Generalized Hardy's theorem for the Jacobi transform

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Abstract

The classical Hardy theorem on $\mathbf{R}$ was generalized by Miyachi [**?**] and Bonami, Demange, and Jaming [**?**]. In this paper we show that Miyachi's theorem and Bonami-Demange-Jaming' one can be reformulated for the Jacobi transform in terms of the heat kernel.

1. Introduction. For $f \in L^1(\mathbf{R})$ we define the Fourier transform $\tilde{f}(\lambda)$, $\lambda \in \mathbf{R}$, of f by

$$\tilde{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx.$$

Let us take two positive numbers a, b which satisfy the relation $ab = 1/4$. Miyachi's theorem in [**?**] states that if $f \in L^1(\mathbf{R})$ satisfies

$$e^{ax^2} f(x) \in L^1(\mathbf{R}) + L^\infty(\mathbf{R})$$

and

$$\int_{-\infty}^{+\infty} \log^+ \frac{|\tilde{f}(\lambda) e^{b\lambda^2}|}{C} d\lambda < \infty$$

for some $C > 0$, then f is a constant multiple of e^{-ax^2}, where $L^1(\mathbf{R}) + L^\infty(\mathbf{R})$ is the set of functions of the form $f = f_1 + f_2$, $f_1 \in L^1(\mathbf{R})$, $f_2 \in L^\infty(\mathbf{R})$, and $\log^+ x = \log x$ if $x > 1$ and $\log^+ x = 0$ if $x \leq 1$. On the other hand, one dimendional case of Bonami-Demange-Jaming's theorem in [**?**] states that $f \in L^2(\mathbf{R})$ satisfies

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|f(x)||\tilde{f}(y)|}{(1 + |x| + |y|)^N} e^{|xy|} dx dy < \infty$$

for some $N \geq 0$ if and only if f is written as $f(x) = P(x) e^{-ax^2}$, where P is a polynomial of degree $< (N-1)/2$. Both theorems are generalizations of the classical Hardy theorem and the Cowling-Price theorem which is an L^p version of the classical Hardy one (see [?] and [?]).

Recently, Hardy's theorem on Lie groups has been investigated by various people. As remarked by V.S. Varadarajan some years ago, Hardy's theorem can be written in terms of the heat kernel of the Laplacian on the groups. Then, considerable attention has been paid to discover a connection between the heat kernel and analogues of Hardy's theorem and Cowling-Price's theorem on Lie groups. For this subject we refer to [?], [?], [?], and [?]. Moreover, N.B. Andersen [?] and the second auther of this article and J. Liu [?] obtained independently an analogue of Hardy's theorem and its L^p version for the Jacobi transform. The aim of this article is to show that the above two theorems can be restated for the Jacobi transform in terms of the heat kernel.

2. Notations. We collect relevant material from the harmonic analysis associated with the Jacobi transform. General references for this section are [?], [?] and [?]. For $\alpha, \beta, \lambda \in \mathbf{C}$ and $x \in \mathbf{R}_+ = [0, \infty)$, the Jacobi function $\phi_\lambda(x)$ of order (α, β), $\alpha \neq -1, -2, \cdots$, is the unique solution on $\mathbf{R}_+$ of the differential equation:

$$L_{\alpha,\beta} u = -(\lambda^2 + \rho^2) u, \quad u(0) = 1, \text{ and } u'(0) = 0,$$

where $\rho = \alpha + \beta + 1$ and

$$L_{\alpha,\beta} = \frac{d^2}{dx^2} + ((2\alpha + 1)\coth x + (2\beta + 1)\tanh x)\frac{d}{dx}.$$

In the following we suppose that $\alpha \geq \beta \geq -1/2$. Then $\phi_\lambda(x)$ is estimated as

$$|\phi_\lambda(x)| \leq \begin{cases} 1 & \text{if } |\Im\lambda| \leq \rho, \\ e^{(|\Im\lambda| - \rho)x} & \text{if } |\Im\lambda| > \rho, \\ \phi_{i\Im\lambda}(x) \end{cases} \tag{1}$$

for all $x \in \mathbf{R}_+$ (see [?, Lemma 11]). For a compactly supported C^∞ function f on $\mathbf{R}_+$ the Jacobi transform $\hat{f}(\lambda)$, $\lambda \in \mathbf{C}$, of f is given by

$$\hat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x)\Delta_{\alpha,\beta}(x)dx, \tag{2}$$

where $\Delta_{\alpha,\beta}(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$. We recall that for all $\lambda \in \mathbf{C}$,

$$(L_{\alpha,\beta} f)^\wedge(\lambda) = -(\lambda^2 + \rho^2)\hat{f}(\lambda). \tag{3}$$

The Abel transform $F_f(x)$, $x \in \mathbf{R}_+$, of f is given as

$$F_f(x) = \int_x^\infty f(s)A(x,s)ds, \quad x \geq 0, \tag{4}$$

where $A(x,s)$ is positive, even with respect to x and moreover, it satisfies

$$\Delta_{\alpha,\beta}(s)\phi_\lambda(s) = c\int_0^s \cos(\lambda x)A(x,s)dx, \quad s \geq 0. \tag{5}$$

We refer to [**?**, (2.16), (3.5)] for the explicit form of $A(x,s)$. We recall that

$$\hat{f}(\lambda) = \tilde{F}_f(\lambda), \quad \lambda \in \mathbf{C}, \tag{6}$$

where f and F_f are regarded as even functions on $\mathbf{R}$ and the right hand side $\tilde{F}_f$ denotes the Euclidean Fourier transform of F_f. We note that the Jacobi transform is extended to functions for which the right hand side of (2) is well-defined. For example, if $f \in L^1(\mathbf{R}_+, \Delta_{\alpha,\beta}(x)dx)$, then $\hat{f}(\lambda)$, $\lambda \in \mathbf{R}$, is well-defined and it has a holomorphic extension on the tube domain $|\Im\lambda| < \rho$ (see (1)). Also the relations (3) and (6) hold for $|\Im\lambda| < \rho$. Moreover, the map $f \to \hat{f}$ extends to an isometry between $L^2(\mathbf{R}_+, \Delta_{\alpha,\beta}(x)dx)$ and $L^2(\mathbf{R}_+, |C_{\alpha,\beta}(\lambda)|^{-2}d\lambda)$, where $C_{\alpha,\beta}(\lambda)$ denotes the Harish-Chandra C-function (cf. [**?**, (2.6)]).

For $t > 0$ let $h_t(x)$, $x \in \mathbf{R}$, denote the heat kernel associated to $L_{\alpha,\beta}$, that is, the even C^∞ function on $\mathbf{R}$ such that

$$\hat{h}_t(\lambda) = e^{-t(\lambda^2+\rho^2)}, \quad \lambda \in \mathbf{R}. \tag{7}$$

We recall that

$$h_t(x) \sim t^{-\alpha-1}e^{\rho^2 t}e^{-\rho x - x^2/4t}(1+t+x)^{\alpha-1/2}(1+x), \quad x \geq 0, \tag{8}$$

where "$\sim$" means that the ratio of the left side and the right side is bounded below and above by positive constants (see [**?**, Corollary 1], cf. [**?**, Theorem 3.1]). Hence (8) and (1) imply that $\hat{h}_t(\lambda)$ is entire and (7) holds for $\lambda \in \mathbf{C}$.

3. Miyachi's theorem. We shall obtain an extension of Miyachi's theorem for the Jacobi transform. We put

$$d_\alpha x = (\tanh x)^{2\alpha+1}(1+x)^{\alpha+1/2}dx \text{ on } \mathbf{R}_+$$

and

$$L^\infty(\mathbf{R}_+) + L^1(\mathbf{R}_+, d_\alpha x) = \{f_1 + f_2 \ ; \ f_1 \in L^\infty(\mathbf{R}_+), f_2 \in L^1(\mathbf{R}_+, d_\alpha x)\}.$$

Theorem 3.1. *Let us take positive constants a, b which satisfy $ab = 1/4$. Suppose f is a measurable function on $\mathbf{R}_+$ satisfying*

$$
\begin{aligned}
(A) &\quad : \quad f(x)h_{1/4a}^{-1}(x) \in L^\infty(\mathbf{R}_+) + L^1(\mathbf{R}_+, d_\alpha x) \\
(B) &\quad : \quad \int_{-\infty}^{\infty} \log^+ \frac{|\hat{f}(\lambda)e^{b\lambda^2}|}{C} d\lambda < \infty \text{ for some } 0 < C < \infty.
\end{aligned}
$$

Then f is a constant multiple of $h_{1/4a}$.

Proof. The first condition (A) implies that $fh_{1/4a}^{-1} = u+v$, where $u \in L^\infty(\mathbf{R}_+)$ and $v \in L^1(\mathbf{R}_+, d_\alpha x)$ and hence, $f = h_{1/4a}u + h_{1/4a}v$. As for the first term, it follows from (1) that for all $\lambda = \xi + i\eta \in \mathbf{C}$,

$$
\begin{aligned}
|(h_{1/4a}u)^\wedge(\lambda)| &\leq \|u\|_\infty \int_0^\infty h_{1/4a}(x)\phi_{i\eta}(x)\Delta_{\alpha,\beta}(x)dx \\
&= c\hat{h}_{1/4a}(i\eta) = ce^{b\eta^2}.
\end{aligned}
$$

As for the second term, it follows from (1) and (7) that, if $|\eta| > \rho$, then

$$
\begin{aligned}
&|(h_{1/4a}v)^\wedge(\lambda)| \\
\leq\ & c\int_0^\infty |v(x)|e^{-\rho x - ax^2}(1+x)^{\alpha-1/2}(1+x)e^{(|\eta|-\rho)x}\Delta_{\alpha,\beta}(x)dx \\
\leq\ & c\int_0^\infty |v(x)|(\tanh x)^{2\alpha+1}(1+x)^{\alpha+1/2}e^{-a(x-|\eta|/2a)^2}dx \cdot e^{\eta^2/4a} \\
\leq\ & c\|v\|_{L^1(\mathbf{R}_+, d_\alpha x)}e^{b\eta^2}
\end{aligned}
$$

and, if $|\eta| \leq \rho$, since $e^{-ax^2} \leq ce^{-\rho x}$ for $x \geq 0$, it follows that

$$
|(h_{1/4a}v)^\wedge(\lambda)| \leq c\|v\|_{L^1(\mathbf{R}_+, d_\alpha x)} \leq ce^{b\eta^2}.
$$

Hence, $\hat{f}(\lambda)$ is entire and it satisfies $|\hat{f}(\lambda)| \leq ce^{b\eta^2}$ for all $\lambda \in \mathbf{C}$ and (B). We here recall the lemma which is used in the proof of Miyachi's theorem (see [**?**, Lemma 4]):

Lemma 3.2. *Suppose $F(\lambda)$ is an entire function and there exist constant $A, B > 0$ such that*

$$
|F(\lambda)| \leq Ae^{B(\Re\lambda)^2} \quad \text{and} \quad \int_{-\infty}^{\infty} \log^+ |F(\lambda)|d\lambda < \infty.
$$

Then F is a constant function.

Therefore, applying this lemma to $\hat{f}(\lambda)e^{-b\lambda^2}/C$, we see that $\hat{f}(\lambda) = ce^{-b\lambda^2}$ and thus, $f(x) = ch_{1/4a}(x)$. ■

4. Bonami-Demange-Jaming's theorem. We shall obtain an extension of Bonami-Demange-Jaming's theorem for the Jacobi transform.

Theorem 4.1 *Let us take a function* $f \in L^2(\mathbf{R}_+, \Delta_{\alpha,\beta}(x)dx)$ *and a non-negative integer* N. *Then the inequality*

$$(A) \quad \int_0^\infty \int_0^\infty \frac{|f(x)||\hat{f}(\lambda)|}{(1+\lambda)^N} \phi_{i\lambda}(x)\Delta_{\alpha,\beta}(x)dxd\lambda < \infty$$

holds if and only if f *can be written as*

$$(B) \quad f(x) = P(L_{\alpha,\beta})h_a(x),$$

where $a > 0$ *and* P *is a polynomial of* $\deg P < (N-1)/4$.

Proof. First we shall prove that (A) implies (B) by reducing the case to the original Bonami-Demange-Jaming theorem on $\mathbf{R}$. Since $\hat{f}(\lambda) = \tilde{F}_f(\lambda)$ (see (6)), it follows from (4), (5) and (A) that

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{|F_f(x)||\tilde{F}_f(\lambda)|}{(1+x+\lambda)^N} e^{x\lambda} dxd\lambda \\ \leq & \int_0^\infty \int_0^\infty |f(s)||\hat{f}(\lambda)|\Big(\int_0^s \frac{A(x,s)e^{x\lambda}}{(1+x+\lambda)^N}dx\Big)dsd\lambda \\ \leq & \; 2\int_0^\infty \int_0^\infty \frac{|f(s)||\hat{f}(\lambda)|}{(1+\lambda)^N}\Big(\int_0^s A(x,s)\cos(ix\lambda)dx\Big)dsd\lambda \\ = & \; 2c\int_0^\infty \int_0^\infty \frac{|f(s)||\hat{f}(\lambda)|}{(1+\lambda)^N}\phi_{i\lambda}(s)\Delta_{\alpha,\beta}(s)dsd\lambda < \infty. \end{aligned}$$

As in the first step of the proof of Proposition 2.2 in [**?**], F_f belongs to $L^1(\mathbf{R}_+)$. Hence $\hat{f} = \tilde{F}_f$ is bounded on $\mathbf{R}$. Since $\tilde{F}_f \in L^2(\mathbf{R}_+, |C_{\alpha,\beta}(\lambda)|^{-2}d\lambda)$ and $|C_{\alpha,\beta}(\lambda)|^{-2}$ is polynomial growth of order $\alpha + 1/2$, it easily follows that $\tilde{F}_f \in L^2(\mathbf{R})$ and thus, $F_f \in L^2(\mathbf{R})$ as an even function on $\mathbf{R}$. Then F_f satisfies the condition of Theorem 1.1 in [**?**], which yields that

$$\tilde{F}_f(\lambda) = Q(\lambda)e^{-a\lambda^2},$$

where $a > 0$ and Q is an even polynomial of degree $< (N-1)/2$. Since Q is even, this relation can be rewritten as

$$\tilde{F}_f(\lambda) = P(-(\lambda^2+\rho^2))e^{-a(\lambda^2+\rho^2)},$$

where P is a polynomial of $\deg P < (N-1)/4$. Since the map $f \to \tilde{F}_f$ is bijectiove on $L^2(\mathbf{R}_+, \Delta_{\alpha,\beta}(x)dx)$, it easily follows from (3) that $f(x) = P(L_{\alpha,\beta})h_a(x)$.

Next we suppose that $f(x) = P(L_{\alpha,\beta})h_a(x)$, where $a > 0$ and P is a polynomial of $\deg P < (N-1)/4$. Then, $\hat{f}(\lambda) = \tilde{F}_f(\lambda)$ is of the form $Q(\lambda)e^{-a\lambda^2}$, where Q is an even polynomial of degree $< (N-1)/2$. We note that, if $f \geq 0$, then

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{|f(x)||\hat{f}(\lambda)|}{(1+\lambda)^N}\phi_{i\lambda}(x)\Delta_{\alpha,\beta}(x)dxd\lambda \\ = &\int_0^\infty \frac{\hat{f}(i\lambda)|\hat{f}(\lambda)|}{(1+\lambda)^N}d\lambda = \int_0^\infty \frac{Q(i\lambda)|Q(\lambda)|}{(1+\lambda)^N}d\lambda < \infty. \end{aligned}$$

We recall that for $x \geq 0$, $f(x) = P(L_{\alpha,\beta})h_a(x) \sim U(x)h_a$, where $U(x)$ is a polynomial of degree $d = 2\deg P$, because $h_a = h_a^{\alpha,\beta}$ is defined by $h_a^{\alpha,\beta} \sim W^2_{-\beta-1/2}W^1_{-\alpha+\beta}(e^{-x^2/4a})$ as a function of x (cf. [?, §3]) and thus, $dh_a^{\alpha,\beta}/dx = \sinh(2x)W^2_{-1}(h_a^{\alpha,\beta}) \sim \sinh(2x)h_a^{\beta+1,\alpha+1} \sim xh_a^{\alpha,\beta}$ (see (8)). Here we may suppose that the coefficient of x^d is positive. Since there exists a positive constant c such that $h_a(x) \geq c(1+x)^{\alpha+1/2}e^{-x^2/4a-\rho x} \geq ce^{-x^2/4a-\rho x}$ for $x \geq 0$ (see (8)), there exists a positive constant A such that $f(x)+Ah_a(x) \geq 0$ for $x \geq 0$. Hence, $|f(x)| = |f(x)+Ah_a(x)-Ah_a(x)| \leq f(x)+2Ah_a(x)$. Then, replacing $|f(x)|$ with $f(x)+2Ah_a(x) \geq 0$, that is, $Q(i\lambda)$ with $Q(i\lambda)+2Ae^{-a\rho^2}$ in the above calculation, we have the desired result. ∎

As an easy consequence of Theorem 4.1, we can deduce the Beurling theorem for the Jacobi transform.

Theorem 4.2. *Suppose that $f \in L^1(\mathbf{R}_+, \Delta_{\alpha,\beta}(x)dx)$ satisifes*

$$\int_0^\infty \int_0^\infty |f(x)||\hat{f}(\lambda)|\phi_{i\lambda}(x)\Delta_{\alpha,\beta}(x)dxd\lambda < \infty.$$

Then $f = 0$.

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Added in proof. After we have accomplished this paper, we were informed that R. P. Sarkar and J. Sengupta also investigated a generalization of Beurling's theorem in the paper titled Beurling's theorem for Riemaniann symmetric spaces of noncompact type.

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