REAL HARDY SPACES ON REAL RANK 1 SEMISIMPLE LIE GROUPS

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Let G be a real rank one connected semisimple Lie group with finite center. We introduce a real Hardy space $H^1(G//K)$ on G as the space consisting of all K-bi-invariant functions f on G whose radial maximal functions $M_\phi f$ are integrable on G. We shall obtain a relation between $H^1(G//K)$ and $H^1(\mathbf{R})$, the real Hardy space on the real line $\mathbf{R}$, via the Abel transform on G and give a characterization of $H^1(G//K)$.

1. Introduction

The study of the classical Hardy spaces on the unit disk and the upper half plane was originated during the 1910's by the complex variable method. In the 1970's the Hardy spaces were completely characterized by various maximal functions of their boundary values and also by atomic decompositions, without using the complex variable method. This is a significant breakthrough in harmonic analysis. Nowadays, the spaces defined by the real variable method – maximal functions and atoms – called *real* Hardy spaces and a fruitful theory of real Hardy spaces has been extended to the spaces of homogeneous type: A topological space X with measure μ and distance d is of homogeneous type if there exists a constant $c > 0$ such that for all $x \in X$ and $r > 0$

$$\mu(B(x, 2r)) \leq c\mu(B(x, r)),$$

where $B(x, r)$ is the ball defined by $\{y \in X \mid d(x, y) < r\}$ and $\mu(B(x, r))$ the volume of the ball (cf. [**1**, §1]). However, when the space X is not of homogeneous type, little work on real Hardy spaces on X has been done.

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Hence, looking at the example of a semisimple Lie group G as a space of non-homogeneous type, we shall introduce a real Hardy space $H^1(G//K)$ by using radial maximal functions on G. In this article we shall overview some results obtained in the previous papers [**5**], [**6**], [**7**] and announce a new characterization of $H^1(G//K)$, which gives a relation between $H^1(G//K)$ and the real Hardy space $H^1(\mathbf{R})$ on $\mathbf{R}$ via the Abel transform on G.

2. Notation

Let G be a real rank one connected semisimple Lie group with finite center, $G = KAN = KAK$ Iwasawa and Cartan decompositions of G. Let $dg = dkdadn = \Delta(a)dkdadk'$ denote the corresponding decompositions of a Haar measure dg on G. In what follows we shall treat only K-bi-invariant functions on G. Since A is identified with $\mathbf{R}$ as $A = \{a_x; x \in \mathbf{R}\}$, all K-bi-invariant functions can be identified with even functions on $\mathbf{R}$ denoted by the same letter as

$$f(g) = f(a_{\sigma(g)}) = f(\sigma(g)) = f(-\sigma(g)).$$

We may regard the weight $\Delta(a_x)$ as an even function given by

$$\Delta(x) = c(\mathrm{sh}|x|)^{2\alpha+1}(\mathrm{sh}2|x|)^{2\beta+1}, \tag{1}$$

where $\alpha = (m_1 + m_2 - 1)/2$, $\beta = (m_2 - 1)/2$ and m_1, m_2 the multiplicities of a simple root γ of (G, A) and 2γ respectively. We note that the one dimensional space $\mathbf{R}$ with normal distance and weighted measure $\Delta(x)dx$ is not of homogeneous type, because $\Delta(x) \sim e^{2\rho x}$ with $\rho = \alpha + \beta + 1 > 0$ as $x \to \infty$. Let $L^p(G//K)$ denote the space of all K-bi-invariant functions on G with finite L^p-norm and $L^1_{\mathrm{loc}}(G//K)$ the space of all locally integrable, K-bi-invariant functions on G.

Let F be the dual space of the Lie algebra of A and for $\lambda \in \mathsf{F}$, φ_λ the normalized zonal spherical function on G:

$$\varphi_\lambda(x) = {}_2F_1\left((\rho + i\lambda)/2, (\rho - i\lambda)/2; \alpha + 1; -\mathrm{sh}^2 x\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function. We recall that, if $\lambda \notin \mathbf{Z}$, then $\varphi_\lambda(x)$ has the so-called Harish-Chandra expansion:

$$\varphi_\lambda(x) = e^{-\rho x}\left(\Phi(\lambda, x)C(\lambda)e^{i\lambda x} + \Phi(-\lambda, x)C(-\lambda)e^{-i\lambda x}\right), \tag{2}$$

where $C(\lambda)$ is Harish-Chandra's C-function. For some basic properties of $\varphi_\lambda(x)$, $\Phi(\lambda, x)$, and $C(\lambda)$ we refer to [**2**, §2, §3] and [**12**, 9.1.4, 9.1.5].

For $f \in L^1(G//K)$ the spherical Fourier transform $\hat{f}(\lambda)$, $\lambda \in \mathsf{F}$, of f is defined by

$$\hat{f}(\lambda) = \int_G f(g)\varphi_\lambda(g)dg = \int_0^\infty f(x)\varphi_\lambda(x)\Delta(x)dx.$$

Since $\varphi_\lambda(x)$ is even with respect to λ, x and uniformly bounded on x if λ is in the tube domain $\mathsf{F}(\rho) = \{\lambda \in \mathsf{F}_c; |\Im\lambda| \le \rho\}$, it follows that $\hat{f}(\lambda)$ is even, continuously extended on $\mathsf{F}(\rho)$, holomorphic in the interior, and

$$|\hat{f}(\lambda)| \le \|f\|_1, \quad \lambda \in \mathsf{F}(\rho).$$

For $f \in C_c^\infty(G//K)$ the Paley-Wiener theorem (cf. [**2**, Theorem 4]) implies that $\hat{f}(\lambda)$ is holomorphic on F_c of exponential type. Furthermore, it satisfies the inversion formula

$$f(x) = \int_{-\infty}^\infty \hat{f}(\lambda)\varphi_\lambda(x)|C(\lambda)|^{-2}d\lambda$$

and the Plancherel formula

$$\int_0^\infty |f(x)|^2\Delta(x)dx = \int_0^\infty |\hat{f}(\lambda)|^2|C(\lambda)|^{-2}d\lambda.$$

Therefore, the spherical Fourier transform $f \mapsto \hat{f}$ of $C_c^\infty(G//K)$ is uniquely extended to an isometry between $L^2(G//K) = L^2(\mathbf{R}_+,\ \Delta(x)dx)$ and $L^2(\mathbf{R}_+,\ |C(\lambda)|^{-2}d\lambda)$ (cf. [**2**, Proposition 3], [**12**, Theorem 9.2.2.13]).

For $f \in C_c^\infty(G//K)$ we define the Abel transform F_f^s, $s \in \mathbf{R}$, of f as

$$F_f^s(x) = e^{\rho(1+s)x}\int_N f(a_x n)dn. \tag{3}$$

Here the Euclidean Fourier transform $(F_f^s)^\sim(\lambda)$ is holomorphic on F_c of exponential type, because $F_f^s(f) \in C_c^\infty(\mathbf{R})$, and it coincides with the spherical Fourier transform of f:

$$\hat{f}(\lambda + is\rho) = (F_f^s)^\sim(\lambda), \quad \lambda \in \mathsf{F}_c \tag{4}$$

(cf. [**9**, §3]). Especially, F_f^0 is even on $\mathbf{R}$. The integral over N in (3) can be explicitly rewritten by using a generalized Weyl type fractional integral operator W_μ^σ: For $\sigma > 0$, $\mu \in \mathbf{C}$ and $y > 0$,

$$W_\mu^\sigma(f)(y) = \frac{(-1)^n}{\Gamma(\mu+n)}\int_y^\infty \frac{d^n f(x)}{d(\mathrm{ch}\sigma x)^n}(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu+n-1}d(\mathrm{ch}\sigma x), \tag{5}$$

where $n = 0$ if $\Re\mu > 0$ and $-n < \Re\mu \le -n+1$, $n = 0, 1, 2, \cdots$, if $\Re\mu \le 0$ (see [**9**, (3.11)]). Then Koornwinder obtains that for $x > 0$,

$$F_f^0(x) = W_{\alpha-\beta}^1 \circ W_{\beta+1/2}^2(f)(x)$$

(see [**9**, (2.18), (2.19), (3.5)]). In the following, for simplicity, we denote $W_+(f)(x) = F_f^1(|x|)$, that is,

$$W_+(f)(x) = e^{\rho x} W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(f)(|x|),\ x \in \mathbf{R} \tag{6}$$

and for a smooth function F on $\mathbf{R}_+$,

$$W_-(F)(x) = W^2_{-(\beta+1/2)} \circ W^1_{-(\alpha-\beta)}(e^{-\rho x}F),\ x \in \mathbf{R}_+. \tag{7}$$

Clearly, $W_- \circ W_+(f) = f$ and $W_+ \circ W_-(F) = F$.

For $f \in L^1(G//K)$, $W_+(f)$ belongs to $L^1(\mathbf{R})$, because the integral formula for the Iwasawa decomposition of G yields that

$$\|W_+(f)\|_{L^1(\mathbf{R})} \le \|f\|_1 \tag{8}$$

(cf. [**9**, (3.5), (2.20)]). Hence $W_+(f)^\sim(\lambda)$, $\lambda \in \mathsf{F}$, is well-defined and it follows from (4) that

$$\hat{f}(\lambda + i\rho) = W_+(f)^\sim(\lambda), \quad \lambda \in \mathsf{F}. \tag{9}$$

Let $f, g \in L^1(G//K)$. Since $f * g \in L^1(G//K)$ and $(f*g)^\wedge(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$ (cf. [**2**, Theorem 5], [**3**, §5]), it follows that

$$W_+(f * g) = W_+(f) * W_+(g). \tag{10}$$

We say that a function F on $\mathbf{R}$ is W_+-smooth if $W_-(F)$ is well-defined and continuous. Then, for W_+-smooth functions F, G on $\mathbf{R}$ with compact support such that $e^{-\rho x}F$ and $e^{-\rho x}G$ are even, it follows that

$$W_-(F * G) = W_-(F) * W_-(G).$$

3. Radial maximal functions

As in the Euclidean case, to define a radial maximal function we need to define a dilation ϕ_t, $t > 0$, of a function ϕ on G. Let ϕ be a positive compactly supported C^∞, K-bi-invariant function on G such that

$$\int_G \phi(g)dg = \int_0^\infty \phi(x)\Delta(x)dx = 1 \tag{11}$$

and furthermore, there exists $M \in \mathbf{N}$ such that

$$\phi(x) = O(x^{2M}). \tag{12}$$

We define the dilation ϕ_t of ϕ as

$$\phi_t(x) = \frac{1}{t}\frac{1}{\Delta(x)}\Delta\left(\frac{x}{t}\right)\phi\left(\frac{x}{t}\right).$$

Clearly, ϕ_t has the same L^1-norm of ϕ: $\|\phi_t\|_1 = \|\phi\|_1$ and, for $1 \le p \le \infty$, it gives an approximate identity in $L^p(G//K)$ (see [**2**, Lemma 16]). We here introduce the radial maximal function $M_\phi f$ on G as follows.

Definition 3.1. *For $f \in L^1_{\rm loc}(G//K)$,*

$$(M_\phi f)(g) = \sup_{0<t<\infty} |(f * \phi_t)(g)|, \quad g \in G.$$

As shown in [**5**, Theorem 3.4 and Theorem 3.5], M_ϕ satisfies the maximal theorem and, for $1 \le p \le \infty$, $\|f\|_p \le c\|M_\phi f\|_p$ if the both sides exist. By using $W_+(\phi_t)$, we shall define a maximal function on $\mathbf{R}$ as follows.

Definition 3.2. *For $F \in L^1_{\rm loc}(\mathbf{R})$,*

$$(M_\phi^{\mathbf{R}} F)(x) = \sup_{0<t<\infty} |(F * W_+(\phi_t)(x)|, \quad x \in \mathbf{R}.$$

Since $W_+(f * \phi_t) = W_+(f) * W_+(\phi_t)$ (see (10)) and W_+ is an integral operator with a positive kernel (see (5), (6)), it follows that

$$\sup_{0<t<\infty} |W_+(f) * W_+(\phi_t)(x)| \le W_+ \left(\sup_{0<t<\infty} |f * \phi_t| \right)(x).$$

Therefore, from (8) we have a relation between M_ϕ and $M_\phi^{\mathbf{R}}$:

Proposition 3.3. *For $f \in L^1_{\rm loc}(G//K)$,*

$$(M_\phi^{\mathbf{R}} W_+(f))(x) \le W_+(M_\phi f)(x), \quad x \in \mathbf{R}.$$

In particular,

$$\|M_\phi^{\mathbf{R}} W_+(f)\|_{L^1(\mathbf{R})} \le c\|M_\phi f\|_1$$

if the both sides exist.

Now we note that $W_+(\phi_t)^\sim(\lambda) = \hat{\phi}_t(\lambda + i\rho)$ (see (9)) has similar properties of the Euclidean Fourier transform of a Euclidean dilation:

(1) *There exists c such that for all $t > 0$, $\lambda \in \mathbf{R}$ and $0 \le k \le M$,*

$$\left| \left(\frac{d}{d\lambda} \right)^n \hat{\phi}_t(\lambda + i\rho) \right| \le ct^n (1+t)^k (1 + |t\lambda|)^{-2k},$$

(2) *There exists c such that for all $t > 1$ and $\lambda \in \mathbf{R}$,*

$$\left| \left(\frac{d}{d\lambda} \right)^n \hat{\phi}_t(\lambda + i\rho) \right| \le ct^n (1 + |t\lambda|)^{-(2M+\alpha+1/2)},$$

(3) $\hat{\phi}_t(\lambda + i\rho) \to 1 \quad \text{as } |t\lambda| \to 0,$

(4) $|\hat{\phi}_t(\lambda + i\rho)| \ge 1/2 \quad \text{if } 0 \le |t\lambda| \le 2,$

where M is the same as in (12). These properties guarantee that $W_+(\phi)$ behaves like a Euclidean dilation on $\mathbf{R}$. Hence, the maximal operator $M_\phi^{\mathbf{R}}$ can characterize $H^1(\mathbf{R})$, that is, $F \in H^1(\mathbf{R})$ if and only if $M_\phi^{\mathbf{R}}(F) \in L^1(\mathbf{R})$:

Theorem 3.4. *Let ϕ be as above and suppose that $M \geq 2$. Then $F \in H^1(\mathbf{R})$ if and only if $M_\phi^{\mathbf{R}} F \in L^1(\mathbf{R})$:*

$$\|F\|_{H^1(\mathbf{R})} \approx \|M_\phi^{\mathbf{R}} F\|_{L^1(\mathbf{R})}.$$

4. Real Hardy spaces

Let ϕ be the same as in the previous section (see (11), (12)) and $M_\phi, M_\phi^{\mathbf{R}}$ the corresponding radial maximal operators on G and $\mathbf{R}$ respectively (see Definitions 3.1 and 3.2). In this section we shall define two real Hardy spaces $H_\phi^1(G//K)$ and $W_-(H^1(\mathbf{R}))$ on G and give a relation between them.

Definition 4.1. *We define*

$$H_\phi^1(G//K) = \{f \in L^1_{\mathrm{loc}}(G//K)\ ;\ M_\phi f \in L^1(G//K)\}$$

and $\|f\|_{H_\phi^1(G)} = \|M_\phi f\|_1$.

Clearly, since $\|f\|_1 \leq c\|M_\phi f\|_1$, it follows that

$$H_\phi^1(G//K) \subset L^1(G//K).$$

Next we shall introduce a pull-back of the real Hardy space $H^1(\mathbf{R})$ on $\mathbf{R}$ to G via W_+ (see (6)). Let M_s, $s \geq 0$, denote the Euclidean Fourier multiplier defined by

$$\mathsf{M}_s(F)^\sim(\lambda) = (\lambda + i\rho)^s F^\sim(\lambda).$$

Definition 4.2. *For $s \geq 0$, we define*

$$W_-(M_{-s}(H^1(\mathbf{R}))) = \{f \in L^1_{\mathrm{loc}}(G//K)\ ;\ M_s \circ W_+(f) \in H^1(\mathbf{R})\}$$

and give the norm by $\|M_s \circ W_+(f)\|_{H^1(\mathbf{R})}$. We denote $W_-(M_0(H^1(\mathbf{R})))$ by $W_-(H^1(\mathbf{R}))$ for simplicity.

Obviously, Proposition 3.3 and Theorem 3.4 yield the following.

Corollary 4.3. *Let $M \geq 2$. There exists a positive constant c such that $\|W_+(f)\|_{H^1(\mathbf{R})} \leq c\|f\|_{H_\phi^1(G)}$ for all $f \in H_\phi^1(G//K)$ and thus,*

$$H_\phi^1(G//K) \subset W_-(H^1(\mathbf{R})).$$

Let $s_\alpha = \alpha + 1/2$. Then we see that

$$W_-(\mathsf{M}_{-s_\alpha}(H^1(\mathbf{R}))) \subset W_-(H^1(\mathbf{R})). \tag{13}$$

Actually, let $f \in W_-(\mathsf{M}_{-s_\alpha}(H^1(\mathbf{R})))$ and put $F = W_+(f)$. By the definition, $\mathsf{M}_{s_\alpha}(F)$ belongs to $H^1(\mathbf{R})$. Since the Fourier multiplier M_{-s_α} satisfies the Hörmander condition (cf. [**11**, §5 in Chap.11]), it is bounded on $H^1(\mathbf{R})$ (cf. [**11**, Theorem 4.4 in Chap.14]). Thereby, $F \in H^1(\mathbf{R})$ and the desired inclusion (13) follows. Similarly, since the Fourier multiplier $\mathsf{M}_{s_\alpha}^{-1} \circ W_{-\gamma}^{\mathbf{R}}$, $0 \le \gamma \le s_\alpha$, which corresponds to $(i\lambda)^\gamma/(\lambda + i\rho)^{s_\alpha}$, satisfies the Hörmander condition, it is bounded on $H^1(\mathbf{R})$. Hence, each $W_{-\gamma}^{\mathbf{R}}(F)$ also belongs to $H^1(\mathbf{R})$: For $0 \le \gamma \le s_\alpha$,

$$\|W_{-\gamma}^{\mathbf{R}}(F)\|_{H^1(\mathbf{R})} = \|M_\phi^{\mathbf{R}}(W_{-\gamma}^{\mathbf{R}}(F))\|_{L^1(\mathbf{R})} \le c\|\mathsf{M}_{s_\alpha}(F)\|_{H^1(\mathbf{R})}.$$

Now we shall characterize the H_ϕ^1-norm of $f \in H_\phi^1(G//K)$ and show that the real Hardy space $H_\phi^1(G//K)$ is located between $W_-(\mathsf{M}_{-s_\alpha}(H^1(\mathbf{R})))$ and $W_-(H^1(\mathbf{R}))$ (see (13)). We recall that

$$f * \phi_t = W_-(W_+(f) * W_+(\phi_t)) = W_-(F * W_+(\phi_t))$$

(see (10)). Therefore, roughly speaking, the H_ϕ^1-norm of f, that is, the L^1-norm of $M_\phi f$ on G (see Definition 4.1) can be characterized in terms of the L^1-norm of $M_\phi^{\mathbf{R}}(W_-(F))$ (see Definition 3.2). We rewrite $W_-(F)$ by using the Weyl type fractional operator $W_\mu^{\mathbf{R}}$ on $\mathbf{R}$:

$$W_\mu^{\mathbf{R}}(F)(y) = \frac{(-1)^n}{\Gamma(\mu+n)} \int_y^\infty \frac{d^n F(x)}{dx^n}(x-y)^{\mu+n-1}dx.$$

Here $W_\mu^{\mathbf{R}}(F)(\mathrm{ch}y) = W_\mu^1(f)(y)$ if $f(x) = F(\mathrm{ch}x)$ and $\alpha = \beta = 1/2$ in (5). Let $\delta = (\alpha - \beta) - [\alpha - \beta]$ and $\delta' = (\beta - 1/2) - [\beta - 1/2]$, where $[\ \ ]$ is the Gauss symbol, and put $\underline{n} = [s_\alpha]$, $\underline{\delta} = \delta + \delta'$ and $\underline{D} = \{\delta, \delta', \delta + \delta'\}$ respectively. Then the local and global forms of $W_-(F)$ can be rewritten as follows:

(1) *If F is W_+-smooth and supported on $0 < x \le 1$, then*

$$|W_-(F)(x)| \le c \sum_{m,\xi} \left(x^{-2s_\alpha + \underline{\delta} + m} W_{-(m+\xi)}^{\mathbf{R}}(F)(x) + \int_x^\infty W_{-(m+\xi)}^{\mathbf{R}}(F)(s) A_m^1(x,s)ds \right),$$

where the sum is taken over $0 \le m \le \underline{n}$ and $\xi \in \underline{D}$ and $A_m^1(x,s)$ satisfies

$$0 \le A_m^1(x,s) \le x^{-2s_\alpha + \underline{\delta} + m - 1} \quad \textit{for all } 0 < x \le s. \tag{14}$$

(2) *If F is W_+-smooth and supported on $x \geq 1$, then*

$$
\begin{aligned}
&|W_-(F)(x)| \\
&\leq c\sum_{m,\xi}\Bigg(\Big(x^{-2s_\alpha+\underline{\delta}+m}W^{\mathbf{R}}_{-(m+\xi)}(F)(x)+\int_x^\infty W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^2_m(x,s)ds \\
&\quad +x^{-2\alpha+\underline{\delta}+m}\int_x^\infty W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^3_m(x,s)ds\Big)\chi_{[0,1]}(x) \\
&\quad +e^{-2\rho x}\Big(W^{\mathbf{R}}_{-(m+\xi)}(F)(x)+\int_x^\infty W^{\mathbf{R}}_{-(m+\xi)}(F)(s)A^4_m(x,s)ds\Big)\chi_{[1,\infty)}(x)\Bigg),
\end{aligned}
$$

where $A^2_m(x,s)$ satisfies (14) and for $j=3,4$, $A^j_m(x,s)\geq 0$ and there exists a positive constant c such that

$$\int_0^s A^j_m(x,s)dx \leq c \quad \text{for all } s>0.$$

By using these local and global forms of $W_-(F)$, we can rewrite the L^1-norm of $M^{\mathbf{R}}_\phi(W_-(F))$ on $\mathbf{R}$ in terms of $M^{\mathbf{R}}_\phi(W^{\mathbf{R}}_{-\gamma}(F))$, $0\leq\gamma\leq s_\alpha$, on $\mathbf{R}$. Finally, we have the following.

Theorem 4.4 *Let $M\geq 2$ and $F=W_+(f)$ for $f\in W_-(\mathsf{M}_{-s_\alpha}(H^1(\mathbf{R})))$. Then there exist c_1,c_2 such that for all $0\leq\gamma\leq s_\alpha$,*

$$
\begin{aligned}
c_1\|M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-\gamma}(F)(x)(\mathrm{th}x)^\gamma\|_{L^1(\mathbf{R})} &\leq \|f\|_{H^1_\phi(G)} \\
\leq c_2\sum_{m=0}^{\underline{n}}\sum_{\xi\in\underline{D}}\|M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-(m+\xi)}(F)(x)(\mathrm{th}x)^{m+\underline{\delta}}\|_{L^1(\mathbf{R})}.
\end{aligned}
$$

Especially,

$$
\begin{aligned}
\|f\|_{H^1_\phi(G)} &\approx \sum_{m=0}^{\underline{n}}\sum_{\xi\in\underline{D}}\|M^{\mathbf{R}}_\phi\circ W^{\mathbf{R}}_{-(m+\xi)}(F)(x)(\mathrm{th}x)^{m+\xi}\|_{L^1(\mathbf{R})} \\
&\leq c\sum_{m=0}^{\underline{n}}\sum_{\xi\in\underline{D}}\|W^{\mathbf{R}}_{-(m+\xi)}(F)\|_{H^1(\mathbf{R})} \\
&\leq c\|\mathsf{M}_{s_\alpha}(F)\|_{H^1(\mathbf{R})}
\end{aligned}
$$

and thus,

$$W_-(\mathsf{M}_{-s_\alpha}(H^1(\mathbf{R})))\subset H^1_\phi(G//K)\subset W_-(H^1(\mathbf{R})).$$

Remark 4.5. Let $C(\lambda)$ be Harish-Chandra's C-function (see (2)) and M_{C_ρ} the Euclidean Fourier multiplier corresponding to $C_\rho(\lambda) = C(\lambda + i\rho)$:

$$\mathsf{M}_{C_\rho}(F)^{\sim}(\lambda) = C(\lambda + i\rho)F^{\sim}(\lambda).$$

We define

$$W_{-}(\mathsf{M}_{C_\rho}(H^1(\mathbf{R}))) = \{f \in L^1_{\mathrm{loc}}(G//K) \; ; \; \mathsf{M}_{C_\rho}^{-1} \circ W_{+}(f) \in H^1(\mathbf{R})\}.$$

Then it easily follows from Theorem 4.4 that

$$W_{-}(\mathsf{M}_{C_\rho}(H^1(\mathbf{R}))) \subset H^1_\phi(G//K) \subset W_{-}(H^1(\mathbf{R})).$$

This is one of the main results in [**6**]. However, the proof in [**6**] was a little bit complicated, because to obtain the first inclusion we used the Harish-Chandra expansion of the zonal spherical function φ_λ and also the Gangolli expansion of Φ_λ (see (2) and [**2**, §3]). Thereby, to sum up the estimates of each expanded terms we required sharp ones. Here we can obtain the desired inclusion as an easy consequence of Theorem 4.4.

5. Atomic Hardy spaces

We introduce *atomic* Hardy spaces on G. In the Euclidean case the atomic Hardy space $H^1_{\infty,0}(\mathbf{R})$ coincides with $H^1(\mathbf{R})$ (cf. [**4**, Theorem 3.30], [**10**, §2 in Chap.3]). However, it may be not true in our setting, because the Lebesgue measure dx is replaced by the weighted measure $\Delta(x)dx$ (see (1)). We denote the interval $[x_0 - r, x_0 + r]$ by $R(x_0, r)$ and set the volume by

$$|R(x_0, r)| = \int_{x_0 - r}^{x_0 + r} \Delta(x)dx.$$

We say that a K-bi-invariant function a on G is a $(1, \infty, 0)$-atom on G provided that there exist $x_0 \geq 0$ and $r > 0$ such that

$$\begin{aligned} &(i) \quad \mathrm{supp}(a) \subset R(x_0, r), \\ &(ii) \quad \|a\|_\infty \leq |R(x_0, r)|^{-1}, \\ &(iii) \quad \int_0^\infty a(x)\Delta(x)dx = 0. \end{aligned} \tag{15}$$

Here a is identified with a function on $\mathbf{R}_+$. Similarly, we shall define a $(1, \infty, 0, \epsilon)$-atom a and a $(1, \infty, +)$-atom a by replacing (ii) with

$$(ii)_\epsilon \quad \|a\|_\infty \leq |R(x_0, r)|^{-1}(1 + r)^{-\epsilon} \tag{16}$$

and (iii) with

$$(iii)_+ \quad \int_0^\infty a(x)\Delta(x)dx = 0 \;\text{ if } r \leq 1 \tag{17}$$

respectively. Then we shall define atomic Hardy spaces $H^1_{\infty,0}(G//K)$, $H^{1,\epsilon}_{\infty,0}(G//K)$, $H^{1,+}_{\infty,0}(G//K)$ and $h^1_{\infty,0}(G//K)$ as follows.

Definition 5.1. *Let notations be as above. We define*

$$H^1_{\infty,0}(G//K) = \{f = \sum_i \lambda_i a_i \; ; \; a_i \textit{ is } (1,\infty,0)\textit{-atom on } G \textit{ and } \textstyle\sum_i |\lambda_i| < \infty\}$$

and denote the norm by

$$\|f\|_{H^1_{\infty,0}(G)} = \inf \sum_i |\lambda_i|,$$

where the infimum is taken over all such representations $f = \sum_i \lambda_i a_i$. We also define $H^{1,\epsilon}_{\infty,0}(G//K)$ $(\epsilon \geq 0)$ and $H^{1,+}_{\infty,0}(G//K)$ by replacing $(1,\infty,0)$-atoms a on G in the above definition with $(1,\infty,0,\epsilon)$-atoms and $(1,\infty,+)$-atoms respectively. Moreover, we define the small Hardy space $h^1_{\infty,0}(G//K)$ on G by restricting $(1,\infty,0)$-atoms in the definition of $H^1_{\infty,0}(G//K)$ to ones with radius ≤ 1.

Clearly, for $\epsilon \geq 0$,

$$h^1_{\infty,0}(G//K) \subset H^{1,\epsilon}_{\infty,0}(G//K) \subset H^1_{\infty,0}(G//K) \subset H^{1,+}_{\infty,0}(G//K).$$

Let χ_1 denote the characteristic function of $B(1) = R(0,1)$ and set

$$\theta(g) = |B(1)|^{-1}\chi_1(g), \quad g \in G.$$

Moreover, for each (not necessarily K-bi-invariant) function f on G, we define a K-bi-invariant function $f^\flat_x$, $x \in G$, as

$$f^\flat_x(g) = \int_K \int_K f(x^{-1}kgk')dkdk', \quad g \in G.$$

Then the difference between $h^1_{\infty,0}(G//K)$ and $H^{1,+}_{\infty,0}(G//K)$ is given as follows.

Proposition 5.2. *For $f \in H^{1,+}_{\infty,0}(G//K)$ there exist $f_0 \in h^1_{\infty,0}(G//K)$ and $x_i \in G$, $\lambda_i \in \mathbf{R}$ such that*

$$f = f_0 + \sum_i \lambda_i \theta^\flat_{x_i},$$

where $\|f_0\|_{H^{1,\mathrm{loc}}_{\infty,0}(G)}$ and $\sum_i |\lambda_i|$ are respectively bounded by $\|f\|_{H^{1,+}_{\infty,0}(G)}$.

As in the Euclidean case, we shall introduce the truncated maximal operator M^{loc}_ϕ on G as

$$(M^{\mathrm{loc}}_\phi f)(g) = \sup_{0<t<1} |(f * \phi_t)(g)|, \quad g \in G.$$

Then M_ϕ^{loc} is bounded from $H^{1,+}_{\infty,0}(G//K)$ to $L^1(G//K)$ (see [**7**]). As for M_ϕ, we see from [**7**, Theorem 5.3] that it is bounded from $H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ to $L^1(G//K)$:

Proposition 5.3. *Let $M \geq 2$. M_ϕ is bounded from $H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ to $L^1(G//K)$, that is, there exists a constant $c > 0$ such that*

$$\|M_\phi f\|_1 \leq c\left(\|f\|_{H^{1,+}_{\infty,0}(G)} + \|W_+(f)\|_{H^1(\mathbf{R})}\right)$$

for all $f \in H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R}))$ and thus,

$$H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R})) \subset H^1_\phi(G//K).$$

Let a be a $(1,\infty,0,1)$-atom on G supported on $R(x_0,r)$. The conditions (15) and (16) imply that $\|a\|_\infty \leq |R(x_0,r)|^{-1}(1+r)^{-1}$ and $\displaystyle\int_G a(g)dg = 0$. Then $A = W_+(a)$ is supported on $R(x_0,r)$ and

$$\int_{-\infty}^{\infty} A(x)dx = A^\sim(0) = \hat{a}(i\rho) = \int_G a(g)dg = 0.$$

Moreover, it follows from (6) and [**8**, Lemma 3.4]) that $|A(x)| \leq ce^{2\rho x}\mathrm{th}(x_0+r)^{2s_\alpha}\|a\|_\infty$. Hence,

Case I: $x_0 - r \geq 1$. Since A is supported on $R(x_0,r)$ and

$$|R(x_0,r)| \sim \int_{x_0-r}^{x_0+r} e^{2\rho x}dx \sim e^{2\rho x_0}\mathrm{sh}r,$$

it follows that $|A(x)| \leq ce^{2\rho(x_0+r)}(e^{2\rho x_0}\mathrm{sh}r)^{-1}(1+r)^{-1} \leq cr^{-1}$.

Case II: $x_0 - r < 1$ and $r \geq 1$. Since $x_0 + r \geq 1$,

$$|R(x_0,r)| \geq c\int_1^{x_0+r} e^{2\rho x}dx \sim e^{2\rho(x_0+r)}.$$

Therefore, as in Case I, we have $\|A\|_\infty \leq cr^{-1}$.

Case III: $x_0 - r < 1$, $r < 1$ and $x_0 > 2r$. Since $x_0 > 2r$, it follows that $x_0 + r \geq 3$ and thus

$$|R(x_0,r)| \sim \int_{x_0-r}^{x_0+r} x^{2s_\alpha}dx \leq c(x_0-r)^{s_\alpha}r.$$

Since $(x_0+r)/(x_0-r) \leq 3$, we have $|A(x)| \leq c\mathrm{th}(x_0+r)^{s_\alpha}((x_0-r)^{s_\alpha}r)^{-1} \leq cr^{-1}$.

Case IV: $x_0 - r < 1$, $r < 1$ and $x_0 \leq 2r$. Since $x_0 + r \leq 3r < 3$ and $|R(x_0,r)| \geq |B(r)| \sim |B(3r)|$, we may suppose that a is a centered atom supported on $B(3r)$. Then $|A(x)| \leq c(\mathrm{th}3r)^{s_\alpha}|B(3r)|^{-1} \leq cr^{-1}$.

These four cases imply that cA is a $(1,\infty,0)$-atom on $\mathbf{R}$, where c is independent of a. Therefore, we obtain the following.

Theorem 5.4. *Let $M \geq 2$. Then*

$$H^{1,1}_{\infty,0}(G//K) \subset H^{1,+}_{\infty,0}(G//K) \cap W_-(H^1(\mathbf{R})).$$

Especially, M_ϕ is bounded from $H^{1,1}_{\infty,0}(G//K)$ to $L^1(G//K)$, that is, there exists a constant $c > 0$ such that

$$\|M_\phi f\|_1 \leq c\|f\|_{H^{1,1}_{\infty,0}(G)}$$

for all $f \in H^{1,1}_{\infty,0}(G//K)$.

6. Characterization of $H^1_\phi(G//K)$

We shall prove that the inclusion in Proposition 5.3 is the equality. Let $s_\alpha = \alpha + 1/2$ as above and put

$$d_\alpha(x_0, r) = \int_{\max\{0,|x_0|-r\}}^{|x_0|+r} (\mathrm{th}x)^{s_\alpha}\, dx.$$

We define a subspace $H^{1,+}_{\infty,0}(\mathbf{R})_\alpha$ of $H^1(\mathbf{R})$ as the space of all $F = \sum_i \lambda_i A_i$ such that $\sum_i |\lambda_i| < \infty$ and each A_i satisfies

$$\begin{aligned} &(i) \quad \mathrm{supp}(A_i) \subset R(x_i, r_i) \\ &(ii) \quad \|W^{\mathbf{R}}_{-s_\alpha}(A_i)\|_\infty \leq d_\alpha(x_i, r_i)^{-1} \\ &(iii) \quad \int_{-\infty}^{\infty} A_i(x)dx = 0 \text{ if } r_i < 1. \end{aligned} \tag{18}$$

Definition 6.1. *We define*

$$W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha) = \{f \in L^1_{\mathrm{loc}}(G//K)\ ;\ W_+(f) \in H^{1,+}_{\infty,0}(\mathbf{R})_\alpha\}.$$

We can construct a $(1,\infty,+)$-atomic decomposition (see (17)) for $f \in W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$. Let $F = W_+(f)$ and $F = \sum_i \lambda_i A_i$ the decomposition of F given by the definition, that is, $\sum_i |\lambda_i| < \infty$ and each A_i satisfies (i) to (iii) in (18). Here we may suppose that $r_i \leq 1$. Actually, when $r_i > 1$, we decompose the support of A_i by using a smooth decomposition of 1, where each piece is supported in the interval with radius ≤ 1 and thus, we have $A_i = \sum_j A_{ij}$ and each A_{ij} satisfies (18) with radius ≤ 1. Moreover, we may suppose that $x_i = 0$ with $r_i \leq 1$ or $|x_i| > 2r_i$, because, if $x_i \neq 0$ and

$|x_i| \le 2r_i$, then we may regard $x_i = 0$ without loss of generality. Hence, we can rearrange the decomposition of F as

$$F = \sum_i \lambda_i A_i + \sum_j \mu_j B_j + \sum_k \gamma_k E_k,$$

where each A_i satisfies (i), (ii) with $x_i = 0$, $r_i \le 1$, $\int A_i(x)dx = 0$; each B_j satisfies (i) to (iii) with $|x_j| \ge 2r_j$, $r_j < 1$; each E_k satisfies (i), (ii) with $|x_k| \ge 2r_k$, $r_k = 1$, and moreover, $\sum_i |\lambda_i| + \sum_j |\mu_j| + \sum_k |\gamma_k| < \infty$. Since F is W_+-smooth, finally, we have

$$f = \sum_i \lambda_i a_i + \sum_j \mu_j b_j + \sum_k \gamma e_k, \tag{19}$$

where $a_i = W_-(A_i)$, $b_j = W_-(B_j)$ and $e_k = W_-(E_k)$. Here it is easy to see that each a_i, b_j, e_k have the same supports of A_i, B_j, E_k respectively.

Now we apply fractional calculus in [**8**] to estimate each a_i, b_j, e_k. For simplicity, we abbreviate the suffices i, j, k and denote the supports of a, b, e by $R(x_0, r)$. Without loss of generality, we may suppose that $x_0 \ge 0$.

As for e, since e is supported on $R(x_0, 1)$ and $x_0 \ge 2$, it follows that $x_0 - 1 \ge 1$ and thus, $d_\alpha(x_0, 1) \sim 1$. Thereby, (ii) and [**8**, Lemma 3.3] imply that on the support of e

$$|e(x)| \le c(\mathrm{th}x)^{-(\alpha+1/2)} e^{-2\rho x} \le c e^{-2\rho x} \le c|R(x,1)|^{-1}.$$

This means that $c^{-1}e$ is a $(1, \infty, +)$-atom on G.

As for b, we recall that $x_0 > 2r$.

Case I. $x_0 - r \ge 1$: Since $x_0 - r \ge 1$, $d_\alpha(x_0, r) \sim r$. Thereby, (ii) and [**8**, Lemma 3.3] imply that on the support of b

$$|b(x)| \le c(\mathrm{th}x)^{-(\alpha+1/2)} e^{-2\rho x} r^{-1} \le c e^{-2\rho x} r^{-1} \le c|R(x,r)|^{-1}.$$

This means that $c^{-1}b$ is a $(1, \infty, 0)$-atom on G.

Case II. $x_0 - r < 1$: Since $r < 1$ and $x_0 > 2r$, it follows that $x_0 < r + 1 < 2$, $x_0 - r > x_0/2$, and $x_0 + r < 3x_0/2 < 3$. Therefore, $d_\alpha(x_0, r) \le c(x_0 - r)^{s_\alpha} r$ and thus, on the support of b

$$|b(x)| \le c(\mathrm{th}x)^{-(\alpha+1/2)} e^{-2\rho x} r^{-1} (x_0 - r)^{-s_\alpha} \le c(x_0 - r)^{-(2\alpha+1)} r^{-1}.$$

Since $(x_0 + r)/(x_0 - r) \le 3$, it follows that

$$|R(x_0, r)| \le c(x_0 + r)^{2\alpha+1} r \le c(x_0 - r)^{2\alpha+1} r.$$

Therefore, $|b(x)| \le c|R(x_0, r)|^{-1}$ on the support. This means that $c^{-1}b$ is a $(1, \infty, 0)$-atom on G.

As for a, since $x_0 = 0$ and $r < 1$, it follows that $d_\alpha(0,r) \sim r^{s_\alpha+1}$ and

$$|a(x)| \le c(\text{th}x)^{-(\alpha+1/2)}e^{-2\rho x}r^{-1}r^{-(s_\alpha+1)} \le c\Delta(x)^{-1}r^{-1}. \tag{20}$$

Hence, if we put

$$a_+(x) = c\Delta(x)^{-1}r^{-1}\chi_{[0,r]}(x),\ x > 0,$$

then $|a(x)| \le a_+(x)$ and a_+ is a non-increasing function on $\mathbf{R}_+$ with finite L^1-norm:

$$\|a_+\|_{L^1(\Delta)} = \int_0^\infty a_+(x)\Delta(x)dx = c_0.$$

Since a is supported on $B(r)$ and $\displaystyle\int_0^\infty a(x)\Delta(x)dx = \int_{-\infty}^\infty A(x)dx = 0$, it follows that $|B(s)|^{-1}\displaystyle\int_s^\infty a(x)\Delta(x)dx$ is also supported on $B(r)$ and

$$\frac{1}{|B(s)|}\int_s^\infty a(x)\Delta(x)dx = \frac{1}{|B(s)|}\int_0^s a(x)\Delta(x)dx \le c\Delta(s)^{-1}r^{-1}.$$

Here we used (20) and $|B(s)| \sim \Delta(s)s$ if $s \le r \le 1$ (see (1)). Hence,

$$\frac{1}{|B(s)|}\int_s^\infty a(x)\Delta(x)dx \le a_+(s). \tag{21}$$

This means that ca_+ is an L^1 non-increasing denominator of a satisfying (21). Then [**5**, Theorem 4.5] yields that a has a centered $(1,\infty,0)$-atomic decomposition $a = \sum_j \gamma_j a_j$ on G such that $\sum_j |\gamma_j| \le c\|a_+\|_{L^1(\Delta)} \le cc_0$. Especially, $a \in H^1_{\infty,0}(G//K)$ and $\|a\|_{H^1_{\infty,0}(G)} \le cc_0$.

These three cases imply that all a_i, b_j, e_k in (19), and thus f belongs to $H^{1,+}_{\infty,0}(G//K)$:

Proposition 6.2. *All functions in $W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$ have $(1,\infty,+)$-atomic decompositions, that is, $W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha) \subset H^{1,+}_{\infty,0}(G//K)$.*

Now we shall prove that $H^1_\phi(G//K) \subset W_-(H^{1,+}_{\infty,0}(\mathbf{R})_\alpha)$. We shall give a sketch of the proof in the case of $s_\alpha = \alpha + 1/2$ is integer. Let $f \in H^1_\phi(G//K)$ and put $F = W_+(f)$. Then it follows from Theorem 4.4 that $\|M^{\mathbf{R}}_\phi \circ W^{\mathbf{R}}_{-s_\alpha}(F)(x)(\text{th}x)^{s_\alpha}\|_{L^1(\mathbf{R})} < \infty$. We recall that $(\text{th}x)^{s_\alpha}$ is an A_1-weight. Therefore, $W^{\mathbf{R}}_{-s_\alpha}(F)$ has a $(1,\infty,s_\alpha)$-atomic decomposition with respect to this weight:

$$W^{\mathbf{R}}_{-s_\alpha}(F) = \sum_i \lambda_i B_i,$$

where B_i is supported on $R(x_i, r_i)$, $\int_{-\infty}^{\infty} B_i(x)x^k dx = 0$, $0 \le k \le s_\alpha$, $\|B_i\|_\infty \le d_\alpha(x_i, r_i)^{-1}$ and $\sum_i |\lambda_i| < \infty$. We set

$$F = \sum_i \lambda_i W_{s_\alpha}^{\mathbf{R}}(B_i) = \sum_i \lambda_i A_i.$$

Since s_α is integer and each B_i satisfies the s_α-th moment condition, it easily follows that A_i is supported on $R(x_i, r_i)$ and $\int_{-\infty}^{\infty} A_i(x)dx = 0$. Moreover, $\|W_{-s_\alpha}^{\mathbf{R}}(A_i)\|_\infty = \|B_i\|_\infty \le d_\alpha(x_i, r_i)^{-1}$. Therefore, each A_i satisfies (18) and thus, $F \in H_{\infty,0}^{1,+}(\mathbf{R})_\alpha$ and f has a $(1, \infty, +)$-atomic decomposition on G by Proposition 6.2. Furthermore, we can drop the assumption that s_α is integer. Therefore, we have $H_\phi^1(G//K) \subset H_{\infty,0}^{1,+}(G//K)$ in general. Finally, as a refinement of Proposition 5.3, we have the following main theorem.

Theorem 6.3. *Let notations be as above. Then*

$$H_\phi^1(G//K) = H_{\infty,0}^{1,+}(G//K) \cap W_-(H^1(\mathbf{R})).$$

As an easy consequence of the previous argument, we have

Theorem 6.4. *Let $\epsilon \ge 0$. Then $H_{\infty,0}^{1;\epsilon}(G//K) \cap W_-(H^1(\mathbf{R}))$ is dense in $W_-(H^1(\mathbf{R}))$. Especially, $H_\phi^1(G//K)$ is dense in $W_-(H^1(\mathbf{R}))$.*

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