_{H^1_{w_\gamma}(\mathbb{R})}. \end{aligned}$$

This completes the proof that $g_{\mathbb{R}}$ is bounded form $H^1_{w_\gamma}(\mathbb{R})$ to $L^1_{w_\gamma}(\mathbb{R})$. Hence we can obtain the following.

Theorem 4.5. *g is bounded from $H^1(G//K)$ to $L^1(G//K)$.*

5. Lusin area function.

We introduce a modified area function $S_+(f)$ as

$$S_+(f)(x) = \Big(\int_0^{\infty}\int_G \Theta(x,y)\chi_t(xy^{-1})\Big|t\frac{\partial}{\partial t}f * p_t(y)\Big|^2 dy\frac{dt}{t}\Big)^{1/2},$$

where $\Theta(x,y)$ is the bi-K-bi-invariant function on $G\times G$ defined by

$$\Theta(x,y)=\begin{cases}\dfrac{\Delta(y)}{\Delta(x)}\Big(\dfrac{\mathrm{th}_+x}{\mathrm{th}_+y}\Big)^{2\gamma_\alpha} & \text{if } \sigma(y)\geq\sigma(x)\\ \dfrac{\Delta(y)^2}{\Delta(x)^2} & \text{if } \sigma(y)<\sigma(x).\end{cases}$$

Clearly, $\Theta(x,y)\geq 1$ if $\sigma(y)\geq\sigma(x)$ and $\Theta(x,y)<1$ if $\sigma(y)\geq\sigma(x)$ and moreover, for $0\leq\gamma\leq\gamma_\alpha$, it follows that

$$\begin{aligned}\Theta(x,y)\frac{\Delta(x)^2}{\Delta(y)}\Big(\frac{\mathrm{th}_+y}{\mathrm{th}_+x}\Big)^{2\gamma} &\leq \begin{cases}\Delta(x)\Big(\dfrac{\mathrm{th}_+x}{\mathrm{th}_+y}\Big)^{2(\gamma_\alpha-\gamma)} & \text{if } \sigma(y)\geq\sigma(x)\\ \Delta(y)\Big(\dfrac{\mathrm{th}_+y}{\mathrm{th}_+x}\Big)^{2\gamma} & \text{if } \sigma(y)<\sigma(x)\end{cases}\\ &\leq \min\{\Delta(x),\Delta(y)\}. \qquad (14)\end{aligned}$$

We shall consider an (H^1,L^1) or an $(H^1_\flat,L^1)$-bound on G for S_+.

Case of $G=SO_0(n,1)$, n odd: Here $\alpha=n/2-1$ and $\beta=-1/2$ are both half-integer and no integral terms appear in (4), because $\alpha-\beta$ and $\beta+1/2$ are both integer. Since $t(\partial/\partial t)f*p_t=W^1_-(F\circledast W_+(K_t))$, if we introduce $S^1_{\mathbb{R}}$ on $\mathbb{R}$ as

$$\begin{aligned}S^1_{\mathbb{R}}(H)(x) &= \Big(\int_0^\infty\int_0^\infty\Theta(x,y)\int_K\chi_t(a_xka_y^{-1})dk \qquad (15)\\ &\quad\times(\mathrm{th}y)^{2\gamma}\Delta(y)^{-1}|H\circledast W_+(K_t)(y)|^2dy\frac{dt}{t}\Big)^{1/2}(\mathrm{th}x)^{-\gamma}\Delta(x),\end{aligned}$$

then we have $\|S_+(f)\|_{L^1(G)}\leq c\sum_{\gamma\in\Gamma}\|S^1_{\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))\|_{L^1_{w_\gamma}(\mathbb{R})}$. As in the case of $g_{\mathbb{R}}$, the (H^1,L^1) bound on G for S_+ is reduced to an $(H^1_{w_\gamma},L^1_{w_\gamma})$ bound on $\mathbb{R}$ for $S^1_{\mathbb{R}}$. Then it is enough to prove that $S^1_{\mathbb{R}}$ is L^2 bounded on $\mathbb{R}$, which yields (10) for $S^1_{\mathbb{R}}$, and it satisfies (11) for each centered $(1,\infty,1)$-atom on $\mathbb{R}$. As before, these estimates yield the $(H^1_{w_\gamma},L^1_{w_\gamma})$ bound on $\mathbb{R}$ for $S^1_{\mathbb{R}}$.

We shall consider L^2 bound on $\mathbb{R}$ for $S^1_{\mathbb{R}}$. First we apply (14) to (15) and then, we use the fact that $\int_0^\infty \int_K \chi_t(a_x k a_y^{-1}) dk \Delta(x) dx = \int_G \chi_t(g a_y^{-1}) dg = \|\chi_t\|_1 = 1$. Then it follows from Proposition 4.2 that

$$\begin{aligned} \|S^1_{\mathbb{R}}(H)\|_{L^2(\mathbb{R})} &\leq c \int_0^\infty \int_0^\infty |H \circledast W_+(K_t)(y)|^2 dy \frac{dt}{t} \\ &= c \int_0^\infty g_{\mathbb{R}}(H)^2(y) dy \leq c \|H\|_{L^2(\mathbb{R})}. \end{aligned} \tag{16}$$

Let A be a $(1,\infty,1)$-atom on $\mathbb{R}$ supported on $[-r,r]$. We suppose that $|x| \geq 2r$. In (15) replaced H by A, it follows that $|x-y| \leq t$, $A \circledast W_+(K_t)(y) = \int_{-\infty}^\infty A(z) W_+(K_t)(y-z) dz$, and $|z| \leq r$. Since $x = (x-y)+(y-z)+z$, $|x| \leq t + |y-z| + r \leq t + |y-z| + |x|/2$ and thus, $|x| \leq 2(t+|y-z|)$. Moreover, $|y-z| \leq |y|+|z| \leq t + |x| + r \leq t + 3|x|/2$. Hence it follows that $t+|x| \sim t+|y-z|$. Then, applying the arguments used in Lemmas 4.3 and 4.4, we can deduce that $|A \circledast W_+(K_t)(y)| \leq c G_r(t,x)$. Hence (14) and the fact that $\int_0^\infty \int_K \chi_t(a_x k a_y^{-1}) dk \Delta(y) dy = 1$ yield that

$$S^1_{\mathbb{R}}(A)(x) \leq c \Big(\int_0^\infty G_r^2(t,x) \frac{dt}{t} \Big)^{1/2} \leq c r^{1/2} |x|^{-3/2} \quad \text{if } |x| \geq 2r. \tag{17}$$

As said before, (16) and (17) implies the $(H^1_{w_\gamma}, L^1_{w_\gamma})$ bound on $\mathbb{R}$ for $S^1_{\mathbb{R}}$. Therefore, S_+ is bounded form $H^1(G//K)$ to $L^1(G//K)$.

Case of $G \neq SO_0(n,1)$, n odd: Here $\beta + 1/2 = 0, \alpha - \beta \in \mathbb{Z} + 1/2$ or $\beta + 1/2 \in \mathbb{Z} + 1/2, \alpha - \beta \in \mathbb{Z}$. Hence γ_α is half-integer and integral terms appear in (4). Therefore, noting (6), we introduce $S^2_{\mathbb{R}}$ on $\mathbb{R}$ as

$$\begin{aligned} S^2_{\mathbb{R}}(H)(x) = \Big(\int_0^\infty \int_0^\infty \Theta(x,y) \int_K \chi_t(a_x k a_y^{-1}) dk (\text{th} y)^{2\gamma - 1} \Delta(y)^{-1} \\ \times \Big| \int_{|y|}^\infty H \circledast W_+(K_t)(s) \frac{A(|y|,s)}{\sqrt{s}} ds \Big|^2 dy \frac{dt}{t} \Big)^{1/2} (\text{th} x)^{-(\gamma - 1/2)} \Delta(x). \end{aligned} \tag{18}$$

Then it follows that

$$\begin{aligned}&\|S_+(f)\|_{L^1(G)}\\ \leq\ & c\sum_{\gamma\in\Gamma}\Big(\|S^1_{\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))\|_{L^1_{w_\gamma}(\mathbb{R})}+\|S^2_{\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))\|_{L^1_{w_{\gamma-1/2}}(\mathbb{R})}\Big).\end{aligned}$$

We shall prove that S_+ is bounded from $H^1_\flat(G//K)$ to $L^1(G//K)$. Since $L^1_{w_{\gamma-1/2}}(\mathbb{R})\subset L^1_{w_\gamma}(\mathbb{R})$, the $(H^1_\flat, L^1)$ bound on G for S_+ is reduced to an $(H^1_{w_{\gamma-1/2}}, L^1_{w_\gamma -1/2})$ bound on $\mathbb{R}$ for $S^1_{\mathbb{R}}$ and $S^2_{\mathbb{R}}$. Clearly, by the previous argument, $S^1_{\mathbb{R}}$ is bounded form $H^1_{w_{\gamma-1/2}}(\mathbb{R})$ to $L^1_{w_\gamma -1/2}(\mathbb{R})$. Therefore, to obtain the $(H^1_\flat, L^1)$ bound on G for S_+, it is enough to prove that $S^2_{\mathbb{R}}$ is L^2 bounded on $\mathbb{R}$ and it satisfies (11) for each centered $(1,\infty,1)$-atom A on $\mathbb{R}$.

First we shall consider the case that $A(y,s)$ is of the form $A(y,s) = \chi_{[0,\infty)}(s-y)\chi_{[0,1]}(s)$. It follows form (14), $\int_0^\infty\int_K \chi_t(a_xka_y^{-1})dk\Delta(x)dx = 1$ and Proposition 4.2 that for $H\in L^2(\mathbb{R})$,

$$\begin{aligned}\int_0^\infty S^2_{\mathbb{R}}(H)^2(x)dx &\leq c\int_0^\infty\int_0^1\Big|\int_y^1 H\circledast W_+(K_t)(s)\frac{ds}{\sqrt{s}}\Big|^2 dy\frac{dt}{t}\\ &\leq c\int_0^1\int_y^1 g_{\mathbb{R}}(H)^2(s)ds\int_y^1 s^{-1}dsdy\\ &\leq c\|g_{\mathbb{R}}(H)\|^2_{L^2(\mathbb{R})}\int_0^1\log ydy\leq c\|H\|_{L^2(\mathbb{R})}.\end{aligned}$$

Let $|x|\geq 2r$. We estimate $A\circledast W_+(K_t)(s)$ in (17) replaced H by A. When $s\geq|x|$, it follows that $s\geq 2r$ and $|A\circledast W_+(K_t)(s)|\leq G_r(t,s)\leq G_r(t,|x|)$ by Lemma 4.4. When $s\leq|x|$, we note that $A\circledast W_+(K_t)(s)=\int_{-\infty}^\infty A(z)\ W_+(K_t)(s-z)dz$ and $t+|s-z|\sim t+|x|$. Actually, we may suppose that $|z|\leq r$, $|x-y|\leq t$, and $0\leq y\leq s\leq x$. Since $x=(x-y)+(y-s)+(s-z)+z$, we see that $x\leq 2t+|s-z|+r\leq 2t+|s-z|+x/2$ and thus, $t+x\leq 4(t+|s-z|)$. Moreover,

$t+|s-z| \le t+s+|z| \le t+3x/2 \le 3(t+x)/2$. Therefore, it follows from the arguments used in Lemmas 4.3 and 4.4 that $A \circledast W_+(K_t)(s) \le cG_r(t,|x|)$ again. Hence $\int_{|y|}^1 A \circledast W_+(K_t)(s)A(|y|,s)/\sqrt{s}ds \le cG_r(t,|x|)$ and thus, (14) and $\int_0^\infty \int_K \chi_t(a_x k a_y^{-1})dk\Delta(y)dy = 1$ imply that

$$S^2_{\mathbb{R},0}(A)(x) \le c\Big(\int_0^\infty G_r^2(t,x)\frac{dt}{t}\Big)^{1/2} \le cr^{1/2}|x|^{-3/2} \quad \text{if } |x| \ge 2r.$$

Next we shall consider the case of $A(y,s) = B(s-y)$, where B is bounded integrable on $\mathbb{R}_+$. As before, it follows that for $H \in L^2(\mathbb{R})$,

$$\begin{aligned}
&\int_0^\infty S^2_{\mathbb{R}}(H)^2(x)dx \\
\le\ & \int_0^\infty \int_{-\infty}^\infty \Big|\int_0^\infty H \circledast W_+(K_t)(s+y)\frac{B(s)}{\sqrt{s+y}}ds\Big|^2 dy\frac{dt}{t} \\
\le\ & c\int_{-\infty}^\infty \Big(\int_0^\infty g_{\mathbb{R}}(H)(s+y)\frac{B(s)}{\sqrt{s}}ds\Big)^2 dy \\
\le\ & c\|g_{\mathbb{R}}(H)\|^2_{L^2(\mathbb{R})}\Big(\int_0^\infty \frac{B(s)}{\sqrt{s}}ds\Big)^2 \le c\|H\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Let $|x| \ge 2r$. Since $|A \circledast W_+(K_t)(s+y)| \le cG_r(t,|x|)$, $s,y \ge 0$, as before, it follows that, if $|x| \ge 2r$, then

$$S^2_{\mathbb{R},0}(A)(x) \le c\Big(\int_0^\infty G_r^2(t,x)\frac{dt}{t}\Big)^{1/2}\int_0^\infty \frac{B(s)}{\sqrt{s}}ds \le cr^{1/2}|x|^{-3/2}.$$

In both cases we can obtain that $S^2_{\mathbb{R}}$ is L^2 bounded on $\mathbb{R}$ and satisfies (12). Hence the $(H^1_\flat, L^1)$ bound on G for S_+ follows. Finally, we can obtain the following.

Theorem 5.1. When $G = SO_0(n,1)$, n odd, S_+ is bounded from $H^1(G//K)$ to $L^1(G//K)$, and otherwise, bounded from $H^1_\flat(G//K)$ to $L^1(G//K)$. Especially, S_+ is bounded from $H^1_\sharp(G//K)$ to $L^1(G//K)$.

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Uncertainty principles for the Jacobi transform

Takeshi KAWAZOE *

Abstract

We obtain some uncertainty inequalities for the Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$, where we suppose $\alpha, \beta \in \mathbb{R}$ and $\rho = \alpha + \beta + 1 \geq 0$. As in the Euclidean case, analogues of the local and global uncertainty principles hold for $\hat{f}_{\alpha,\beta}$. In this paper, we shall obtain a new type of an uncertainty inequality and its equality condition: When $\beta \leq 0$ or $\beta \leq \alpha$, the L^2-norm of $\hat{f}_{\alpha,\beta}(\lambda)\lambda$ is estimated below by the L^2-norm of $\rho f(x)(\cosh x)^{-1}$. Otherwise, a similar inequality holds. Especially, when $\beta > \alpha + 1$, the discrete part of f appears in the Parseval formula and it influences the inequality. We also apply these uncertainty principles to the spherical Fourier transform on $SU(1,1)$. Then the corresponding uncertainty principle depends, not uniformly on the K-types of f.

1. Introduction. The uncertainty principle on $\mathbb{R}$ says that if a function $f(x)$ is concentrated around $x = 0$, then its Fourier transform $\hat{f}(\lambda)$ cannot be concentrated around $\lambda = 0$ unless f is identically zero. As surveyed in [7] and [9], there are various generalizations of this principle on locally compact groups G; the Heisenberg group, motion groups, and semisimple Lie groups, and so on. In this paper we shall obtain a generalization of this principle for the Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$ (see (7)).

On semisimple Lie groups G the local and global uncertainty principles for the spherical Fourier transform of K-finite functions are obtained in [7]. When the real rank of G equals to one, these inequalities correspond to the ones for the Jacobi transforms with specialized α and β. Hence, the results

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in [7] are easily generalized for the Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$. However, it is not clear how the constants appeared in the inequalities depend on α, β, and moreover, how the discrete part of f (see (10)) contributes the uncertainty principles. Hence in §2 and §3, arguing exactly as in the Euclidean case, we shall give the proofs of local and global uncertainty inequalities for the Jacobi transform (see Theorems 3.1, 3.2, 4.2 and 4.3).

On the Euclidean space $\mathbb{R}$, to figure a concentration of $f(x)$ around $x = 0$, we consider a multiplication of x; $f(x)x$, and similarly, for the Fourier transform side, we do a multiplication of λ; $\hat{f}(\lambda)\lambda$. On the other hand, for the global uncertainty inequality for $\hat{f}_{\alpha,\beta}(\lambda)$ (see Theorem 4.1) these x and λ are respectively replaced by

$$V(x) = \int_0^x \Delta(t)dt \quad \text{and} \quad W(\lambda) = \int_{D(\lambda)} d\nu,$$

where $\Delta(t)$ is the weight function on $\mathbb{R}_+$ (see (2)), $D(\lambda) = \{z \in \mathbb{C}; |z| \leq |\lambda|\}$, and $d\nu$ the Plancherel measure for the Jacobi transform (see (13)). In Theorem 4.2 we modify $V(x)$ and $W(\lambda)$ respectively as

$$V_\delta(x) = \min(V(x), \delta^{-1}) \quad \text{and} \quad w_\alpha(\lambda) = (\lambda^2 + \rho^2)^{\alpha+1}$$

for $\delta > 0$. Furthermore, in §5 we shall give a refinement of Theorem 4.2 by replacing $V_\delta(x)$ as

$$v(x) = \frac{V(x)}{\Delta(x)}.$$

We shall obtain a global uncertainty inequality, which figures concentrations of f and $\hat{f}_{\alpha,\beta}$ by the multiplications of $v(x)$ and $w_{-1/2}(\lambda)$ respectively. Especially, we can obtain the equality condition (see Theorem 5.1). We note that functions satisfy the equality condition are neither Gaussian nor heat kernels for the Jacobi transform (see (21b)). In §6, using these inequalities, we shall consider some uncertainty principles for f and $\hat{f}_{\alpha,\beta}$.

In §7 we shall apply these global uncertainty inequalities for the Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$ to the spherical Fourier transform $\tilde{f}(\lambda)$ on $G = SU(1,1)$. Then we can deduce a uncertainty principle for general functions, not K-finite, on G. As in the Euclidean case, to deduce a non-concentration of $\tilde{f}(\lambda)$ around $\lambda = 0$, a concentration of $f(g)$ around $g = e$ is sufficient (see Theorem 7.1). In particular, we see that this sufficient condition depends on the K-types of f and is not uniform on the K-types (see Remark 7.2).

2. Notation. Let $\alpha, \beta \in \mathbb{C}$, $\Re\alpha > -1$ and $\rho = \alpha+\beta+1$. For $\lambda \in \mathbb{C}$, let $\phi_\lambda(x)$ denote the Jacobi function of the first kind, that is, the unique solution of

$$(L + \lambda^2 + \rho^2)f = 0 \tag{1}$$

satisfying $f(0) = 1$ and $f'(0) = 0$, where $L = \Delta(x)^{-1}\frac{d}{dx}\Big(\Delta(x)\frac{d}{dx}\Big)$ and

$$\Delta(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}. \tag{2}$$

For $\lambda \neq -i, -2i, -3i, \ldots$, let $\Phi_\lambda(x)$ denote the Jacobi function of the second kind which satisfies

$$2\pi^{1/2}\Gamma(\alpha+1)^{-1}\phi_\lambda(x) = C(\lambda)\Phi_\lambda(x) + C(-\lambda)\Phi_{-\lambda}(x), \tag{3}$$

where $C(\lambda)$ is Harish-Chandra's C-function (cf. [3, §2]). For convenience, we suppose that $\alpha, \beta \in \mathbb{R}$ and $\rho \geq 0$ in the following. Then the following estimates are well-known (cf. [3, 4]): For $x \geq 0$ and $\lambda \in \mathbb{C}$ with $|\Im\lambda| \leq \rho$

$$|\phi_\lambda(x)| \leq 1, \tag{4}$$

and for each $\delta > 0$ there exist a positive constant K_δ such that for all $x \geq \delta$ and $\lambda \in \mathbb{C}$ with $\Im\lambda \geq 0$

$$|\Phi_\lambda(x)| \leq K_\delta e^{-(\Im\lambda+\rho)x}, \tag{5}$$

where K_δ is independent of α, β, and for each $r > 0$ there exist positive constants $K^1_{r,\alpha}$, $K^2_{r,\alpha}$ such that if $\lambda \in \mathbb{C}$ with $\Im\lambda \geq 0$ is at distance larger than r from the poles of $C(-\lambda)^{-1}$ then

$$K^1_{r,\alpha}2^{-\rho}(\rho + |\lambda|)^{\alpha+1/2} \leq |C(-\lambda)|^{-1} \leq K^2_{r,\alpha}2^{-\rho}(\rho + |\lambda|)^{\alpha+1/2}, \tag{6}$$

where $K^i_{r,\alpha}$, $i = 1, 2$, are independent of β.

Let $L^p(\Delta)$, $1 \leq p < \infty$, denote the space of all p-th integrable functions on $\mathbb{R}_+$ with respect to $\Delta(x)dx$ and $C^\infty_{c,e}(\mathbb{R})$ the space of all even C^∞ functions on $\mathbb{R}$ with compact support. For $f \in C^\infty_{c,e}(\mathbb{R})$, the Jacobi transform $\hat{f}(\lambda)$ is defined as

$$\hat{f}(\lambda) = \frac{\sqrt{2}}{\Gamma(\alpha+1)}\int_0^\infty f(x)\phi_\lambda(x)\Delta(x)dx. \tag{7}$$

Clearly (1) and (4) imply that for $\lambda \in \mathbb{C}$,

$$(Lf)^\wedge(\lambda) = -(\lambda^2 + \rho^2)\hat{f}(\lambda) \tag{8}$$

and for $|\Im\lambda| \le \rho$,

$$|\hat{f}(\lambda)| \le \frac{\sqrt{2}}{\Gamma(\alpha+1)}\|f\|_{L^1(\Delta)}. \tag{9}$$

This transform $f \to \hat{f}$ satisfies analogous properties of the classical cosine Fourier transform; the inversion formula, the Paley-Wiener theorem, and the Plancherel formula were obtained in [3, 4]: We set

$$D_{\alpha,\beta} = \{i(\beta - \alpha - 1 - 2m); m = 0, 1, 2, \cdots, \beta - \alpha - 1 - 2m > 0\}.$$

Then the inversion formula is given as follows: For $f \in C^\infty_{c,e}(\mathbb{R})$,

$$\begin{aligned} f(x) &= \frac{\sqrt{2}}{\Gamma(\alpha+1)}\Big(\int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda + \sum_{\mu \in D_{\alpha,\beta}} a_\mu \phi_\mu(x) d(\mu)\Big) \\ &= f_P(x) + {}^\circ f(x), \end{aligned} \tag{10}$$

where $a_\mu = \hat{f}(\mu)$ and $d(\mu) = -2\pi i C(\mu)^{-1}\mathrm{Res}_{\lambda=\mu}C(-\lambda)^{-1}$. We call f_P and ${}^\circ f$ the principal part and the discrete part of f respectively. We note that since $\rho \ge 0$, $|\beta| \le \alpha + 1$ if $\beta \le 0$ and hence $D_{\alpha,\beta} = \emptyset$ if $\beta \le 0$. Moreover, there exists a positive constant K_μ such that

$$|\phi_\mu(x)| \le K_\mu e^{-(\rho+|\mu|)x}, \ x \ge 0 \tag{11}$$

and thereby

$$d(\mu)^{-1} = \frac{2}{\Gamma(\alpha+1)^2}\int_0^\infty |\phi_\mu(x)|^2 \Delta(x) dx > 0. \tag{12}$$

We denote by $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\mu\})$ a function on $\mathbb{R}_+ \cup D_{\alpha,\beta}$ defined by

$$\boldsymbol{F}(\nu) = \begin{cases} F(\lambda) & \text{if } \nu = \lambda \in \mathbb{R}_+ \\ a_\mu & \text{if } \nu = \mu \in D_{\alpha,\beta}. \end{cases}$$

We put $\overline{\boldsymbol{F}}(\nu) = (\overline{F(\lambda)}, \{\overline{a_\mu}\})$ and define a product of $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\mu\})$ and $\boldsymbol{G}(\nu) = (G(\lambda), \{b_\mu\})$ as

$$(\boldsymbol{F}\boldsymbol{G})(\nu) = (F(\lambda)G(\lambda), \{a_\mu b_\mu\}).$$

Moreover, for a function $h(\lambda)$ on $\mathbb{C}$, we define a multiplication of h as $h(\nu)\boldsymbol{F}(\nu) = (h(\lambda)F(\lambda), \{h(\mu)a_\mu\})$. Let $d\nu$ denote the measure on $\mathbb{R}_+ \cup D_{\alpha,\beta}$ defined by

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} \boldsymbol{F}(\nu) d\nu = \int_0^\infty F(\lambda)|C(\lambda)|^{-2} d\lambda + \sum_{\mu \in D_{\alpha,\beta}} a_\mu d(\mu). \tag{13}$$

For $f \in C^\infty_{c,e}(\mathbb{R})$, we put

$$\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{\hat{f}(\mu)\}).$$

Then the Parseval formula for the Jacobi transform on $C^\infty_{c,e}(\mathbb{R})$ can be stated as follows (see [4, Theorem 2.4] and cf. [2]): For $f, g \in C^\infty_{c,e}(\mathbb{R})$

$$\int_0^\infty f(x)\overline{g(x)}\Delta(x) dx = \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} \hat{\boldsymbol{f}}(\nu)\overline{\hat{\boldsymbol{g}}(\nu)} d\nu. \tag{14}$$

The map $f \to \hat{\boldsymbol{f}}$, $f \in C^\infty_{c,e}(\mathbb{R})$, can be extended to an isometry between $L^2(\Delta)$ and $L^2(\nu) = L^2(\mathbb{R}_+ \cup D_{\alpha,\beta}, d\nu)$. Actually, each function f in $L^2(\Delta)$ is of the form $f = f_P + {}^\circ f$ (see (10)) and their L^2-norms are given as

$$\int_0^\infty |f_P(x)|^2 \Delta(x) dx = \int_0^\infty |\hat{f}_P(\lambda)|^2 |C(\lambda)|^{-2} d\lambda, \tag{15a}$$

$$\int_0^\infty |{}^\circ f(x)|^2 \Delta(x) dx = \sum_{\mu \in D_{\alpha,\beta}} |a_\mu|^2 d(\mu). \tag{15b}$$

Therefore, if we define $\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{a_\mu\})$, (14) implies that

$$\|f\|_{L^2(\Delta)} = \|\hat{\boldsymbol{f}}\|_{L^2(\nu)}.$$

3. Local uncertainty principles. We define a function $V(x)$ on $\mathbb{R}_+$ by

$$V(x) = \int_0^x \Delta(t) dt \tag{16}$$

and for a measurable subset E of $\mathbb{R}_+ \cup D_{\alpha,\beta}$ we put

$$\sigma(E) = \int_E d\nu.$$

Then as in the Euclidean case, we can deduce the local uncertainty principle (see [5, §3] for semisimple Lie groups and motion groups).

Theorem 3.1. *Let $0 \leq \theta < 1/2$. Then there exists a constant $C_{\theta,\alpha}$ such that for all $f \in L^1(\Delta) \cap L^2(\Delta)$ and $E \subset \mathbb{R}_+ \cup D_{\alpha,\beta}$ with $\sigma(E) < \infty$,*

$$\int_E |\hat{\boldsymbol{f}}(\nu)|^2 d\nu \leq C_{\theta,\alpha}\sigma(E)^{2\theta} \int_0^\infty |f(x)|^2 V(x)^{2\theta}\Delta(x)dx.$$

In order to clear the fact that $C_{\theta,\alpha}$ is independent of β we shall give a sketch of the proof. Let χ_r, $r > 0$, denote the characteristic function of the interval $[0, r]$. We set $g = f\chi_r$ and $h = f - g$. Then

$$\int_E |\hat{\boldsymbol{f}}(\nu)|^2 d\nu \leq 2\Big(\int_E |\hat{\boldsymbol{g}}(\nu)|^2 d\nu + \int_E |\hat{\boldsymbol{h}}(\nu)|^2 d\nu\Big).$$

It follows from (9) and Schwarz' inequality that

$$\begin{aligned}
&\int_E |\hat{\boldsymbol{g}}(\nu)|^2 d\nu \\
\leq\ & \frac{2}{\Gamma(\alpha+1)^2}\|g\|^2_{L^1(\Delta)}\sigma(E) \\
\leq\ & \frac{2}{\Gamma(\alpha+1)^2}\sigma(E)\int_0^r V(x)^{-2\theta}\Delta(x)dx \int_0^r |g(x)|^2 V(x)^{2\theta}\Delta(x)dx \\
=\ & \frac{2}{\Gamma(\alpha+1)^2}\frac{1}{-2\theta+1}\sigma(E)V(r)^{-2\theta+1}\int_0^r |g(x)|^2 V(x)^{2\theta}\Delta(x)dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_E |\hat{\boldsymbol{h}}(\nu)|^2 d\nu &\leq \int_r^\infty |h(x)|^2 \Delta(x)dx \\
&\leq V(r)^{-2\theta}\int_r^\infty |h(x)|^2 V(x)^{2\theta}\Delta(x)dx.
\end{aligned}$$

Here we take an r such that $\sigma(E) = V(r)^{-1}$. Then

$$\int_E |\hat{\boldsymbol{f}}(\nu)|^2 d\nu \leq C_{\theta,\alpha}\sigma(E)^{2\theta} \int_0^\infty |f(x)|^2 V(x)^{2\theta}\Delta(x)dx,$$

where $C_{\theta,\alpha} = 2\max\Big(\frac{2}{\Gamma(\alpha+1)^2}\frac{1}{1-2\theta}, 1\Big)$.

We shall modify the above local uncertainty inequality. For each $\delta > 0$ we denote by x_δ the point satisfying $V(x_\delta) = \delta^{-1}$ and we let

$$V_\delta(x) = \begin{cases} V(x) & \text{if } 0 \leq x < x_\delta, \\ \delta^{-1} & \text{if } x \geq x_\delta. \end{cases} \tag{17}$$

Theorem 3.2. *Let $\delta > 0$ and $0 \leq \theta < 1/2$. Then there exists a constant $C_{\theta,\alpha}$ such that for all $f \in L^1(\Delta) \cap L^2(\Delta)$ and $E \subset \mathbb{R}_+ \cup D_{\alpha,\beta}$ with $\sigma(E) \geq \delta$,*

$$\int_E |\hat{\boldsymbol{f}}(\nu)|^2 d\nu \leq C_{\theta,\alpha}\sigma(E)^{2\theta} \int_0^\infty |f(x)|^2 V_\delta(x)^{2\theta}\Delta(x)dx.$$

Proof. Since $\sigma(E) \geq \delta$ and δ is the minimum value of $V_\delta(x)^{-1}$, we can take an r such that $\sigma(E) = V(r)^{-1}$. Therefore, we can repeat the above sketch of the proof replacing V by V_δ. ■

4. global uncertainty principles. As in the Euclidean case, we can deduce the global uncertainty principles from the local ones. We denote

$$W(r) = \sigma(\{\lambda \in \mathbb{C}; |\lambda| \leq r\}).$$

Then the following global uncertainty inequality follows from Theorem 3.1 (see [5, §4] for symmetric spaces).

Theorem 4.1. *Let $0 \leq \theta < 1/2$. Then there exists a constant $C_{\theta,\alpha}$ such that for all $f \in L^1(\Delta) \cap L^2(\Delta)$*

$$\|f\|^4_{L^1(\Delta)} \leq C_{\theta,\alpha} \int_0^\infty |f(x)|^2 V(x)^{2\theta}\Delta(x)dx \int_{\mathbb{R}_+\cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 W(\nu)^{2\theta} d\nu.$$

We now deduce a global uncertainty inequality from Theorem 3.2. We set $E_r = \{\lambda \in \mathbb{C}; |\lambda|^2 \leq r^2 + \rho^2\}$. Since $\sigma(E_r \cap \mathbb{R}_+) = \int_0^{\sqrt{r^2+\rho^2}} |C(\lambda)|^{-2} d\lambda$, substituting the estimate of $C(-\lambda)^{-1}$ (see (6)), we see that there exist positive constants C^i_α, $i = 1, 2$, such that for $\lambda \in \mathbb{R}$

$$C^1_\alpha 2^{-2\rho}(r^2 + \rho^2)^{(\alpha+1)} \leq \sigma(E_r \cap \mathbb{R}_+) \leq C^2_\alpha 2^{-2\rho}(r^2 + \rho^2)^{(\alpha+1)}. \tag{18}$$

Therefore, if we take $\delta > 0$ as $\delta = C_\alpha^1 2^{-2\rho}\rho^{2(\alpha+1)}$, then $\sigma(E_r \cap \mathbb{R}_+) \geq \delta$. For $\gamma \geq 0$, we define the fractional power of $-L$ as

$$((-L)^\gamma f)(\lambda) = (\lambda^2+\rho^2)^\gamma \hat{f}(\lambda)$$

(cf. (8)). Then we have the following.

Theorem 4.2. *Let δ, V_δ be as above and let $0 \leq \theta < 1/2$. Then there exists a positive constant $C_{\theta,\alpha}$ such that for all $f = f_P \in L^1(\Delta) \cap L^2(\Delta)$*

$$\|f\|_{L^1(\Delta)}^4 \leq C_{\theta,\alpha} 2^{-4\rho\theta} \int_0^\infty |f(x)|^2 V_\delta(x)^{2\theta}\Delta(x)dx \int_0^\infty |(-L)^{(\alpha+1)\theta} f(x)|^2 \Delta(x)dx.$$

Proof. Let $\gamma = 2(\alpha+1)\theta$ and $f = f_P$. By using the Plancherel formula (16a), we obtain that

$$\begin{aligned} \|f\|_{L^2(\Delta)}^2 &= \int_{\mathbb{R}_+} (\lambda^2+\rho^2)^{-\gamma}(\lambda^2+\rho^2)^\gamma |\hat{f}(\lambda)|^2 |C(\lambda)|^{-2} d\lambda \\ &\leq \rho^{-2\gamma} \int_0^\infty |((-L)^{\gamma/2} f(x)|^2 \Delta(x)dx. \qquad (19) \end{aligned}$$

Moreover, if $\hat{f}(\lambda)$ is supported on $E_r^c \cap \mathbb{R}_+$, then $\rho^{-2\gamma}$ can be replaced by $(r^2+\rho^2)^{-\gamma}$, because $\lambda^2+\rho^2 \geq \lambda^2 \geq r^2+\rho^2$ for $\lambda \in E_r^c \cap \mathbb{R}_+$. Then it follows from Theorem 3.2 and (18) that for each $r > 0$

$$\begin{aligned} \|f\|_{L^2(\Delta)}^2 &= \int_{E_r\cap\mathbb{R}_+} |\hat{\boldsymbol{f}}(\nu)|^2 d\nu + \int_{E_r^c\cap\mathbb{R}_+} |\hat{\boldsymbol{f}}(\nu)|^2 d\nu \\ &\leq C_{\theta,\alpha}\sigma(E_r\cap\mathbb{R}_+)^{2\theta} \int_0^\infty |f(x)|^2 V_\delta(x)^{2\theta}\Delta(x)dx \\ &\quad + (r^2+\rho^2)^{-\gamma} \int_0^\infty |((-L)^{\gamma/2} f(x)|^2 \Delta(x)dx \qquad (20) \\ &\leq (r^2+\rho^2)^\gamma 2^{-4\rho\theta} C_{\theta,\alpha}(C_\alpha^2)^{2\theta} \int_0^\infty |f(x)|^2 V_\delta(x)^{2\theta}\Delta(x)dx \\ &\quad + (r^2+\rho^2)^{-\gamma} \int_0^\infty |((-L)^{\gamma/2} f(x)|^2 \Delta(x)dx \\ &= (r^2+\rho^2)^\gamma 2^{-4\rho\theta} C_{\theta,\alpha}(C_\alpha^2)^{2\theta} I_1 + (r^2+\rho^2)^{-\gamma} I_2. \end{aligned}$$

Especially, since $C_{\theta,\alpha} \geq 2$ and $C_\alpha^1 \leq C_\alpha^2$, it follows that

$$\begin{aligned} \|f\|_{L^2(\Delta)}^2 &\leq (r^2+\rho^2)^\gamma 2^{-4\rho\theta} C_{\theta,\alpha}(C_\alpha^2)^{2\theta} I_1 + (r^2+\rho^2)^{-\gamma} C_{\theta,\alpha}(C_\alpha^1/C_\alpha^2)^{-2\theta} I_2 \\ &= (r^2+\rho^2)^\gamma A + (r^2+\rho^2)^{-\gamma} B. \end{aligned}$$

As a function of x on $\mathbb{R}_+$, $x^\gamma A + x^{-\gamma} B$ attains the minimum value $2\sqrt{AB}$ at $x_0 = (B/A)^{1/2\gamma}$. Therefore, it follows from (17) with $\delta = C_\alpha^1 2^{-2\rho}\rho^{2(\alpha+1)}$ and (19) that

$$x_0 = \Big(\frac{C_{\theta,\alpha}(C_\alpha^1/C_\alpha^2)^{-2\theta} I_2}{2^{-4\rho\theta} C_{\theta,\alpha}(C_\alpha^2)^{2\theta} I_1}\Big)^{1/2\gamma} \geq \rho^2.$$

Hence we can take an r such that $x_0 = r^2 + \rho^2$ and therefore,

$$\|f\|_{L^2(\Delta)}^4 \leq 2^{-4\rho\theta+2} C_{\theta,\alpha}^2 (C_\alpha^2)^{2\theta} (C_\alpha^1/C_\alpha^2)^{-2\theta} I_1 I_2.$$

This completes the proof. ■

For a general $f \in L^1(\Delta) \cap L^2(\Delta)$ we must pay attention to the discrete part ${}^\circ f$ of f. Let ${}^\circ f \neq 0$ and thus, $D_{\alpha,\beta} \neq \emptyset$ and $\beta > 0$. In (19) $\mathbb{R}_+$ must be replaced by $\mathbb{R}_+ \cup D_{\alpha,\beta}$ and when $\nu \in D_{\alpha,\beta}$, we see that

$$(\nu^2 + \rho^2)^{-\gamma} \leq (\rho^2 - (\beta - \alpha - 1)^2)^{-\gamma} = (4\beta(\alpha+1))^{-\gamma}.$$

Since $\beta - \alpha - 1 < \rho$, it follows that $E_r^c \cap D_{\alpha,\beta} = \emptyset$. Moreover, in (20) $\sigma(E_r \cap \mathbb{R}_+)$ must be replaced by $\sigma(E_r) = \sigma(E_r \cap \mathbb{R}_+) + \sigma(D_{\alpha,\beta})$. We note that

$$\sigma(D_{\alpha,\beta}) \leq (r^2 + \rho^2)^{\alpha+1} \frac{\sigma(D_{\alpha,\beta})}{(r^2+\rho^2)^{\alpha+1}} \leq (r^2 + \rho^2)^{\alpha+1} \frac{\sigma(D_{\alpha,\beta})}{\rho^{2(\alpha+1)}}.$$

Hence, applying the same argument, we can deduce the following.

Theorem 4.3. *Let $\delta > 0$ and $0 \leq \theta < 1/2$. Then there exists a positive constant $C_{\theta,\alpha,\beta}$ such that for all $f \in L^1(\Delta) \cap L^2(\Delta)$*

$$\|f\|_{L^1(\Delta)}^4 \leq C_{\theta,\alpha,\beta} \int_0^\infty |f(x)|^2 V_\delta(x)^{2\theta} \Delta(x) dx \int_0^\infty |(-L)^{(\alpha+1)\theta} f(x)|^2 \Delta(x) dx.$$

5. Main theorem. We retain the notations in the previous sections. We shall obtain a refinement of Theorem 4.3 with $\theta = 1/2(\alpha+1)$. For $x \geq 0$ we put

$$v(x) = \frac{V(x)}{\Delta(x)}$$

and for $\lambda \in \mathbb{C}$

$$w(\lambda) = (\lambda^2 + \rho^2)^{1/2}.$$

Theorem 5.1. *For all* $f \in L^1(\Delta) \cap L^2(\Delta)$,

$$\|fv\|^2_{L^2(\Delta)} \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \geq \frac{1}{4} \|f\|^4_{L^2(\Delta)}, \tag{21a}$$

where the equality holds if and only if f *is of the form*

$$f(x) = ce^{\gamma \int_0^x v(t)dt} \tag{21b}$$

for some $c, \gamma \in \mathbb{C}$ *and* $\Re\gamma < 0$.

Proof. Without loss of generality we may suppose that $f \in C^\infty_{c,e}(\mathbb{R})$. Since $(-Lf)^\wedge(\lambda) = \hat{f}(\lambda)(\lambda^2 + \rho^2) = \hat{f}(\lambda) w(\lambda)^2$ (see (8)) and $w(\lambda)$ is positive on $\mathbb{R}_+ \cup D_{\alpha,\beta}$, the Parseval formula (14) yields that

$$\begin{aligned}
\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu &= \int_0^\infty (-Lf)(x)\overline{f(x)}\Delta(x)dx \\
&= \int_0^\infty |f'(x)|^2 \Delta(x)dx.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
&\int_0^\infty |f(x)|^2 v(x)^2 \Delta(x)dx \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \\
= &\int_0^\infty |f(x)|^2 v(x)^2 \Delta(x)dx \int_0^\infty |f'(x)|^2 \Delta(x)dx \\
\geq &\Big(\int_0^\infty \Re(f(x)f'(x))v(x)\Delta(x)dx\Big)^2 \\
= &\frac{1}{4}\Big(\int_0^\infty (|f(x)|^2)' V(x)dx\Big)^2 = \frac{1}{4}\Big(\int_0^\infty |f(x)|^2 \Delta(x)dx\Big)^2.
\end{aligned}$$

Here we used the fact that $V' = \Delta$ (see (16)). Clearly, the equality holds if and only if $fv = cf'$ for some $c \in \mathbb{C}$, that is, $f'/f = c^{-1}v$. This means that $\log(f) = c^{-1} \int_0^x v(t)dt + C$ and thus, the desired result follows. ■

Since $w^2(\lambda) = \lambda^2 + \rho^2$, (21) and the Parseval formula (15) yield the following.

Corollary 5.2. *Let f be the same as in Theorem 5.1.*

$$\|fv\|^2_{L^2(\Delta)} \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \nu^2 d\nu \geq \frac{1}{4} \|f\|^2_{L^2(\Delta)} \int_0^\infty |f(x)|^2 (1 - 4\rho^2 v(x)^2) \Delta(x) dx.$$

We shall estimate v and $1 - 4\rho^2 v^2$. Since $\alpha > -1$, it follows that

$$\begin{aligned} V(x) &= \int_0^x (2\sinh s)^{2\alpha+1} (2\cosh s)^{2\beta+1} ds \\ &= 2^{2\rho} \int_0^{\sinh x} t^{2\alpha+1} (1+t^2)^\beta dt \\ &= 2^{2\rho} (\sinh x)^{2\alpha+2} \int_0^1 t^{2\alpha+1} (1 + (\sinh x)^2 t^2)^\beta dt \\ &= 2^{2\rho-1} (\sinh x)^{2\alpha+2} (\cosh x)^{2\beta} \int_0^1 (1-s)^\alpha (1 - (\tanh x)^2 s)^\beta ds \\ &= 2^{2\rho-1} (\sinh x)^{2\alpha+2} (\cosh x)^{2\beta} \frac{1}{\alpha+1} F(1, -\beta, 2+\alpha; (\tanh x)^2) \end{aligned}$$

and thus,

$$v(x) = \frac{1}{2(\alpha+1)} F(1, -\beta, 2+\alpha; (\tanh x)^2) \tanh x. \tag{22}$$

Lemma 5.3. *Let notation be as above. If $\beta \leq 0$ or $\beta \leq \alpha$, then*

$$0 \leq v(x) \leq \frac{1}{2\rho}$$

and if $\beta \geq 0$, then

$$0 \leq v(x) \leq \frac{1}{2(\alpha+1)},$$

and if $\beta > 0, \alpha \geq 0$, then

$$0 \leq v(x) \leq \frac{1}{\sqrt{2\rho - 1}}.$$

Proof. We recall Euler's integral expression of the hypergeometric function:

$$F(1,-\beta,2+\alpha,x^2) = (\alpha+1)\int_0^1 (1-t)^\alpha (1-tx^2)^\beta dt. \tag{23}$$

Thereby, $v(x) \geq 0$. If $\beta \leq 0$, then it is easy to see that $F(1,-\beta,2+\alpha;x)$ is increasing on $0 \leq x \leq 1$. Hence $H(x) = xF(1,-\beta,2+\alpha;x^2)$ is dominated by $H(1) = \Gamma(2+\alpha)\Gamma(\rho)/\Gamma(1+\alpha)\Gamma(\rho+1) = (\alpha+1)/\rho$ and thus $v(x) \leq 1/2\rho$. Let $0 < \beta \leq \alpha$. We shall prove that $H(x)$ is also increasing and $H(x) \leq H(1)$ as before. In order to prove that $H(x)$ is increasing, we shall show that its derivative is positive. We put $H_k(\alpha,\beta,x) = x^{2k+1}F(k+1,k-\beta,k+2+\alpha;x^2)$ and we note that

$$\begin{aligned} H'(x) &= x^{-1}H_0(\alpha,\beta,x) - \frac{2\beta}{2+\alpha}x^{-1}H_1(\alpha,\beta,x) \\ &= x^{-1}H_0(\alpha,\beta,x) + 2(1+\alpha)x^{-1}\Big(H_0(\alpha-1,\beta,x) - H_0(\alpha,\beta,x)\Big) \\ &= K(x), \end{aligned}$$

where $K(x) = F(1,-\beta,2+\alpha;x^2) + 2(1+\alpha)(F(1,-\beta,1+\alpha;x^2) - F(1,-\beta,2+\alpha,x^2))$. Then

$$\begin{aligned} K'(x) &= -2\beta x^{-2}\Big(\frac{1}{2+\alpha}H_1(\alpha,\beta,x) \\ &\quad +2(1+\alpha)\Big(\frac{H_1(\alpha-1,\beta,x)}{1+\alpha} - \frac{H_1(\alpha,\beta,x)}{2+\alpha}\Big)\Big). \end{aligned}$$

Since $\beta > 0$, $H_1(\alpha,\beta,x) = x^3F(2,1-\beta,3+\alpha;x) \leq x^3F(2,1-\beta,2+\alpha;x) = H_1(\alpha-1,\beta,x)$ and $1/(1+\alpha) - 1/(2+\alpha) > 0$, it follows that $K'(x) < 0$. Therefore, $H'(x) = K(x)$ is decreasing and

$$H'(x) \geq H'(1) = \frac{(\alpha-\beta)(\alpha+1)}{\rho(\alpha+\beta)} \geq 0$$

under the assumption on β. Hence $H(x)$ is increasing.

Next let $\beta \geq 0$. Then it follows from (23) that

$$\frac{1}{2(\alpha+1)}xF(1,-\beta,2+\alpha;x^2) \leq \frac{1}{2}\int_0^1 (1-t)^\alpha dt = \frac{1}{2(\alpha+1)}.$$

Last let $\beta > 0$ and $\alpha \geq 0$. Then it follows from (23) that

$$\begin{aligned}\frac{1}{2(\alpha+1)} x F(1,-\beta,2+\alpha;x^2) &\leq \frac{x}{2}\int_0^1 (1-x^2t)^{\alpha+\beta}dt \\ &= \frac{1}{2\rho x}(1-(1-x^2)^\rho).\end{aligned}$$

We suppose that the last function takes the maximum at $x = x_0$. Then $2\rho(1-x_0^2)^{\rho-1}x_0^2 = 1-(1-x_0^2)^\rho$ and thereby, the last function is dominated by $(1-x_0^2)^{\alpha+\beta}x_0$. Since $(1-x^2)^{\alpha+\beta}x$ takes the maximum at $x = 1/\sqrt{2(\alpha+\beta)+1}$ and $\alpha+\beta > 0$, we see that $(1-x^2)^{\alpha+\beta}x$ is dominated by

$$\Big(\frac{2(\alpha+\beta)}{2(\alpha+\beta)+1}\Big)^{\alpha+\beta}\frac{1}{\sqrt{2(\alpha+\beta)+1}} \leq \frac{1}{\sqrt{2\rho-1}}.$$

Hence the desired estimate follows. ■

Lemma 5.4. *Let $\Upsilon(x) = 1-4\rho^2 v(x)^2$. If $\beta \leq 0$ or $\beta \leq \alpha$, then $\Upsilon(x) \geq (\cosh x)^{-2}$. Generally,*

$$\Upsilon(x) = \begin{cases} O((\cosh x)^{-2}) & \textit{if } x \to \infty, \\ O(1) & \textit{if } x \to 0. \end{cases}$$

Proof. Since $F(1,-\beta,2+\alpha;0) = 1$ and $F(1,-\beta,2+\alpha;1) = (\alpha+1)/\rho$, the asymptotic behavior easily follows. As in the proof of Lemma 5.3, if $\beta \leq 0$ or $\beta \leq \alpha$, then $F(1,-\beta,2+\alpha;x)$ is increasing with respect to x. Hence $v(x) \leq F(1,-\beta,2+\alpha;1)\tanh x\ /2(\alpha+1) \leq (1/2\rho)\tanh x$ and thus, $\Upsilon(x) \geq (\cosh x)^{-2}$. ■

We put

$$\tau_{\alpha,\beta} = \begin{cases} 1 & \text{if } \beta \leq 0 \text{ or } \beta \leq \alpha, \\ \dfrac{\rho}{\alpha+1} & \text{if } \beta > 0 \text{ and } \alpha < 0, \\ \min\Big(\dfrac{\rho}{\alpha+1}, \dfrac{2\rho}{\sqrt{2\rho-1}}\Big) & \text{if } \beta > \alpha \geq 0. \end{cases} \tag{24}$$

Lemma 5.3 implies that

$$0 \leq v(x) \leq \frac{\tau_{\alpha,\beta}}{2\rho}. \tag{25}$$

The following assertion follows from Theorem 5.1, Corollary 5.2, Lemma 5.3 and Lemma 5.4.

Corollary 5.5. *Let $\rho > 0$ and f be the same as in Theorem 5.1.*

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \geq \rho^2 \tau_{\alpha,\beta}^{-2} \|f\|^2_{L^2(\Delta)}, \tag{26}$$

and if $f = f_P$, then

$$\int_0^\infty |\hat{f}(\lambda)|^2 |C(\lambda)|^{-2} d\lambda \geq \rho^2 \tau_{\alpha,\beta}^{-2} \int_0^\infty |f_P(x)|^2 \Upsilon(x) \Delta(x) dx.$$

The shapes of $v(t)$ and $\Upsilon(t)$, $t = \operatorname{arctanh}\sqrt{x}$, $x \geq 0$, are respectively given as follows.

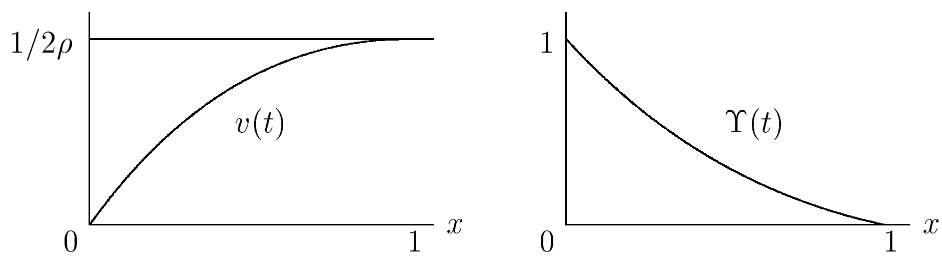

Figure 1: The case of $\beta \leq 0$ or $\beta \leq \alpha$.

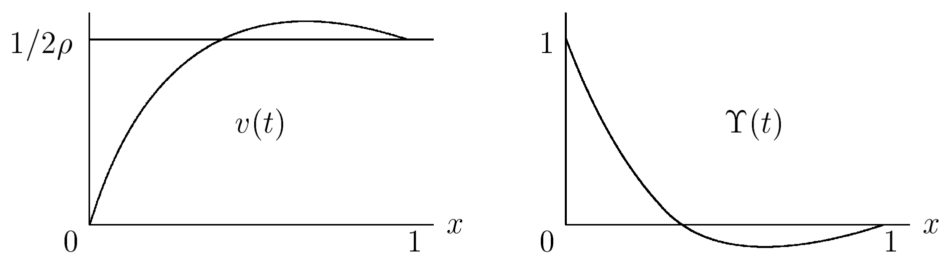

Figure 2: The case of $\beta > 0$ and $\beta > \alpha$.

In (26) we set

$$f(g) = \phi_\mu(g) = \frac{\sqrt{2}}{\Gamma(\alpha+1)} \Big(\frac{\Gamma(\alpha+1)}{\sqrt{2}} d(\mu)^{-1} \Big) \phi_\mu(g) d(\mu)$$

for $\mu \in D_{\alpha,\beta}$. Then it follows from (12) that

$$\|\phi_\mu v\|^2_{L^2(G)}(-|\mu|^2 + \rho^2) \geq \frac{1}{4}\|\phi_\mu\|^2_{L^2(G)}.$$

Especially,

$$\int_0^\infty |\phi_\mu(x)|^2 \Upsilon(x)\Delta(x)dx \leq -4\|\phi_\mu v\|^2_{L^2(G)}|\mu|^2 < 0.$$

Moreover, if we denote the maximum value of v by $v_{\max}$, then for $\mu \in D_{\alpha,\beta}$,

$$v^2_{\max} \geq \frac{1}{4(-|\mu|^2 + \rho^2)}$$

and hence

$$v^2_{\max} \geq \frac{1}{16\beta(\alpha+1)}.$$

6. Uncertainty principles. We shall apply the inequalities obtained in the previous section to deduce some information on the concentration of f and $\hat{f}$. Let f be a non-zero function in $L^2(\Delta)$. We recall that

$$f = f_P + {}^\circ f, \quad {}^\circ f(x) = \frac{2}{\Gamma(\alpha+1)} \sum_{\mu \in D_{\alpha,\beta}} a_\mu \phi_\mu(x) d(\mu)$$

and $\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{a_\mu\})$ (see (10)).

Definition 6.1. *Let $0 < \epsilon < 1/2\rho$ and $M > 0$.*

(1) We say that a function $f(x)$ on $\mathbb{R}_+$ is (v, ϵ)-concentrated at $x = 0$ if

$$\|fv\|_{L^2(\Delta)} \leq \epsilon \|f\|_{L^2(\Delta)} \tag{27a}$$

and is (v, M)-nonconcentrated at $x = 0$ if the reverse replaced ϵ by M holds.

(2) We say that a function $\hat{f}(\lambda)$ on $\mathbb{R}_+$ is (λ, ϵ)-concentrated at $\lambda = 0$ if

$$\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda \leq \epsilon^2 \|f\|^2_{L^2(\Delta)} \tag{27b}$$

and is (λ, M)-nonconcentrated at $\lambda = 0$ if the reverse replaced ϵ by M holds.

(3) We say that a function $f(x)$ on $\mathbb{R}_+$ has an ϵ-small discrete part if

$$\|{}^\circ f\| \leq \epsilon \|f\|_{L^2(\Delta)}. \tag{27c}$$

(4) We say that a function $f(x)$ on $\mathbb{R}_+$ is (Υ, ϵ)-nonconcentrated at $x = 0$ if

$$\Big| \int_0^\infty |f(x)|^2 \Upsilon(x) \Delta(x) dx \Big| \leq \epsilon^2 \|f\|^2_{L^2(\Delta)}.$$

(5) We say that a function $f(x)$ on $\mathbb{R}_+$ is (x_0, ϵ)-bounded if

$$|f(x)| \leq \epsilon e^{-\rho x} \|f\|_{L^2(\Delta)} \text{ if } x \geq x_0.$$

Now we suppose that $f(x)$ is (v, ϵ)-concentrated at $x = 0$. Since

$$\begin{aligned} &\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \\ = &\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda - \sum_{D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) + \rho^2 \|f\|^2_{L^2(\Delta)} \end{aligned}$$

(see (15)), it follows from (21) and (27a)

$$\begin{aligned} &\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda \\ \geq &\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda - \sum_{D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) \\ = &\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu - \rho^2 \|f\|^2_{L^2(\Delta)} \\ \geq &(1/4\epsilon^2 - \rho^2) \|f\|^2_{L^2(\Delta)}. \end{aligned} \tag{28}$$

Therefore, $\hat{f}(\nu)$ is $(\lambda, (1/4\epsilon^2 - \rho^2)^{1/2})$-nonconcentrated at $\lambda = 0$.

Conversely, we suppose that $\hat{f}(\nu)$ is (λ, ϵ)-concentrated at $\lambda = 0$. Since $\Upsilon(x) = 1 - 4\rho^2 v(x)^2 \geq 1 - \tau^2_{\alpha,\beta}$ (see (25)), it follows that

$$\int_0^\infty |f_P(x)|^2 \Upsilon(x) \Delta(x) dx \geq (1 - \tau^2_{\alpha,\beta}) \|f_P\|^2_{L^2(\Delta)}. \tag{29}$$

We recall that $1-\tau_{\alpha,\beta}^2 \le 0$. Moreover, letting $A = \displaystyle\int_0^\infty |f_P(x)|^2 \Upsilon(x)\ \Delta(x)dx$ and $B = \|f_P\|^2_{L^2(\Delta)}$, we see from Corollary 5.2 for $f = f_P$ and (27b) that

$$(B-A)\epsilon^2 B \ge \rho^2 AB$$

and thus, $A \le \dfrac{\epsilon^2 B}{\rho^2+\epsilon^2} \le \dfrac{\epsilon^2}{\rho^2}B$, that is,

$$\int_0^\infty |f_P(x)|^2 \Upsilon(x)\Delta(x)dx \le \frac{\epsilon^2}{\rho^2}\|f_P\|^2_{L^2(\Delta)}. \tag{30}$$

Therefore, (29) and (30) imply that $f_P(x)$ is (Υ, δ)-nonconcentrated at $x = 0$, where

$$\delta = \max\{(\tau_{\alpha,\beta}^2 - 1)^{1/2}, \rho^{-1}\epsilon\}.$$

Moreover, letting $\delta = 1$ in (5), we see from (10), (3) and (27b) that for $x \ge 1$,

$$\begin{aligned}
|f_P(x)| &\le c\Big|\int_0^\infty \hat{f}(\lambda)\Phi_\lambda(x)C(\lambda)^{-1}d\lambda\Big| \\
&\le ce^{-\rho x}K_1\Big(\int_0^\epsilon |\hat{f}(\lambda)||C(-\lambda)|^{-1}d\lambda + \int_\epsilon^\infty |\hat{f}(\lambda)||C(-\lambda)|^{-1}d\lambda\Big) \\
&\le ce^{-\rho x}K_1\Big(\epsilon^{1/2}\|f_P\|_{L^2(\Delta)} \\
&\qquad + \Big(\int_\epsilon^\infty |\hat{f}(\lambda)|^2\lambda^2|C(\lambda)|^{-2}d\lambda\Big)^{1/2}\Big(\int_\epsilon^\infty \lambda^{-2}d\lambda\Big)^{1/2}\Big) \\
&\le 2cK_1\epsilon^{1/2}e^{-\rho x}\|f_P\|_{L^2(\Delta)}. \qquad (31)
\end{aligned}$$

Hence we have the following.

Theorem 6.2 *Let $\rho > 0$ and $f \in L^2(\Delta)$. If $f(x)$ is (v, ϵ)-concentrated at $x = 0$, then $\hat{f}(\lambda)$ is $(\lambda, (1/4\epsilon^2 - \rho^2)^{1/2})$-nonconcentrated at $\lambda = 0$. Conversely, if $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$, then $f_P(x)$ is (Υ, δ)-nonconcentrated at $x = 0$, where $\delta = \max\{(\tau_{\alpha,\beta}^2 - 1)^{1/2}, \rho^{-1}\epsilon\}$, and there exists a positive constant $c = c_{\alpha,\beta}$ such that $f_P(x)$ is $(1, c\epsilon^{1/2})$-bounded.*

When $\beta \le \alpha$, we recall that $D_{\alpha,\beta} = \emptyset$, $f = f_P$ and $\tau_{\alpha,\beta} = 1$. Hence, the above theorem implies that, if $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$, then $f(x)$ is $(\Upsilon, \rho^{-1}\epsilon)$-nonconcentrated at $x = 0$ and $(1, c\epsilon^{1/2})$-bounded. Therefore, $f(x)$ is spread if ϵ goes to 0.

When $\beta > \alpha$, then $\tau_{\alpha,\beta} > 1$ and it is not clear that $f(x)$ is spread if ϵ goes to 0. We must pay attention to the discrete part of f. We suppose that $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$ and moreover, $f(x)$ has an ϵ_d-small discrete part. Of course, if $\beta < \alpha + 1$, then we can take $\epsilon_d = 0$, because $D_{\alpha,\beta} = \emptyset$. We shall prove that $f(x)$ is spread if ϵ and ϵ_d go to 0. First we note that (30) replaced f_P by f holds as before:

$$\int_0^\infty |f(x)|^2 \Upsilon(x)\Delta(x)dx \leq \frac{\epsilon^2}{\rho^2}\|f\|_{L^2(\Delta)}. \tag{32}$$

Let $x_0 > 0$ be the point such that $\Upsilon(x_0) = 0$ (see Fig. 2). In (31), replacing $\delta = 1$ in (5) by $\delta = x_0$, we see that for $x \geq x_0$,

$$|f_P(x)| \leq cK_{x_0}\epsilon^{1/2}e^{-\rho x}\|f_P\|_{L^2(\Delta)}.$$

On the other hand, it follows from (11), (15b) and (27c) that

$$\begin{aligned} |{}^\circ f(x)| &\leq c \sum_{\mu \in D_{\alpha,\beta}} |a_\mu||\phi_\mu(x)|d(\mu) \\ &\leq ce^{-\rho x}\Big(\sum_{\mu \in D_{\alpha,\beta}} e^{-2|\mu|x_0}d(\mu)\Big)^{1/2}\|{}^\circ f\|_{L^2(\Delta)} \leq c\epsilon_d e^{-\rho x}\|f\|_{L^2(\Delta)}. \end{aligned}$$

Hence, for $x \geq x_0$, we see that there exists a positive constant c_0 such that

$$|f(x)| \leq c_0 e^{-\rho x}(\epsilon^{1/2} + \epsilon_d)\|f\|_{L^2(\Delta)}. \tag{33}$$

Since $\Upsilon(x) \leq 0$ if $x \geq x_0$, it follows that

$$\begin{aligned} \int_0^\infty |f(x)|^2\Upsilon(x)\Delta(x)dx &\geq c\int_{x_0}^\infty |f(x)e^{\rho x}|^2\Upsilon(x)dx \\ &\geq cc_0^2(\epsilon^{1/2} + \epsilon_d)^2\|f\|^2_{L^2(\Delta)}\int_{x_0}^\infty \Upsilon(x)dx \\ &= -c_\Upsilon(\epsilon^{1/2} + \epsilon_d)^2\|f\|^2_{L^2(\Delta)}, \end{aligned} \tag{34}$$

where $c_\Upsilon \geq 0$. Then (32), (33) and (34) imply the following.

Theorem 6.3 *Let $\rho > 0$, $\beta > \alpha$ and $f \in L^2(\Delta)$. We suppose that $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$ and $f(x)$ has an ϵ_d-small discrete part. We take a sufficiently small ϵ such that $\delta^2 = c_\Upsilon(\epsilon^{1/2} + \epsilon_d)^2 \geq \rho^{-2}\epsilon^2$. Then $f(x)$ is*

(Υ, δ)-nonconcentrated at $x = 0$ and there exists a positive constant $c = c_{\alpha,\beta}$ such that $f(x)$ is $(x_0, c\delta)$-bounded.

We suppose that f is supported on $[R, \infty)$. Then there exists a constant $0 < \delta(R) \leq 1$ such that

$$0 \leq v(x) \leq \frac{1}{2\rho\delta(R)}, \ x \geq R$$

and $\delta(R) \to 1$ if $R \to \infty$. Since $1 - 4\rho^2 v(x)^2 \geq 1 - \delta(R)^{-2}$, it follows from Corollary 5.2 that

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \nu^2 d\nu \geq \rho^2(\delta(R)^2 - 1)\|f\|^2_{L^2(\Delta)}.$$

Then we obtain the following.

Proposition 6.4. *Let $\rho > 0$ and suppose that $f \in L^2(\Delta)$ is supported on $[R, \infty)$. Then*

$$\sum_{\mu \in D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) \leq \int_0^\infty |\hat{f}_P(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda + \rho^2(1 - \delta(R)^2)\|f\|^2_{L^2(\Delta)}.$$

Remark 6.5. When $\beta = 0$ and $\alpha \geq 0$, it follows from (22) that $v(x) = (2\rho)^{-1} \tanh x$ and $1 - 4\rho^2 v(x)^2 = (\cosh x)^{-2}$. Therefore, the inequalities in Theorem 5.1 and Corollary 5.2 became

$$\|f(x) \tanh x\|^2_{L^2(\Delta)} \|\hat{f}(\lambda)(\lambda^2 + \rho^2)^{1/2}\|^2_{L^2(|C|^{-2})} \geq \rho^2 \|f\|^4_{L^2(\Delta)},$$

where the equality holds if and only if f is of the form $c(\cosh x)^\gamma$, $c, \gamma \in \mathbb{C}$, $\Re\gamma < 0$, and

$$\|f(x) \tanh x\|^2_{L^2(\Delta)} \|\hat{f}(\lambda)\lambda\|^2_{L^2(|C|^{-2})} \geq \rho^2 \|f\|^2_{L^2(\Delta)} \|f(x)(\cosh x)^{-1}\|^2_{L^2(\Delta)}.$$

Since the Jacobi transform of $(\cosh \lambda)^\gamma$ is explicitly calculated in [1], we can directly check the above equality condition for these inequalities.

7. Uncertainty principles on $SU(1,1)$. We briefly give some basic notations to introduce the spherical Fourier transform on $G = SU(1,1)$. For the precise definitions we refer to [6] and [8]. We denote ϕ_λ, $\Delta(x)$ and $C(\lambda)$ in §1 respectively by $\phi_\lambda^{\alpha,\beta}$, $\Delta_{\alpha,\beta}(x)$ and $C_{\alpha,\beta}(\lambda)$.

Let A, K denote the subgroups of G of the matrices

$$a_x = \begin{pmatrix} \cosh x/2 & \sinh x/2 \\ \sinh x/2 & \cosh x/2 \end{pmatrix} \quad \text{and} \quad k_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix},$$

where $x \in \mathbb{R}$ and $0 \le \phi \le 4\pi$ respectively. According to the Cartan decomposition of G, each $g \in G$ can be written uniquely as $g = k_\phi a_x k_\psi$ where $0 \le x$, $0 \le \phi, \psi \le 4\pi$. Let $\pi_{j,\lambda}$ ($j = 0, 1/2, \lambda \in \mathbb{R}$) denote the principal series representation of G. Then the (operator-valued) spherical Fourier transform $\pi_{j,\lambda}(f)$ of f on G is defined as $\pi_{j,\lambda}(f) = \displaystyle\int_G f(g)\pi_{j,\lambda}(g)dg$, where dg a Haar measure on G. In the following, we normalize dg as $dg = \Delta_{0,0}(x)dxd\phi d\psi$ and we treat only functions f on G whose K-types are supported on $\mathbb{Z} \times \mathbb{Z}$. Under this restriction, $\pi_{j,\lambda}(f)$ is supported on $j = 0$ and $\lambda > 0$ (cf. [6] and [8, §8]) and

$$f(a_x) = f(a_{-x}), \quad x \in \mathbb{R}.$$

Let $n, m \in \mathbf{N}$ and $\psi_\lambda^{n,m}(g)$ ($\lambda \in \mathbb{R}$, $g \in G$) denote the matrix coefficient of $\pi_{0,\lambda}(g)$ with K-type (n, m). Let f be a compactly supported C^∞ function on G whose K-type is (n, m). Then the scalar-valued spherical Fourier transform $\tilde{f}_{n,m}(\lambda)$ of type (n, m) is defined by

$$\tilde{f}_{n,m}(\lambda) = \int_G f(g)\psi_\lambda^{(n,m)}(g)dg. \tag{35}$$

Since the K-type of $\psi_\lambda^{n,m}(g)$ is of (n, m), this integral is determined on $A_+ \cong \mathbb{R}_+$. We recall that the explicit form of $\psi_\lambda^{n,m}(a_x)$ is given by using the Jacobi function (cf. [4, (4.17)] and [6, (3.4.10)]): For $g = k_\phi a_x k_\psi \in G$,

$$\psi_\lambda^{n,m}(g) = (\cosh x)^{n+m}(\sinh x)^{|n-m|}Q_{n,m}(\lambda)\phi_\lambda^{|n-m|,n+m}(x)e^{in\phi}e^{im\psi}, \tag{36}$$

where

$$Q_{n,m}(\lambda) = \binom{-1/2 - i\lambda/2 \mp m}{|n-m|}$$

and $\mp m$ is equal to $-m$ if $n \geq m$ and m if $n \leq m$. Hence, compared with (7) and (35), we see from (36) that

$$\begin{aligned}\tilde{f}_{n,m}(\lambda) &= 2^{-(|n-m|+n+m)-1/2}\Gamma(|n-m|+1)Q_{n,m}(\lambda)\\ &\quad\times\Big(f(x)(2\sinh x)^{-|n-m|}(2\cosh x)^{-(n+m)}\Big)^{\wedge}_{|n-m|,n+m}(\lambda).\end{aligned}$$

We here fix the K-type of f as (n,m) and we define a compactly supported C^∞ even function F on $\mathbb{R}$ as

$$F(x) = f(x)(2\sinh x)^{-|n-m|}(2\cosh x)^{-(n+m)}.$$

Then it follows that

$$\|f\|^2_{L^2(G)} = \int_0^\infty |f(x)|^2\Delta_{0,0}(x)dx = \|F\|_{L^2(\Delta_{|n-m|,n+m})}$$

and

$$\tilde{f}_{n,m}(\lambda) = 2^{-(|n-m|+n+m)-1/2}\Gamma(|n-m|+1)Q_{n,m}(\lambda)\hat{F}_{|n-m|,n+m}(\lambda).$$

Therefore, since

$$Q_{n,m}(\lambda)^{-2}|C_{|n-m|,n+m}(\lambda)|^{-2} = 2^{-2(|n-m|+n+m)}\Gamma(|n-m|+1)^2|C_{0,0}(\lambda)|^{-2},$$

the Plamcherel formula for the Jacobi transform for F (see (10) and (15)) implies that

$$\|f\|^2_{L^2(G)} = 2\Big(\int_0^\infty |\tilde{f}_{n,m}(\lambda)|^2|C_{0,0}(\lambda)|^{-2}d\lambda + \sum_{\mu\in D^{n,m}}|\tilde{f}_{n,m}(\mu)|^2 d^{n,m}(\mu)\Big),$$

where $D^{n,m} = D_{|n-m|,n+m}$ in §1 and $d^{n,m}(\mu) = 2^{2(|n-m|+n+m)}\Gamma(|n-m|+1)^{-2}Q_{n,m}(\mu)^{-2}d_{|n-m|,n+m}(\mu)$. This is nothing but the Plancherel formula for the spherical Fourier transform of type (n,m) on G (see [4, (4.21)] and [8, Theorem 8.2]). As before, this transform can be extended to the one for L^2-functions on G with K-type (n,m). According to the decomposition (10) for F, each L^2-function f on G with K-type (n,m) is of the form

$$f = f_P + {}^\circ f,$$

where ${}^\circ f(g) = 2\sum_{\mu\in D^{n,m}} a_\mu \psi^{n,m}_\mu(g)d^{n,m}(\mu)$, and then $\tilde{\boldsymbol{f}} = (\tilde{f}_{n,m}, \{a_\mu\})$. We call f_P and ${}^\circ f$ the principal part and the discrete part of f respectively. We here introduce $v_{n,m}, w_{n,m}$ and $\rho_{n,m}$ respectively corresponding to v, w and ρ with $\alpha = |n-m|, \beta = n+m$ in §1. Then for $\tilde{\boldsymbol{f}} = (\tilde{f}_{n,m}, \{a_\mu\})$ it follows that

$$\int_{\mathbb{R}_+ \cup D^{n,m}} \tilde{\boldsymbol{f}}(\nu) d_{m,n}\nu = \int_{-\infty}^{\infty} \tilde{f}(\lambda)|C_{0,0}(\lambda)|^{-2} d\lambda + \frac{1}{2} \sum_{\mu \in D^{n,m}} a_\mu d^{n,m}(\mu).$$

Hence the inequality in Theorem 5.1 can be rewritten as

$$\|f v_{n,m}\|^2_{L^2(G)} \int_{\mathbb{R}_+ \cup D^{n,m}} |\tilde{\boldsymbol{f}}(\nu)|^2 w_{n,m}(\nu)^2 d_{n,m}\nu \geq \frac{1}{4}\|f\|^4_{L^2(G)}.$$

We now suppose that $f(g)$ is concentrated at $g = e$: There exists a positive constant $\epsilon_{n,m}$ such that

$$\|f v_{n,m}\|^2_{L^2(G)} \leq \epsilon_{n,m}\|f\|^2_{L^2(G)}. \tag{37}$$

As in the same argument in §5 (see (28)), it follows that

$$\int_0^{\infty} |\tilde{f}_{n,m}(\lambda)|^2 \lambda^2 |C_{0,0}(\lambda)|^{-2} d\lambda \geq \left(\frac{1}{4\epsilon_{n,m}} - \rho^2_{n,m}\right)\|f\|^2_{L^2(G)}. \tag{38}$$

In particular, if $\epsilon_{n,m}$ is of the form

$$\epsilon_{n,m} = \frac{\epsilon}{8\rho^2_{n,m}}.$$

for $0 < \epsilon < 1$, then

$$\epsilon_{n,m} = \frac{\epsilon}{8\rho^2_{n,m}} \leq \frac{\epsilon}{4(1+\epsilon)\rho^2_{n,m}} \leq \frac{\epsilon}{4(1+\epsilon\rho^2_{n,m})}$$

and thus,

$$\left(\frac{1}{4\epsilon_{n,m}} - \rho^2_{n,m}\right) \geq \frac{1}{\epsilon}.$$

Therefore, (37) and (38) are respectively rewritten as

$$\|f \rho_{n,m} v_{n,m}\|^2_{L^2(G)} \leq \frac{\epsilon}{8}\|f\|^2_{L^2(G)}.$$

and

$$\int_0^{\infty} |\tilde{f}_{n,m}(\lambda)|^2 \lambda^2 |C_{0,0}(\lambda)|^{-2} d\lambda \geq \frac{1}{\epsilon}\|f\|^2_{L^2(G)}.$$

Let $f = \sum_{n,m\in\mathbb{N}} f^{n,m}$ denote the K-type decomposition of an L^2-function f on G whose K-types are supported on $\mathbb{N}\times\mathbb{N}$. Since

$$\|f\|^2_{L^2(G)} = \sum_{n,m\in\mathbb{N}} \|f^{n,m}\|_{L^2(G)}$$

and the Hilbert-Schmidt norm of $\pi_{0,\lambda}(f) = \Big((f^{n,m})^\wedge(\lambda)\Big)_{n,m\in\mathbb{N}}$ is given by

$$\|\pi_{0,\lambda}(f)\|^2_{\mathrm{HS}} = \sum_{n,m\in\mathbb{N}} |f^{n,m}(\lambda)|^2,$$

we can obtain the following.

Theorem 7.1. *Let $\epsilon > 0$ and $f = \sum_{n,m\in\mathbb{N}} f^{n,m}$ be an L^2-function on $SU(1,1)$. We suppose that each $f^{n,m}$ is concentrated at $x = 0$ such as*

$$\|f^{n,m}\rho_{n,m}v_{n,m}\|^2_{L^2(G)} \leq \frac{\epsilon}{8}\|f^{n,m}\|^2_{L^2(G)}. \tag{39}$$

Then

$$\int_0^\infty \|\pi_{0,\lambda}(f)\|^2_{\mathrm{HS}}\lambda^2|C_{0,0}(\lambda)|^{-2}d\lambda \geq \frac{1}{\epsilon}\|f\|^2_{L^2(G)},$$

where $\|\ \|_{\mathrm{HS}}$ is the Hilbert-Schmid norm. In particular, $\|\pi_{0,\lambda}(f)\|_{\mathrm{HS}}$ does not concentrate at $\lambda = 0$.

Remark 7.2. It easily follows from (24) and (25) that

$$\rho_{n,m}v_{n,m} = O\Big(\min\Big(\frac{|n-m|+n+m}{|n-m|+1}, \sqrt{|n-m|+n+m}\Big)\Big).$$

Therefore, if the right or left K-types of f are finite, then $\{\rho_{n,m}v_{n,m}\}$ in (39) are uniformly bounded. However, for example, if $n = m$, then $\{\rho_{n,n}v_{n,n}\}$ are not uniformly bounded.

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