UNCERTAINTY PRINCIPLES FOR THE DUNKL TRANSFORM

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Abstract. The Dunkl transform satisfies some uncertainty principles similar to the Euclidean Fourier transform. A generalization and a variant of Cowling-Price's theorem, Beurling's theorem and Donoho-Stark's uncertainty principle are obtained for the Dunkl transform.

Contents

1. Introduction

There are many theorems known which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot both be sharply localized. That is, it is impossible for a nonzero function and its Fourier transform to be simultaneously small. Here a concept of the smallness had taken different interpretations in different contexts.

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Hardy [13], Morgan [21], Cowling and Price [6], Beurling [2], Miyachi [20] for example interpreted the smallness as sharp pointwise estimates or integrable decay of functions. Benedicks [1], Slepian and Pollak [27], Landau and Pollacl [15], and Donoho and Stark [7] paid attention to the supports of functions and gave qualitative uncertainty principles for the Fourier transforms.

Hardy's theorem [13] for the classical Fourier transform $\mathcal{F}$ on $\mathbb{R}$ asserts that f and its Fourier transform $\widehat{f} = \mathcal{F}(f)$ can not both be very small. More precisely, let a and b be positive constants and assume that f is a measurable function on $\mathbb{R}$ such that $|f(x)| \leq Ce^{-ax^2}$ a.e. and $|\hat{f}(y)| \leq Ce^{-by^2}$ for some positive constant C. Then $f = 0$ a.e. if $ab > \frac{1}{4}$, f is a constant multiple of e^{-ax^2} if $ab = \frac{1}{4}$, and there are infinitely many nonzero functions satisfying the assumptions if $ab < \frac{1}{4}$. Considerable attention has been devoted for discovering generalizations to new contexts for the Hardy's theorem. In particular, Cowling and Price [6] have studied an L^p version of Hardy's theorem which states that for $p, q \in [1, +\infty]$, at least one of them is finite, if $||e^{ax^2} f||_p < +\infty$ and $||e^{by^2} \hat{f}||_q < +\infty$, then $f = 0$ a.e. if $ab \geq \frac{1}{4}$. Another generalization of Hardy's theorem is given by Miyachi [20], which states that, if f is a measurable function on $\mathbb{R}$ such that $e^{ax^2} f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ and

$$\int_{\mathbb{R}} \log^+ \frac{|\hat{f}(\xi) e^{\frac{\xi^2}{4a}}|}{\lambda} d\xi < \infty$$

for some positive constants a and λ, then f is a constant multiple of e^{-ax^2}. Furthermore, Beurling's theorem, which was found by Beurling and his proof was published much later by Hörmander [14], says that for any non trivial function f in $L^2(\mathbb{R})$, the product $f(x)\hat{f}(y)$ is never integrable on $\mathbb{R}^2$ with respect to the measure $e^{|x||y|}dxdy$. A far reaching generalization of this result has been recently proved by Bonami, Demange and Jaming [3]. They proved that, if $f \in L^2(\mathbb{R}^d)$ satisfies for an integer N

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)||\mathcal{F}(f)(y)|}{(1 + ||x|| + ||y||)^N} e^{||x|| ||y||} dxdy < +\infty,$$

then f is of the form $f(x) = P(x)e^{-\beta ||x||^2}$ where P is a polynomial of degree strictly lower than $\frac{N-d}{2}$ and β is a positive constant.

As a generalization of these Euclidean uncertainty principles for the classical Fourier transform $\mathcal{F}$, recently, Gallardo and Trimèche [12] and Trimèche [31] have proved Hardy's theorem, Cowling-Price's theorem and Beurling's theorem for the Dunkl transform $\mathcal{F}_D$. The purpose of this paper is, as further generalizations, to obtain variants of their results and Donoho-Stark's uncertainty principles for $\mathcal{F}_D$.

The structure of this paper is the following. In §2, we recall the basic properties of the Dunkl operators; the Dunkl intertwining operator and its dual, the Dunkl transform $\mathcal{F}_D$ and related harmonic analysis. §3 is devoted to generalize Cowling-Price's theorem for $\mathcal{F}_D$. In §4 and §5 we give variants of Cowling-Price's theorem. We state Miyachi's theorem in §6 and we generalize Beurling's theorem for $\mathcal{F}_D$ in §7. §8 is devoted to Donoho-Stark's uncertainty principle for $\mathcal{F}_D$.

Throughout this paper, the letter C indicates a positive constant not necessarily the same in each occurrence.

2. Preliminaries

In order to confirm the basic and standard notations we briefly overview the theory of Dunkl operators and related harmonic analysis. Main references are [8, 9, 10, 11, 16, 17, 22, 23, 28, 29, 30].

2.1. **Root system, reflection group, and multiplicity function.** Let $\mathbb{R}^d$ be the Euclidean space equipped with a scalar product $\langle , \rangle$ and the norm $||x|| = \sqrt{\langle x, x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e. for $x \in \mathbb{R}^d$,

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{||\alpha||^2}\alpha.$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $R\cap\mathbb{R}\alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R. We fix a $\beta \in \mathbb{R}^d\backslash\cup_{\alpha\in R}H_\alpha$ and define a positive root system $R_+ = \{\alpha \in R \mid \langle\alpha, \beta\rangle > 0\}$. We normalize each $\alpha \in R_+$ as $\langle\alpha, \alpha\rangle = 2$. A function $k : R \longrightarrow \mathbb{C}$ on R is called a multiplicity function if it is invariant under the action of W. We introduce the index γ as

$$\gamma = \gamma(k) = \sum_{\alpha\in R_+} k(\alpha).$$

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in R$. We denote by ω_k the weight function on $\mathbb{R}^d$ given by

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle\alpha, x\rangle|^{2k(\alpha)},$$

which is invariant under the action of W and homogeneous of degree 2γ, and by c_k the Mehta-type constant defined by

$$c_k = \int_{\mathbb{R}^d} e^{-\frac{||x||^2}{2}}\omega_k(x)dx.$$

Let $d \geq 2$. For an integrable function f on $\mathbb{R}^d$ with respect to a measure $\omega_k(x)dx$ we have

$$\int_{\mathbb{R}^d} f(x)\omega_k(x)dx = \int_0^{+\infty}\Big(\int_{S^{d-1}} f(r\beta)\omega_k(\beta)d\sigma(\beta)\Big)r^{2\gamma+d-1}dr, \tag{2.1}$$

where $d\sigma$ is the normalized surface measure on the unit sphere S^{d-1} of $\mathbb{R}^d$. In particular, if f is radial (i.e. $SO(d)$-invariant), then there exists a function F on $[0, +\infty[$ such that $f(x) = F(||x||) = F(r)$ with $||x|| = r$ and

$$\int_{\mathbb{R}^d} f(x)\omega_k(x)dx = d_k\int_0^{+\infty} F(r)r^{2\gamma+d-1}dr, \tag{2.2}$$

where

$$d_k = \int_{S^{d-1}} \omega_k(\beta)d\sigma(\beta).$$

We denote by $L^p(\mathbb{R}^d)$, $1 \leq p < +\infty$, the space of measurable functions f on $\mathbb{R}^d$ with finite L^p-norm $\| \ \|_p$ with respect to the Lebesgue measure dx and by $L^p_k(\mathbb{R}^d)$ the one with respect to the weighted measure $\omega_k(x)dx$:

$$\|f\|_{k,p} = \Big(\int_{\mathbb{R}^d} |f(x)|^p \omega_k(x)dx\Big)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty,$$
$$\|f\|_{k,\infty} = \text{ess} \sup_{x\in\mathbb{R}^d} |f(x)| < +\infty.$$

In the following we denote by

- $C(\mathbb{R}^d)$ the space of continuous functions on $\mathbb{R}^d$.
- $C^p(\mathbb{R}^d)$ the space of functions of class C^p on $\mathbb{R}^d$.
- $C^p_b(\mathbb{R}^d)$ the space of bounded functions of class C^p.
- $\mathcal{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$.
- $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.
- $D(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ with compact support.
- $\mathcal{S}'(\mathbb{R}^d)$ the space of temperate distributions on $\mathbb{R}^d$.
- $\mathcal{P}(\mathbb{R}^d)$ the set of polynomials on $\mathbb{R}^d$ and $\mathcal{P}_m(\mathbb{R}^d)$ the one of degree m.

2.2. **The Dunkl operators and the Dunkl kernel.** The Dunkl operators T_j, $j = 1, 2, \dots, d$, on $\mathbb{R}^d$ associated with the positive root system R_+ and the multiplicity function k are given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha\in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}$$

for $f \in C^1(\mathbb{R}^d)$. Then each T_j satisfies the following:

i) For all f and g in $C^1(\mathbb{R}^d)$, if at least one of them is W-invariant, then

$$T_j(fg) = (T_j f)g + f(T_j g).$$

ii) For all f in $C^1_b(\mathbb{R}^d)$ and g in $\mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} T_j f(x)g(x)\omega_k(x)dx = -\int_{\mathbb{R}^d} f(x)T_j g(x)\omega_k(x)dx.$$

We define the Dunkl-Laplace operator $\triangle_k$ on $\mathbb{R}^d$ by

$$\begin{aligned}\triangle_k f(x) &= \sum_{j=1}^d T_j^2 f(x)\\ &= \triangle f(x) + 2\sum_{\alpha\in R^+} k(\alpha)\Big(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}\Big),\end{aligned}$$

where $\triangle$ and ∇ are the usual Euclidean Laplacian and nabla operators on $\mathbb{R}^d$ respectively. Then for each $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x,y) = y_j u(x,y), \quad j = 1, ..., d,\\ u(0,y) = 1\end{cases}$$

admits a unique analytic solution $K(x,y)$, $x \in \mathbb{R}^d$, called the Dunkl kernel. This kernel has a holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$ and possesses the following properties (cf. [22]):

i) For all $z, t, \lambda \in \mathbb{C}^d$, $K(z,t) = K(t,z)$, $K(z,0) = 1$ and

$$K(\lambda z, t) = K(z, \lambda t). \tag{2.3}$$

ii) For all $\nu \in \mathbb{N}^d, x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$,

$$|D_z^\nu K(x,z)| \leq ||x||^{|\nu|} \exp(||x|| \, ||\mathrm{Re} z||), \tag{2.4}$$

where

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \cdots \partial z_d^{\nu_d}} \quad \text{and} \quad |\nu| = \nu_1 + \cdots + \nu_d.$$

In particular, $|K(x, -iy)| \leq 1$ for all $x, y \in \mathbb{R}^d$.

iii) For all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$,

$$K(x,z) = \int_{R^d} e^{\langle y, z\rangle} d\mu_x(y), \tag{2.5}$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, ||x||)$ of center 0 and radius $||x||$.

The Dunkl intertwining operator V_k on $C(\mathbb{R}^d)$ is defined by

$$V_k f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y),$$

where $d\mu_x$ is the same measure in (2.5). Then for all $x \in \mathbb{R}^d, z \in \mathbb{C}^d$, we have

$$K(x,z) = V_k(e^{\langle ., z\rangle})(x).$$

Let ${}^tV_k$ denote the operator on $D(\mathbb{R}^d)$ satisfying for all $f \in D(\mathbb{R}^d)$ and $g \in C(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} {}^tV_k(f)(y) g(y) dy = \int_{\mathbb{R}^d} V_k(g)(x) f(x) \omega_k(x) dx.$$

Then there exists a positive measure ν_y on $\mathbb{R}^d$ with support in the set $\{x \in \mathbb{R}^d, ||x|| \geq ||y||\}$ for which

$${}^tV_k(f)(y) = \int_{\mathbb{R}^d} f(x) d\nu_y(x). \tag{2.6}$$

This operator ${}^tV_k$ is called the dual Dunkl intertwining operator. The operators V_k and ${}^tV_k$ satisfy the following properties (cf. [29]):

i) V_k is a topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself satisfying the permutation relations: For all $f \in \mathcal{E}(\mathbb{R}^d)$,

$$T_j V_k(f)(x) = V_k(\frac{\partial}{\partial y_j} f)(x).$$

ii) ${}^tV_k$ is a topological isomorphism from $D(\mathbb{R}^d)$(resp. $\mathcal{S}(\mathbb{R}^d)$) onto itself satisfying the permutation relations: For all $f \in D(\mathbb{R}^d)$,

$${}^tV_k(T_j f)(y) = \frac{\partial}{\partial y_j} {}^tV_k(f)(y).$$

Proposition 1. ([12]) Let $(\nu_y)_{y\in\mathbb{R}^d}$ be the family of measures defined by (2.6) and f be in $L^1_k(\mathbb{R}^d)$. Then for almost all $y \in \mathbb{R}^d$ with respect to Lebesgue measure on $\mathbb{R}^d$, f is ν_y-integrable and the function

$$y \mapsto \int_{\mathbb{R}^d} f(x)d\nu_y(x),$$

which will be also denoted by ${}^tV_k(f)$, is Lebesgue integrable on $\mathbb{R}^d$. Moreover for all $g \in C_b(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} {}^tV_k(f)(y)g(y)dy = \int_{\mathbb{R}^d} V_k(g)(x)f(x)\omega_k(x)dx. \tag{2.7}$$

Remark 1. By taking $g \equiv 1$ in (2.7) we can deduce that for all $f \in L^1_k(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} {}^tV_k(f)(y)dy = \int_{\mathbb{R}^d} f(x)\omega_k(x)dx. \tag{2.8}$$

2.3. **The Dunkl transform.** The Dunkl transform $\mathcal{F}_D$ on $L^1_k(\mathbb{R}^d)$ is given by

$$\mathcal{F}_D(f)(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)K(x,-iy)\omega_k(x)dx. \tag{2.9}$$

Some basic properties of this transform are the following (cf. [10] and [11]):

i) For all $f \in L^1_k(\mathbb{R}^d)$,

$$||\mathcal{F}_D(f)||_{k,\infty} \leq \frac{1}{c_k}||f||_{k,1}. \tag{2.10}$$

ii) For all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{F}_D(T_j f)(y) = iy_j\mathcal{F}_D(f)(y). \tag{2.11}$$

iii) For all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{F}_D(f) = \mathcal{F} \circ {}^tV_k(f), \tag{2.12}$$

where $\mathcal{F}$ is the classical Fourier transform on $\mathbb{R}^d$.

iv) For all $f \in L^1_k(\mathbb{R}^d)$, if $\mathcal{F}_D(f)$ belongs to $L^1_k(\mathbb{R}^d)$, then

$$f(y) = \int_{\mathbb{R}^d} \mathcal{F}_D(f)(x)K(ix,y)\omega_k(x)dx. \tag{2.13}$$

v) For $f \in \mathcal{S}(\mathbb{R}^d)$, if we define $\overline{\mathcal{F}_D}(f)(y) = \mathcal{F}_D(f)(-y)$, then

$$\mathcal{F}_D\overline{\mathcal{F}_D} = \overline{\mathcal{F}_D}\mathcal{F}_D = Id. \tag{2.14}$$

Proposition 2. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^d)$ onto itself and for all f in $\mathcal{S}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |f(x)|^2\omega_k(x)dx = \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2\omega_k(\xi)d\xi. \tag{2.15}$$

In particular, the Dunkl transform $f \to \mathcal{F}_D(f)$ can be uniquely extended to an isometric automorphism on $L^2_k(\mathbb{R}^d)$.

2.4. **The Dunkl convolution.** By using the Dunkl kernel in 2.2, we introduce a generalized translation and an associated convolution structure on $\mathbb{R}^d$. For $f \in \mathcal{S}(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$ the Dunkl translation $\tau_y f$ is defined by

$$\mathcal{F}_D(\tau_y f)(x) = K(ix, y)\mathcal{F}_D(f)(x)$$

(cf. [30]). This transform is related to the usual translation as

$$\tau_y f(x) = (V_k)_x (V_k)_y [(V_k)^{-1}(f)(x+y)]. \tag{2.16}$$

Hence, τ_y can also be defined for $f \in \mathcal{E}(\mathbb{R}^d)$. If $f \in \mathcal{E}(\mathbb{R}^d)$ is radial, i.e. $f(x) = F(||x||)$, then it follows that

$$\tau_y f(x) = V_k\Big(F(\sqrt{||x||^2 + ||y||^2 + 2\langle x, .\rangle})\Big)(x)$$

(cf. [23]). For example, for $t > 0$, we see that

$$\tau_y(e^{-t||\xi||^2})(x) = e^{-t(||x||^2+||y||^2)}K(2ty, x). \tag{2.17}$$

We define the Dunkl convolution product $f *_D g$ of $f, g \in \mathcal{S}(\mathbb{R}^d)$ as

$$f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y) g(y) \omega_k(y) dy \tag{2.18}$$

(cf. [28] and [30]). This convolution is commutative and associative and moreover, it satisfies the following (cf. [28]):

i) For all $f, g \in D(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$), $f *_D g$ belongs to $D(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$) and

$$\mathcal{F}_D(f *_D g)(y) = \mathcal{F}_D(f)(y)\mathcal{F}_D(g)(y). \tag{2.19}$$

ii) Let $1 \le p, q, r \le \infty$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If $f \in L_k^p(\mathbb{R}^d)$ and $g \in L_k^q(\mathbb{R}^d)$ is radial, then $f *_D g \in L_k^r(\mathbb{R}^d)$ and

$$\|f *_D g\|_{k,r} \le \|f\|_{k,p} \|g\|_{k,q}. \tag{2.20}$$

2.5. **The Sobolev space $H_k^s(\mathbb{R}^d)$.** Let $s \in \mathbb{R}$. We define the Dunkl-Sobolev space $H_k^s(\mathbb{R}^d)$ as the set of distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ satisfying $(1 + ||\xi||^2)^{\frac{s}{2}}\mathcal{F}_D(u) \in L_k^2(\mathbb{R}^d)$, equipped with the scalar product

$$\langle u, v\rangle_{H_k^s} = \int_{\mathbb{R}^d} (1 + ||\xi||^2)^s \mathcal{F}_D(u)(\xi)\overline{\mathcal{F}_D(v)(\xi)}\omega_k(\xi) d\xi$$

and the norm

$$||u||^2_{H_k^s} = \langle u, u\rangle_{s,k}.$$

As shown in [17], if $p \in \mathbb{N}$ and $s \in \mathbb{R}$ satisfy $s > \frac{d}{2} + \gamma + p$, then the following embedding is continuous (i.e. the inclusion is in the sense of topology)

$$H_k^s(\mathbb{R}^d) \hookrightarrow C^p(\mathbb{R}^d). \tag{2.21}$$

Lemma 1. *Let $f \in \mathcal{S}(\mathbb{R}^d)$ and assume that for all $n \in \mathbb{N}$, there exist a positive constant c_n such that*

$$||\triangle_k^n f||_{k,2} \le c_n.$$

Then for all $n \in \mathbb{N}$,

$$|\triangle_k^n f(x)| \le C(c_n + c_{n+m}),$$

where $m=[\frac{d+2\gamma}{4}]+1$ *and* C *is independent of* n*.*

Proof. Since $|\triangle_k^n f(x)|\le C_m\|\triangle_k^n f\|_{H_k^{2m}}$ by (2.21) and $\|\triangle_k^n f\|_{H_k^{2m}}\le C_m(\|\triangle_k^n f\|_{k,2}+\|\triangle_k^{n+m}f\|_{k,2})$ by the definition of $H_k^{2m}(\mathbb{R}^d)$, the desired result follows. ■

2.6. **Mean value property associate with the Dunkl Laplacian.** Let $d\ge 2$. The mean value operator $M_{r,x}^D$, $r>0, x\in\mathbb{R}^d$, associated with the Dunkl Laplacian $\triangle_k$ is defined by for $u\in\mathcal{E}(\mathbb{R}^d)$,

$$M_{r,x}^D(u)=\frac{1}{d_k}\int_{S^{d-1}}\tau_x u(ry)\omega_k(y)d\sigma(y).$$

To give a development formula for $M_{r,x}^D$, we define a sequence of functions $\{v_p(t)\}_{p\ge 0}$, $0<t\le r$, and a sequence of numbers $\{b_p(r)\}_{p\ge 0}$ as follows. We put

$$v_0(t)=\int_t^r\frac{ds}{s^{2\gamma+d-1}}$$

and inductively, let $v_p(t), p\ge 1$ denote a unique solution of the differential equation:

$$\begin{cases}L_{\gamma+\frac{d}{2}-1}v_p(t)=v_{p-1}(t),\\ v_p(r)=\dfrac{d}{dr}v_p(r)=0,\end{cases}$$

where $L_{\gamma+\frac{d}{2}-1}$ is the Bessel operator given by

$$L_{\gamma+\frac{d}{2}-1}=\frac{d^2}{dt^2}+\frac{2\gamma+d-1}{t}\frac{d}{dt}.$$

We put $b_0(r)=1$ and

$$b_p(r)=\int_0^r v_{p-1}(t)t^{2\gamma+d-1}dt. \tag{2.22}$$

Then we see that

$$b_p(r)=\frac{r^{2p}}{d_p(\gamma)} \tag{2.23}$$

with

$$d_p(\gamma)=\frac{2^{2p}p!\Gamma(\gamma+\frac{d}{2}+p)}{\Gamma(\gamma+\frac{d}{2})}.$$

Proposition 3. ([16]) For $u\in C^{2n+2}(\mathbb{R}^d)$ and $x_0\in\mathbb{R}^d$, it follows that

$$M_{r,x_0}^D(u)=\sum_{p=0}^n b_p(r)\triangle_k^p u(x_0)+\frac{1}{d_k}\int_{B(x_0,r)}v_n(||x||)\triangle_k^{n+1}(\tau_{x_0}u)(x)\omega_k(x)dx,$$

where $B(x_0,r)$ is the closed ball of center x_0 and radius r.

2.7. **Heat functions related to the Dunkl operators.** The heat kernel $N_k(x,s)$, $x \in \mathbb{R}^d$, $s > 0$, associated with the Dunkl-Laplace operator $\triangle_k$ is given by

$$N_k(x,s) = \frac{1}{c_k(2s)^{\gamma+\frac{d}{2}}} e^{-\frac{||x||^2}{4s}}, \tag{2.24}$$

which is a solution of the generalized heat equation:

$$\frac{\partial}{\partial s} N_k(x,s) - \triangle_k N_k(x,s) = 0.$$

Some basic properties of $N_k(x,s)$ are the following:

i) $\mathcal{F}_D\left(N_k(\cdot,s)\right)(x) = \frac{1}{c_k} e^{-s||x||^2}$ and

$$N_k(x,s) = \frac{1}{c_k^2} \int_{\mathbb{R}^d} e^{-r||y||^2} K(ix,y)\omega_k(y)dy. \tag{2.25}$$

ii) For all $\lambda > 0$,

$$N_k(\lambda^{\frac{1}{2}} x, \lambda s) = \lambda^{-(\gamma+\frac{d}{2})} N_k(x,s).$$

iii)

$$\|N_k(\cdot,s)\|_{k,1} = 1. \tag{2.26}$$

iv) For all $s, t > 0$,

$$N_k(\cdot,t) *_D N_k(\cdot,s)(x) = N_k(x,t+s).$$

By noting (2.25) and (2.11), we define the heat functions $W_l^k(x,s)$, $l \in \mathbb{N}^d$, as

$$\begin{aligned} W_l^k(x,s) &= T^l N_k(x,s) \\ &= \frac{i^{|l|}}{c_k^2} \int_{\mathbb{R}^d} {y_1}^{l_1} \cdots {y_d}^{l_d} e^{-r||y||^2} K(ix,y)\omega_k(y)dy, \end{aligned} \tag{2.27}$$

where $T^l = T_1^{l_1} \circ T_2^{l_2} \circ \cdots \circ T_d^{l_d}$. Then $W_0^k(x,s) = N_k(x,s)$ and

$$\mathcal{F}_D(W_l^k(\,.\,,s))(x) = \frac{i^{|l|}}{c_k} {y_1}^{l_1} \cdots {y_d}^{l_d} e^{-s||x||^2}. \tag{2.28}$$

Proposition 4. ([31]) Let $\psi \in \mathcal{P}_m(\mathbb{R}^d)$ be homogeneous. Then for all $\delta > 0$, there exists a homogeneous $Q \in \mathcal{P}_m(\mathbb{R}^d)$ such that

$$\mathcal{F}_D(\psi(\cdot)e^{-\delta||\cdot||^2})(x) = Q(x)e^{-\frac{\|x\|^2}{4\delta}}. \tag{2.29}$$

3. Cowling-Price's theorem for the Dunkl transform

We shall prove a generalization of Cowling-Price's theorem for the Dunkl transform $\mathcal{F}_D$.

Theorem 1. *Let f be a measurable function on $\mathbb{R}^d$ such that*

$$\int_{\mathbb{R}^d} \frac{e^{ap||x||^2} |f(x)|^p}{(1+||x||)^n} \omega_k(x)dx < \infty \tag{3.30}$$

and

$$\int_{\mathbb{R}^d} \frac{e^{bq||\xi||^2}|\mathcal{F}_D(f)(\xi)|^q}{(1+||\xi||)^m}d\xi < \infty, \tag{3.31}$$

for some constants $a, b > 0$, $n > 0$, $m > 1$ *and* $1 \leq p, q < +\infty$.

i) If $ab > \frac{1}{4}$, *then* $f = 0$ *almost everywhere.*

ii) If $ab = \frac{1}{4}$, *then* f *is of the form* $f(x) = Q_b(x)N_k(x, b)$ *where* Q_b *is a polynomial with* $\deg Q_b \leq \min\{\frac{n}{p} + \frac{2\gamma+d-1}{p'}, \frac{m-d}{q}\}$. *Especially, if*

$$n \leq d + 2\gamma + p\min\Big\{\frac{n}{p} + \frac{2\gamma+d-1}{p'}, \frac{m-d}{q}\Big\},$$

then $f = 0$ *almost everywhere. Furthermore, if* $m \in]d, d+q]$ *and* $n > d + 2\gamma$, *then* f *is a constant multiple of* $N_k(., b)$.

iii) If $ab < \frac{1}{4}$, *then for all* $\delta \in]b, \frac{1}{4a}[$, *all functions of the form* $f(x) = P(x)N_k(x, \delta)$, $P \in \mathcal{P}$, *satisfy (3.30) and (3.31).*

Proof. Clearly (3.30) implies that f belongs to $L^1_k(\mathbb{R}^d)$ and thus, $\mathcal{F}_D(f)(\xi)$ exists for all $\xi \in \mathbb{R}^d$. Moreover, it has an entire holomorphic extension on $\mathbb{C}^d$ satisfying for some $s > 0$,

$$|\mathcal{F}_D(f)(z)| \leq Ce^{\frac{||\mathrm{Im}z||^2}{4a}}(1 + ||\mathrm{Im}z||)^s. \tag{3.32}$$

Actually, it follows from (2.9) and (2.4) that for all $z = \xi + i\eta \in \mathbb{C}^d$,

$$\begin{aligned}|\mathcal{F}_D(f)(\xi + i\eta)| &\leq \int_{R^d} |f(x)||K(x, -i\xi + \eta)|\omega_k(x)dx \\ &\leq e^{\frac{||\eta||^2}{4a}} \int_{\mathbb{R}^d} \frac{e^{a||x||^2}|f(x)|}{(1+||x||)^{\frac{n}{p}}}(1+||x||)^{\frac{n}{p}}e^{-a(||x||-||\frac{\eta}{2a}||)^2}\omega_k(x)dx.\end{aligned}$$

Then by using the Hölder inequality, (3.30) and (2.2) we can obtain that

$$\begin{aligned}|\mathcal{F}_D(f)(\xi + i\eta)| &\leq Ce^{\frac{||\eta||^2}{4a}}\Big(\int_{R^d}(1+||x||)^{\frac{np'}{p}}e^{-ap'(||x||-||\frac{\eta}{2a}||)^2}\omega_k(x)dx\Big)^{\frac{1}{p'}} \\ &\leq Ce^{\frac{||\eta||^2}{4a}}\Big(\int_0^\infty(1+r)^{\frac{np'}{p}+2\gamma+d-1}e^{-ap'(r-||\frac{\eta}{2a}||)^2}dr\Big)^{\frac{1}{p'}} \\ &\leq Ce^{\frac{||\eta||^2}{4a}}(1+||\eta||)^{\frac{n}{p}+\frac{2\gamma+d-1}{p'}}.\end{aligned}$$

If $ab = \frac{1}{4}$, then

$$|\mathcal{F}_D(f)(\xi + i\eta)| \leq Ce^{b||\eta||^2}(1+||\eta||)^{\frac{n}{p}+\frac{2\gamma+d-1}{p'}}.$$

Therefore, if we let $g(z) = e^{b(z_1^2+z_2^2+\cdots+z_d^2)}\mathcal{F}_D(f)(z)$, then

$$|g(z)| \leq Ce^{b\|\mathrm{Re}z\|^2}(1+||\mathrm{Im}z||)^{\frac{n}{p}+\frac{2\gamma+d-1}{p'}}.$$

Hence it follows from (3.31) that

$$\int_{\mathbb{R}^d}\frac{|g(\xi)|^q}{(1+||\xi||)^m}d\xi < \infty.$$

Here we use the following lemma.

Lemma 2. ([25]) *Let h be an entire function on $\mathbb{C}^d$ such that*

$$|h(z)| \leq Ce^{a||\mathrm{Re}z||^2}(1+||\mathrm{Im}z||)^l$$

for some $l>0$, $a>0$ and

$$\int_{\mathbb{R}^d} \frac{|h(x)|^q}{(1+||x||)^m}|Q(x)|dx < \infty$$

for some $q \geq 1$, $m>1$ and $Q \in \mathcal{P}_M(\mathbb{R}^d)$. Then h is a polynomial with $\deg h \leq \min\{l, \frac{m-M-d}{q}\}$ and, if $m \leq q+M+d$, then h is a constant.

Hence by this lemma g is a polynomial, we say P_b, with $\deg P_b \leq \min\{\frac{np'}{p} + \frac{2\gamma+d-1}{p'}, \frac{m-d}{q}\}$. Then $\mathcal{F}_D(f)(x) = P_b(x)e^{-b\|x\|^2}$ and thus, $f(x) = Q_b(x)N_k(x,b) = C_bQ_b(x)e^{-a\|x\|^2}$ for $x \in \mathbb{R}^d$, where Q_b is a polynomial with $\deg Q_b = \deg P_b$ (see (2.29)). Therefore, nonzero f satisfies (3.30) provided that

$$n > d + 2\gamma + p\min\Big\{\frac{np'}{p} + \frac{2\gamma+d-1}{p'}, \frac{m-d}{q}\Big\}.$$

Furthermore, if $m \leq d+q$, then g is a constant by Lemma 2 and thus, $\mathcal{F}_D(f)(x) = Ce^{-b\|x\|^2}$ and $f(x) = CN_k(x,b) = C_be^{-a\|x\|^2}$. When $n > d+2\gamma$ and $m > d$, these functions satisfy (3.31) and (3.30) respectively. This proves ii).

If $ab > \frac{1}{4}$, then we can choose positive constants, a_1, b_1 such that $a > a_1 = \frac{1}{4b_1} > \frac{1}{4b}$. Then f and $\mathcal{F}_D(f)$ also satisfy (3.30) and (3.31) with a and b replaced by a_1 and b_1 respectively. Therefore, it follows that $\mathcal{F}_D(f)(x) = P_{b_1}(x)e^{-b_1\|x\|^2}$. But then $\mathcal{F}_D(f)$ cannot satisfy (3.31) unless $P_{b_1} \equiv 0$, which implies $f \equiv 0$. This proves i).

If $ab < \frac{1}{4}$, then for all $\delta \in]b, \frac{1}{4a}[$, the functions of the form $f(x) = P(x)N_k(x,\delta)$, where $P \in \mathcal{P}$, satisfy (3.30) and (3.31). This proves iii). ■

The following is an immediate consequence of Theorem 1.

Corollary 1. *Let f be a measurable function on $\mathbb{R}^d$ such that*

$$|f(x)| \leq Me^{-a||x||^2}(1+||x||)^r \ \text{a.e.} \tag{3.33}$$

and for all $\xi \in \mathbb{R}^d$,

$$|\mathcal{F}_D(f)(\xi)| \leq Me^{-b||\xi||^2} \tag{3.34}$$

for some constants $a, b > 0$, $r \geq 0$ and $M > 0$.

i) If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$, then f is of the form $f(x) = CN_k(x,b)$.

iii) If $ab < \frac{1}{4}$, then there are infinity many nonzero f satisfying (3.33) and (3.34).

4. Cowling-Price's theorem via the D-spherical harmonics coefficients

We suppose that $d \geq 2$. We replace the assumption (3.31) by the D-spherical harmonics coefficients of f. For a non-negative integer l, we put

$$\mathcal{H}_l^k = \Big\{ P \in \mathcal{P}_l \mid P \text{ is homogeneous and } \triangle_k P = 0 \Big\},$$

which is called the space of D-spherical harmonics of degree l. We fix a $P_l \in \mathcal{H}_l^k$ and define the Dunkl coefficients of $f \in L_k^1(\mathbb{R}^d)$ in the angular variable by

$$f_{l,k}(\lambda) = \int_{S^{d-1}} f(\lambda t) P_l(t) \omega_k(t) d\sigma(t). \tag{4.35}$$

Moreover, we define the Dunkl spherical harmonic coefficients of $f \in L_k^1(\mathbb{R}^d)$ by

$$F_{l,k}(\lambda) = \lambda^{-l} \int_{S^{d-1}} \mathcal{F}_D(f)(\lambda, t) P_l(t) \omega_k(t) d\sigma(t), \tag{4.36}$$

where

$$\mathcal{F}_D(f)(\lambda, t) = \frac{1}{c_k} \int_{\mathbb{R}^d} f(x) K(\lambda x, -it) \omega_k(x) dx \tag{4.37}$$

for $t \in S^{d-1}$. The relation between $f_{l,k}$ and $F_{l,k}$ is given by the following.

Proposition 5. *Let notations be as above. Then for $z \in S^{d+2m-1}$,*

$$\begin{aligned} F_{l,k}(\lambda) &= C \int_{\mathbb{R}^{d+2l}} f_{l,k}(||x||) ||x||^{-l} K_l(\lambda x, -iz) \omega_k(x) dx \\ &= C \mathcal{F}_{D_l}(f_{l,k}(||\cdot||) ||\cdot||^{-l})(\lambda z), \end{aligned} \tag{4.38}$$

where $\mathcal{F}_{D_l}$ and K_l are the Dunkl transform and the Dunkl kernel on $\mathbb{R}^{d+2l}$ respectively.

Proof. From (2.3), (4.37) and (4.36) it follows that

$$F_{l,k}(\lambda) = \lambda^{-l} \frac{1}{c_k} \int_{\mathbb{R}^d} \Big(\int_{S^{d-1}} K(t, -i\lambda x) P_l(t) \omega_k(t) dt \Big) f(x) \omega_k(x) dx.$$

Here we recall the formula for the Dunkl coefficients of the Dunkl kernel.

Lemma 3. ([10]) *Let $H \in \mathcal{H}_n^k$. Then for all $x \in \mathbb{R}^d$,*

$$\int_{S^{d-1}} K(t, ix) H(t) \omega_k(t) d\sigma_d(t) = C_{n,k} H(x) j_{\gamma + l + \frac{d}{2} - 1}(||x||), \tag{4.39}$$

where j_α, $\alpha \geq -\frac{1}{2}$, is the normalized Bessel function defined by

$$j_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n! \Gamma(\alpha + 1 + n)}.$$

Therefore, we see that

$$F_{l,k}(\lambda) = C_{l,k} \int_{\mathbb{R}^d} P_l(x) j_{\gamma+l+\frac{d}{2}-1}(\lambda ||x||) f(x)\omega_k(x)dx.$$

Then by using (2.1) and (4.39) replaced d by $d+2l$, we can obtain that for all $z \in S^{d+2l-1}$,

$$\begin{aligned}
F_{l,k}(\lambda) &= C_{l,k} \int_0^\infty \int_{S^{d-1}} j_{\gamma+\frac{d}{2}+l-1}(\lambda r) r^{2\gamma+l+d-1} P_l(t) f(rt)\omega_k(t)dtdr \\
&= C_{l,k} \int_0^\infty f_{l,k}(r) j_{\gamma+\frac{d}{2}+l-1}(\lambda r) r^{2\gamma+l+d-1} dr \\
&= C \int_0^\infty \Big(\int_{S^{d+2l-1}} K_l(t, -i\lambda r z)\omega_k(t) d\sigma_{d+2l}(t)\Big) f_{l,k}(r) r^{2\gamma+l+d-1} dr \\
&= C \int_{\mathbb{R}^{d+2l}} f_{l,k}(||x||)||x||^{-l} K_l(x, -i\lambda z)\omega_k(x)dx.
\end{aligned}$$

This established the lemma. ∎

Theorem 2. *Let $p, q \in [1, \infty[$, $a, b > 0$, $n \in]d+2\gamma, d+2\gamma+p]$ and $m > 1$. Let f be a measurable function on $\mathbb{R}^d$ such that*

$$\int_{\mathbb{R}^d} \frac{e^{ap||x||^2}|f(x)|^p}{(1+||x||)^n}\omega_k(x)dx < \infty \tag{4.40}$$

and

$$\int_{\mathbb{R}^+} \frac{e^{bq\lambda^2}|F_{l,k}(\lambda)|^q}{(1+\lambda)^m} d\lambda < \infty \tag{4.41}$$

for all non-negative integer l.

i) If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$, then $f = CN_k(., b)$.

iii) If $ab < \frac{1}{4}$, then for all $\delta \in]b, \frac{1}{4a}[$, all functions of the form $f(x) = P(x)N_k(x, \delta)$, where $P \in \mathcal{P}$, satisfy (4.40) and (4.41).

Proof. (4.40) implies that $f \in L^1_k(\mathbb{R}^d)$ and thus, each $f_{l,k}$ is well-defined. Moreover, it follows from (4.35), (2.1) and (4.38) that

$$\begin{aligned}
I_l &= \int_0^\infty \frac{e^{apr^2}|f_{l,k}(r)|^p}{(1+r)^n} r^{2\gamma+d-1} dr \\
&\le \Big(\int_{S^{d-1}} \Big(\int_0^\infty \frac{e^{apr^2}|f(rt)|^p}{(1+r)^n} r^{2\gamma+d-1} dr \Big)^{\frac{1}{p}} P_l(t)\omega_k(t)dt \Big)^p \\
&\le C \int_{\mathbb{R}^d} \frac{e^{ap||x||^2}|f(x)|^p}{(1+||x||)^n}\omega_k(x)dx < \infty.
\end{aligned}$$

Here we used Hölder's inequality and the compactness of S^{d-1} to obtain the last inequality. Then, by applying this estimate in the polar coordinate (2.2) of (4.38), the same argument in the proof of Theorem 1 yields that $F_{l,k}(\lambda)$ has an entire holomorphic extension on $\mathbb{C}$ and there exists $N \geq 0$ such that

$$|F_{l,k}(u+iv)| \leq Ce^{\frac{v^2}{4a}}(1+|v|)^N.$$

If $ab \geq \frac{1}{4}$, then $|F_{l,k}(u+iv)| \leq Ce^{bv^2}(1+|v|)^N$. Therefore, if we put $G_{l,k}(z) = F_{l,k}(z)e^{bz^2}$, then $|G_{l,k}(z)| \leq Ce^{bu^2}(1+|v|)^N$ and $\displaystyle\int_{\mathbb{R}} \frac{|G_{l,k}(x)|^q}{(1+|x|)^m}dx < \infty$ by (4.41). Hence, Lemma 2 for $d=1$ yields that $F_{l,k}(\lambda) = C_{l,k}e^{-b\lambda^2}P(\lambda)$, where $\lambda \in \mathbb{R}$ and P is polynomial whose degree depends on N and l. By noting (4.38) and (2.29), the injectivity of the Dunkl transform on $\mathbb{R}^{d+2l}$ implies that for all $x \in \mathbb{R}^{d+2l}$, $f_{l,k}(||x||) = C_{l,k}||x||^l Q(x) N_{l,k}(x,b)$, where $N_{l,k}$ is the heat kernel on $\mathbb{R}^{d+2l}$.

If $ab > \frac{1}{4}$, then I_l is finite provided $f_{l,k} = 0$ for all l. Therefore, $f = 0$ almost everywhere. If $ab = \frac{1}{4}$, then I_l is finite provided $n - (l + \deg Q)p - (2\gamma + d - 1) > 1$, that is, $n > d + 2\gamma + (l + \deg Q)p$. Therefore, the assumption on n implies that $l = 0$ and $\deg Q = 0$. Clearly, $f = CN_{0,k}(x,b)$ satisfies (4.40) and (4.41). If $ab < \frac{1}{4}$, then for a given family of functions, we see that $\mathcal{F}_D(f)(y) = Q(y)e^{-\delta||y||^2}$ for some $Q \in \mathcal{P}$. These functions clearly satisfy (4.40) and (4.41) for all $\delta \in]b, \frac{1}{4a}[$. ■

5. A variant of Cowling-Price's theorem for the Dunkl transform

Let us suppose that $d \geq 2$. The aim of this section is to give a d-dimensional extension of a theorem in [19], which is a variant of Cowling-Price's theorem for the Dunkl transform . Our approach is different from [19].

Theorem 3. *Let $a, b > 0$. If $f \in \mathcal{S}(\mathbb{R}^d)$ satisfies for all $\xi \in \mathbb{R}^d$,*

$$|\mathcal{F}_D(f)(\xi)| \leq Ce^{-2b||\xi||^2}$$

and for all $n \in \mathbb{N}$,

$$||\triangle_k^n \mathcal{F}_D(f)||_{k,2}^2 \leq C(2n)!(2a)^{-2n}, \tag{5.42}$$

then $f = 0$ whenever $ab > \frac{1}{4}$.

Let $m = [\frac{d+2\gamma}{4}] + 1$. Then Lemma 1 and (5.42) imply that for all $x \in \mathbb{R}^d$,

$$|\triangle_k^n f(x)|^2 \leq C(2n+2m)!(2a)^{-2n}.$$

Therefore, Theorem 3 follows from the following.

Theorem 4. *Let $a, b > 0$. If $f \in \mathcal{S}(\mathbb{R}^d)$ satisfies for all $\xi \in \mathbb{R}^d$,*

$$|\mathcal{F}_D(f)(\xi)| \leq Ce^{-2b||\xi||^2} \tag{5.43}$$

and for all $n \in \mathbb{N}$,

$$|\triangle_k^n \mathcal{F}_D(f)(\xi)|^2 \leq C(2n+2m)!(2a)^{-2n} \tag{5.44}$$

with $m = [\frac{d+2\gamma}{4}] + 1$, then $f = 0$ whenever $ab > \frac{1}{4}$.

In order to prove Theorem 4 we need the following lemmas.

Lemma 4. *Let a, m be as above. If $F \in \mathcal{S}(\mathbb{R}^d)$ satisfies for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$,*

$$|\triangle_k^n F(x)|^2 \leq C(2n+2m)!(2a)^{-2n}, \tag{5.45}$$

then for all $x_0 \in \mathbb{R}^d$, the function $r \mapsto M^D_{r,x_0}(F)$ extends to $\mathbb{C}$ as an entire function, which satisfies for all $z \in \mathbb{C}$,

$$|M^D_{z,x_0}(F)| \leq Ce^{|z|^2/(2a)}. \tag{5.46}$$

Proof. It follows from Proposition 3 that for all $r \geq 0$, $x_0 \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $M^D_{r,x_0}(F)$ satisfies

$$M^D_{r,x_0}(F) = \sum_{p=0}^{n} b_p(r)\triangle_k^p F(x_0) + \frac{1}{d_k}\int_{B(0,r)} v_n(||t||)\triangle_k^{n+1}(\tau_{x_0}F)(t)\omega_k(t)dt. \tag{5.47}$$

Then from (2.22) and (2.23) we can deduce that

$$\begin{aligned}&\Big|\frac{1}{d_k}\int_{B(0,r)} v_n(||t||)\triangle_k^{n+1}(\tau_{x_0}F)(t)\omega_k(t)dt\Big|\\ \leq &\frac{r^{2n+2}\Gamma(\gamma+\frac{d}{2})}{d_k 2^{2n+2}(n+1)!\Gamma(\gamma+\frac{d}{2}+n+1)}\Big(\sup_{t\in B(0,r)}|\triangle_k^{n+1}(\tau_{x_0}F)(t)|\Big).\end{aligned}$$

Furthermore, from (5.45) we have

$$|\triangle_k^{n+1}(\tau_{x_0}F)(t)| = |\tau_{x_0}(\triangle_k^{n+1}F)(t)| \leq C\sqrt{2(n+m+1)!(2a)^{-2(n+1)}}.$$

Hence the remainder term of (5.47) tends to zero as n goes to infinity. Therefore, $M^D_{r,x_0}(F)$ admits the series development

$$M^D_{r,x_0}(F) = \sum_{n=0}^{\infty} b_n(r)\triangle_k^n F(x_0) = \sum_{n=0}^{\infty}\frac{r^{2n}}{d_n(\gamma)}\triangle_k^n F(x_0).$$

Thus for all $x_0 \in \mathbb{R}^d$ the function $r \mapsto M^D_{r,x_0}(F)$ can be extended to an entire function on $\mathbb{C}$ as

$$M^D_{z,x_0}(F) = \sum_{n=0}^{\infty}\frac{z^{2n}}{d_n(\gamma)}\triangle_k^n F(x_0). \tag{5.48}$$

For all $z \in \mathbb{C}$ and $x_0 \in \mathbb{R}^d$, (5.48) and (5.45) imply that

$$\begin{aligned}|M^D_{z,x_0}(F)| &\leq \sum_{n=0}^{\infty} |b_n(z)||\triangle_k^n F(x_0)| \\ &\leq \sum_{n=0}^{\infty} \frac{|z|^{2n}\Gamma(\gamma+\frac{d}{2})}{2^{2n}n!\Gamma(n+\gamma+\frac{d}{2})}|\triangle_k^n F(x_0)| \\ &\leq C\sum_{n=0}^{\infty} \frac{(2a)^{-n}|z|^{2n}}{n!}|\triangle_k^n F(x_0)|\frac{(2a)^n}{2^{2n}\Gamma(n+\gamma+\frac{d}{2})} \\ &\leq C\Big(\sum_{n=0}^{\infty}\frac{((2a)^{-1}|z|^2)^n}{n!}\Big)\sup_{n\in\mathbb{N}}\Big(|\triangle_k^n F(x_0)|\frac{(2a)^n}{2^{2n}\Gamma(n+\gamma+\frac{d}{2})}\Big) \\ &\leq C\Big(\sup_{n\in\mathbb{N}}\frac{\sqrt{(2n+2m)!}}{2^{2n}\Gamma(n+\gamma+\frac{d}{2})}\Big)e^{|z|^2/(2a)} = C_{\gamma,a}e^{|z|^2/(2a)},\end{aligned}$$

because $m = [\frac{d+2\gamma}{4}] + 1$. This completes the proof of the lemma. ∎

Lemma 5. ([24]). *Let $c, d > 0$ and F be an entire function on $\mathbb{C}$ satisfying for all $z \in \mathbb{C}$,*

$$|F(z)| \leq Ce^{c|\mathrm{Im}z|^2}$$

and for all $x \in \mathbb{R}$,

$$|F(x)| \leq Ce^{-dx^2}.$$

Then $F = 0$ whenever $c < d$.

Proof. of Theorem 4.

Let $x_0 \in \mathbb{R}^d$. For $z \in \mathbb{C}$, we put

$$F_{x_0}(z) = e^{-z^2/(2a)}M^D_{z,x_0}(\mathcal{F}_D(f)).$$

By Lemma 4 with $F = \mathcal{F}_D(f)$ we see that $|M^D_{z,x_0}(\mathcal{F}_D(f))| \leq Ce^{|z|^2/(2a)}$ and therefore, for all $z \in \mathbb{C}$,

$$|F_{x_0}(z)| \leq Ce^{|\mathrm{Im}z|^2/a}.$$

On the other hand, the positivity of the mean value $M^D_{x,x_0}(.)$ and the relation (5.43) give

$$|M^D_{x,x_0}(\mathcal{F}_D(f))| \leq CM^D_{x,x_0}\Big(e^{-2b\|\cdot\|^2}\Big).$$

Then, using (2.17) and (2.4), we obtain

$$\begin{aligned}M^D_{x,x_0}\Big(e^{-2b\|\cdot\|^2}\Big) &= \frac{1}{d_k}\int_{S^{d-1}}\tau_{x_0}\Big(e^{-2b\|\cdot\|^2}\Big)(xy)\omega_k(y)d\sigma(y) \\ &= \frac{1}{d_k}\int_{S^{d-1}} e^{-2b(x^2+\|x_0\|^2)}K(2bx_0,xy)\omega_k(y)d\sigma(y) \\ &\leq Ce^{-2b(\|x_0\|-x)^2} = Ce^{-2b(\|x_0\|^2-2x\|x_0\|+x^2)}.\end{aligned}$$

Hence, for all $x \leq 0$,

$$|M^D_{x,x_0}(\mathcal{F}_D(f))| \leq Ce^{-2bx^2}.$$

But, as a function of x, $x \mapsto M^D_{x,x_0}(\mathcal{F}_D(f))$ is even, it follows that for all $x \in \mathbb{R}$,

$$|M^D_{x,x_0}(\mathcal{F}_D(f))| \leq Ce^{-2bx^2}.$$

Therefore, we see that for $x \in \mathbb{R}$,

$$|F_{x_0}(x)| \leq Ce^{-(\frac{1}{2a}+2b)x^2}.$$

Then, for all $z \in \mathbb{C}$,

$$|F_{x_0}(z)| \leq Ce^{|\mathrm{Im} z|^2/a}$$

and for all $x \in \mathbb{R}$,

$$|F_{x_0}(x)| \leq Ce^{-(\frac{1}{2a}+2b)x^2}.$$

By Lemma 5 we can conclude that $\mathcal{F}_D(f) = 0$ and thus, $f = 0$. ■

As an application of Theorem 3, we can obtain the following.

Corollary 2. *Let $a, b > 0$ and $p \in [1, +\infty[$. If $f \in \mathcal{S}(\mathbb{R}^d)$ satisfies for all $\xi \in \mathbb{R}^d$,*

$$|\mathcal{F}_D(f)(\xi)| \leq Ce^{-2b||\xi||^2} \tag{5.49}$$

and for all $n \in \mathbb{N}$,

$$||\triangle^n_k \mathcal{F}_D(f)||^2_{k,p} \leq C(2n+2m)!(2a)^{-2n} \tag{5.50}$$

with $m = [\frac{d+2\gamma}{4}] + 1$, then $f = 0$ for $ab > \frac{1}{4}$.

Proof. We put $F(x) = (\mathcal{F}_D(f) *_D N_k(., 1/(8b)))(x)$ where $N_k(., t)$ is the heat kernel given by (2.24). Then by (2.20), it follows that for all $x \in \mathbb{R}^d$,

$$|\triangle^n_k F(x)| \leq ||\triangle^n_k \mathcal{F}_D(f)||_{k,p} ||N_k(., 1/(8b)||_{k,p'},$$

where p' is the conjugate exponent of p. (5.50) implies that

$$|\triangle^n_k F(x)|^2 \leq C(2n+2m)!(2a)^{-2n}.$$

On the other hand, it follows from (2.18) and (2.17) that for all $x \in \mathbb{R}^d$,

$$|F(x)| \leq Ce^{-2b||x||^2}.$$

Therefore, by Theorem 4 we can obtain that $F(x) = 0$ and thus, $\overline{\mathcal{F}_D}(F) = 0$. (2.19) and (2.14) imply that $f = 0$. ■

6. Miyachi's theorem for the Dunkl transform

For the sake of the readers, in this section we state Miyachi's theorem for the Dunkl transform, which is obtained in [4] and [5].

Theorem 5. ([4], [5]) *Let f be a measurable function on $\mathbb{R}^d$ such that*

$$e^{a||x||^2} f \in L^p_k(\mathbb{R}^d) + L^q_k(\mathbb{R}^d) \tag{6.51}$$

and

$$\int_{\mathbb{R}^d} \log^+ \frac{|\mathcal{F}_D(f)(\xi)e^{b||\xi||^2}|}{\lambda} d\xi < \infty, \tag{6.52}$$

for some constants $a, b, \lambda > 0$ and $1 \leq p, q \leq +\infty$.

i) If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$*, then* $f = CN_k(.,b)$ *with* $|C| \leq \lambda$.

iii) If $ab < \frac{1}{4}$*, then for all* $\delta \in]b, \frac{1}{4a}[$*, all functions of the form* $f(x) = P(x)N_k(x,\delta)$*,* $P \in \mathcal{P}$*, satisfy (6.51) and (6.52).*

Corollary 3. ([4]) *Let* f *be a measurable function on* $\mathbb{R}^d$ *such that*

$$e^{a||x||^2} f \in L^p_k(\mathbb{R}^d) + L^q_k(\mathbb{R}^d) \tag{6.53}$$

and

$$\int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^r e^{br||\xi||^2} d\xi < \infty, \tag{6.54}$$

for some constants $a, b > 0$*,* $1 \leq p, q \leq +\infty$ *and* $0 < r \leq \infty$.

i) If $ab \geq \frac{1}{4}$*, then* $f = 0$ *almost everywhere.*

ii) If $ab < \frac{1}{4}$*, then for all* $\delta \in]b, \frac{1}{4a}[$*, all functions of the form* $f(x) = P(x)N_k(x,\delta)$*,* $P \in \mathcal{P}$*, satisfy (6.53) and (6.54).*

Remark 2. In (6.51) and (6.53), $L^1_k(\mathbb{R}^d) + L^\infty_k(\mathbb{R}^d)$ is essential, because $L^p_k(\mathbb{R}^d) + L^q_k(\mathbb{R}^d) \subset L^1_k(\mathbb{R}^d) + L^\infty_k(\mathbb{R}^d)$. Indeed, for $f = f_1 + f_2 \in L^p_k(\mathbb{R}^d) + L^q_k(\mathbb{R}^d)$, we put $f_{i,\infty}(x) = f_i(x)$ if $|f_i(x)| \leq 1$ and 0 otherwise, and $f_{i,+} = f_i - f_{i,\infty}$. Then $f = (f_{1,\infty} + f_{2,\infty}) + (f_{1,+} + f_{2,+}) = f_\infty + f_+$. Since $f_{i,+} \geq 1$, $\|f_{1,+}\|_{k,1} \leq \|f_{1,+}\|^p_{k,p} \leq \|f_1\|^p_{k,p}$ and $\|f_{2,+}\|_{k,1} \leq \|f_{2,+}\|^q_{k,q} \leq \|f_2\|^q_{k,q}$ respectively. Therefore, $f_\infty \in L^\infty_k(\mathbb{R}^d)$ and $f_+ \in L^1_k(\mathbb{R}^d)$.

7. Beurling's theorem for the Dunkl transform

Beurling's theorem and Bonami, Demange, and Jaming's extension are generalized for the Dunkl transform as follows.

Theorem 6. *Let* $N \in \mathbb{N}$*,* $\delta > 0$ *and* $f \in L^2_k(\mathbb{R}^d)$ *satisfy*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)||\mathcal{F}_D(f)(y)||P(y)|^\delta}{(1 + ||x|| + ||y||)^N} e^{||x||||y||} \omega_k(x) dx dy < +\infty, \tag{7.55}$$

where P *is a polynomial of degree* m*. If* $N \geq d + m\delta + 2$*, then*

$$f(y) = \sum_{|s| < \frac{N-d-m\delta}{2}} a^k_s W^k_s(r,y) \text{ a.e.}, \tag{7.56}$$

where $r > 0$*,* $a^k_s \in \mathbb{C}$ *and* $W^k_s(r,.)$ *is given by (2.27). Otherwise,* $f(y) = 0$ *almost everywhere.*

Proof. We start with the following lemma.

Lemma 6. *We suppose that* $f \in L^2_k(\mathbb{R}^d)$ *satisfies (7.55). Then* $f \in L^1_k(\mathbb{R}^d)$.

Proof. We may suppose that $f \neq 0$ in $L^2_k(\mathbb{R}^d)$. (7.55) and the Fubini theorem imply that for almost every $y \in \mathbb{R}^d$,

$$\frac{|\mathcal{F}_D(f)(y)||P(y)|^\delta}{(1 + ||y||)^N} \int_{\mathbb{R}^d} \frac{|f(x)|}{(1 + ||x||)^N} e^{||x||||y||} \omega_k(x) dx < +\infty.$$

Since $\mathcal{F}_D(f) \neq 0$, there exist $y_0 \in \mathbb{R}^d$, $y_0 \neq 0$, such that $\mathcal{F}_D(f)(y_0)P(y_0) \neq 0$. Therefore,

$$\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+||x||)^N} e^{||x|| ||y_0||} \omega_k(x)dx < +\infty.$$

Since $\dfrac{e^{||x|| ||y_0||}}{(1+||x||)^N} \geq 1$ for large $||x||$, it follows that $\displaystyle\int_{\mathbb{R}^d} |f(x)|\omega_k(x)dx < +\infty.$ ∎

This lemma and Proposition 1 imply that ${}^tV_k(f)$ is well-defined almost everywhere on $\mathbb{R}^d$. By the same techniques used in [18], we can deduce that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{e^{||x|| ||y||} |{}^tV_k(f)(x)||\mathcal{F}({}^tV_k)(f)(y)||P(y)|^\delta}{(1+||x||+||y||)^N} dydx < +\infty.$$

According to Theorem 2.3 in [26], we can deduce that for all $x \in \mathbb{R}^d$,

$${}^tV_k(f)(x) = Q(x)e^{-\frac{||x||^2}{4r}},$$

where $r > 0$ and Q is a polynomial of degree strictly lower than $\frac{N-d-m\delta}{2}$. Then it follows from (2.12) that

$$\mathcal{F}_D(f)(y) = \mathcal{F} \circ {}^tV_k(f)(y) = \mathcal{F}\Big(Q(x)e^{-\frac{||x||^2}{4r}}\Big)(y) = R(y)e^{-r||y||^2},$$

where R is a polynomial of degree $\deg Q$. Hence, applying (2.28), we can find constants a_s^k such that

$$\mathcal{F}_D(f)(y) = \mathcal{F}_D\Big(\sum_{|s| < \frac{N-d-m\delta}{2}} a_s^k W_s^k(r,.)\Big)(y).$$

Then the injectivity of $\mathcal{F}_D$ yields the desired result. ∎

As an application of Theorem 6, we can deduce a Gelfand-Shilov type theorem for the Dunkl transform by using the same techniques in [18],

Corollary 4. *Let $N, m \in \mathbb{N}$, $\delta > 0$, $a, b > 0$ with $ab \geq \frac{1}{4}$, and $1 < p, q < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L_k^2(\mathbb{R}^d)$ satisfy*

$$\int_{\mathbb{R}^d} \frac{|f(x)|e^{\frac{(2a)^p}{p}||x||^p}}{(1+\|x\|)^N}\omega_k(x)dx < +\infty \tag{7.57}$$

and

$$\int_{\mathbb{R}^d} \frac{|\mathcal{F}_D(f)(y)|e^{\frac{(2b)^q}{q}||y||^q}|P(y)|^\delta}{(1+||y||)^N} dy < +\infty \tag{7.58}$$

for some $P \in \mathcal{P}_m$.

i) If $ab > \frac{1}{4}$ or $(p,q) \neq (2,2)$, then $f(x) = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$ and $(p,q) = (2,2)$, then f is of the form (7.56) whenever $N \geq \frac{d+m\delta}{2}+1$ and $r = 2b^2$. Otherwise, $f(x) = 0$ almost everywhere.

Proof. Since

$$4ab\|x\|\|y\| \le \frac{(2a)^p}{p}\|x\|^p + \frac{(2b)^q}{q}\|y\|^q,$$

it follows from (7.57) and (7.58) that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)||\mathcal{F}_D(f)(y)||P(y)|^\delta}{(1+||x||+||y||)^{2N}} e^{4ab||x||||y||}\omega_k(x)dxdy < +\infty.$$

Then (7.55) is satisfied, because $4ab \ge 1$. Especially, according to the proof of Theorem 6, we can deduce that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{e^{4ab||x||||y||}|{}^tV_k(f)(x)||\mathcal{F}({}^tV_k)(f)(y)||P(y)|^\delta}{(1+||x||+||y||)^{2N}} dydx < +\infty,$$

and ${}^tV_k(f)$ and f are of the forms

$${}^tV_k(f)(x) = Q(x)e^{-\frac{||x||^2}{4r}} \quad \text{and} \quad \mathcal{F}_D(f)(y) = R(y)e^{-r||y||^2},$$

where $r > 0$ and Q, R are polynomials of the same degree strictly lower than $\frac{2N-d-m\delta}{2}$. Therefore, substituting these, we can deduce that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{e^{-(\sqrt{r}\|y\|-\frac{1}{2\sqrt{r}}\|x\|)^2} e^{(4ab-1)||x||||y||}|Q(x)||R(x)||P(y)|^\delta}{(1+||x||+||y||)^{2N}} dydx < +\infty.$$

When $4ab > 1$, this integral is not finite unless $f = 0$ almost everywhere. Moreover, it follows from (7.57) and (7.58) that

$$\int_{\mathbb{R}^d} \frac{|Q(x)|e^{-\frac{1}{4r}\|x\|^2} e^{\frac{(2a)^p}{p}||x||^p}}{(1+\|x\|)^N}\omega_k(x)dx < +\infty$$

and

$$\int_{\mathbb{R}^d} \frac{|R(y)|e^{-r\|y\|^2} e^{\frac{(2b)^q}{q}||y||^q}|P(y)|^\delta}{(1+||y||)^N} dy < +\infty.$$

Hence, one of these integrals is not finite unless $(p,q) = (2,2)$. When $4ab = 1$ and $(p,q) = (2,2)$, the finiteness of above integrals implies that $r = 2b^2$ and the rest follows from Theorem 6. ■

8. Donoho-Stark uncertainty principle for the Dunkl transform

We shall investigate the case where f and $\mathcal{F}_D(f)$ are close to zero outside measurable sets. Here the notion of "close to zero" is formulated as follows. We say $f \in L^2_k(\mathbb{R}^d)$ is ε-concentrated on a measurable sets $E \subset \mathbb{R}^d$ if there is a measurable function g vanishing outside E such that $||f-g||_{k,2} \le \varepsilon\|f\|_{k,2}$. Therefore, if we introduce a projection operator P_E as

$$P_Ef(x) = \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E, \end{cases}$$

then f is ε-concentrated on E if and only if $||f - P_E f||_{k,2} \leq \varepsilon\|f\|_{k,2}$. We define a projection operator Q_E as

$$Q_E f(x) = \mathcal{F}_D^{-1}\Big(P_E(\mathcal{F}_D(f))\Big)(x).$$

Then $\mathcal{F}_D(f)$ is ε-concentrated on W if and only if $\|f - Q_W f\|_{k,2} \leq \varepsilon\|f\|_{k,2}$. We note that, for measurable sets $E, W \subset \mathbb{R}^d$,

$$Q_W P_E f(x) = \int_{\mathbb{R}^d} q(t,x) f(t)\omega_k(t)dt,$$

where

$$q(t,x) = \begin{cases} \int_W K(-it,\xi)K(ix,\xi)\omega_k(\xi)d\xi & \text{if } t \in E \\ 0 & \text{if } t \notin E. \end{cases}$$

Indeed, by the Fubini theorem we see that

$$\begin{aligned} Q_W P_E f(x) &= \int_W \mathcal{F}_D(P_E f)(\xi)K(\xi, ix)\omega_k(\xi)d\xi \\ &= \int_W \Big(\int_E f(t)K(\xi,-it)\omega_k(t)dt\Big)K(\xi,ix)\omega_k(\xi)d\xi \\ &= \int_E f(t)\Big(\int_W K(\xi,-it)K(\xi,ix)\omega_k(\xi)d\xi\Big)\omega_k(t)dt. \end{aligned}$$

The Hilbert-Schmidt norm $\|Q_W P_E\|_{HS}$ is given by

$$||Q_W P_E||_{HS} = \Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |q(s,t)|^2\omega_k(t)dsdt\Big)^{\frac{1}{2}}.$$

We denote by $\|T\|_2$ the operator norm on $L^2_k(\mathbb{R}^d)$. Since P_E and Q_W are projections, it is clear that $\|P_E\|_2 = \|Q_W\|_2 = 1$. Moreover, it follows that

$$||Q_W P_E||_{HS} \geq ||Q_W P_E||_2. \tag{8.59}$$

Lemma 7. *If E and W are sets of finite measure, then*

$$||Q_W P_E||_{HS} \leq \sqrt{mes_k(E)mes_k(W)}.$$

Proof. For each $t \in E$, we define $g_t(s) = q(s,t)$. (2.13) implies that $\mathcal{F}_D(g_t)(w) = P_W(K(-iw,t))$. Then by Parseval's identity (2.15) and (2.4) it follows that

$$\int_{\mathbb{R}^d} |q(s,t)|^2 ds = \int_{\mathbb{R}^d} |g_t(s)|^2 ds = \int_{\mathbb{R}^d} |\mathcal{F}_D(g_t)(w)|^2\omega_k(w)dw \leq mes_k(W).$$

Hence, integrating over $t \in E$, we see that $||Q_W P_E||^2_{HS} \leq mes_k(E)mes_k(W)$. ∎

Proposition 6. *Let E, W be measurable sets and suppose that $\|f\|_{k,2} = \|\mathcal{F}_D(f)\|_{k,2} = 1$. Assume that $\varepsilon_E + \varepsilon_W < 1$, f is ε_E-concentrated on E and $\mathcal{F}_D(f)$ is ε_W-concentrated on W. Then*

$$mes_k(E)mes_k(W) \geq (1 - \varepsilon_E - \varepsilon_W)^2.$$

Proof. Since $||f||_{k,2} = \|\mathcal{F}_D(f)\|_{k,2} = 1$ and $\varepsilon_E + \varepsilon_W < 1$, the measures of E and W must both be non-zero. Indeed, if not, then the ε_E-concentration of f implies that $||f - P_E f||_{k,2} = ||f||_{k,2} = 1 \leq \varepsilon_E$, which contradicts with $\varepsilon_E < 1$, likewise for $\mathcal{F}_D(f)$. If at least one of $mes_k(E)$ and $mes_k(W)$ is infinity, then the inequality is clear. Therefore, it is enough to consider the case where both E and W have finite positive measures. Since $||Q_W||_2 = 1$, it follows that

$$\begin{aligned} ||f - Q_W P_E f||_{k,2} &\leq ||f - Q_W f||_{k,2} + ||Q_W f - Q_W P_E f||_{k,2} \\ &\leq \varepsilon_W + ||Q_W||_2 ||f - P_E f||_{k,2} \\ &\leq \varepsilon_E + \varepsilon_W \end{aligned}$$

and thus,

$$||Q_W P_E f||_{k,2} \geq ||f||_{k,2} - ||f - Q_W P_E f||_{k,2} \geq 1 - \varepsilon_E - \varepsilon_W.$$

Then $||Q_W P_E||_2 \geq 1 - \varepsilon_E - \varepsilon_W$. (8.59) and Lemma 7 yield the desired inequality. ∎

In the following we shall consider the case of $f \in L^1_k(\mathbb{R}^d)$. As in the L^2_k case, we say that $f \in L^1_k(\mathbb{R}^d)$ is ε-concentrated to E if $||f - P_E f||_{k,1} \leq \varepsilon \|f\|_{k,1}$. Let $B_{k,1}(W)$ denote the subspace of $L^1_k(\mathbb{R}^d)$ which consists of all $g \in L^1_k(\mathbb{R}^d)$ such that $P_W f = f$. We say that f is ε-bandlimited to W if there is a $g \in B_{k,1}(W)$ with $||f - g||_{k,1} < \varepsilon \|f||_{k,1}$. Here we denote by $\|P_E\|_1$ the operator norm of P_E on $L^1_k(\mathbb{R}^d)$ and by $\|P_E\|_{1,W}$ the operator norm of $P_E : B^1_k(W) \to L^1_k(\mathbb{R}^d)$. Corresponding to (8.59) and Lemma 7 in the L^2_k case, we can obtain the following.

Lemma 8. $\|P_E\|_{1,W} \leq mes_k(E) mes_k(W)$.

Proof. For $f \in B_{k,1}(W)$ we see that

$$\begin{aligned} f(t) &= \int_W \mathcal{F}_D(f)(\xi) K(t, i\xi) \omega_k(\xi) d\xi \\ &= \int_W K(t, i\xi) \Big(\int_{\mathbb{R}^d} f(x) K(x, -i\xi) \omega_k(x) \omega_k(\xi) dx \Big) d\xi \\ &= \int_{\mathbb{R}^d} f(x) \Big(\int_W K(t, i\xi) K(x, -i\xi) \omega_k(\xi) d\xi \Big) \omega_k(x) dx. \end{aligned}$$

Therefore, $||f||_{k,\infty} \leq mes_k(W) ||f||_{k,1}$ by (2.4) and thus,

$$||P_E f||_{k,1} = \int_E |f(x)| \omega_k(x) dx \leq mes_k(E) ||f||_{k,\infty} \leq mes_k(E) mes_k(W) ||f||_{k,1}.$$

Then, for $f \in B_{k,1}(W)$,

$$\frac{||P_E f||_{k,1}}{||f||_{k,1}} = mes_k(E) mes_k(W),$$

which implies the desired inequality. ∎

Proposition 7. *Let $f \in L^1_k(\mathbb{R}^d)$. If f is ε_E-concentrated to E and ε_W-bandlimited to W, then*

$$mes_k(E)mes_k(W) \geq \frac{1-\varepsilon_E-\varepsilon_W}{1+\varepsilon_W}.$$

Proof. Without loss of generality, we may suppose that $\|f\|_{k,1} = 1$, Since f is ε_E-concentrated to E, it follows that $\|P_E f\|_{k,1} \geq \|f\|_{k,1} - \|f - P_E f\|_{k,1} \geq 1 - \varepsilon_E$. Moreover, since f is ε_W-bandlimited, there is a $g \in B_{k,1}(W)$ with $||g - f||_{k,1} \leq \varepsilon_W$. Therefore, it follows that

$$||P_E g||_{k,1} \geq ||P_E f||_{k,1} - ||P_E(g-f)||_{k,1} \geq 1 - \varepsilon_E - \varepsilon_W$$

and $||g||_{k,1} \leq ||f||_{k,1} + \varepsilon_W = 1 + \varepsilon_W$. Therefore, for $g \in B_{k,1}(W)$,

$$\frac{||P_E g||_{k,1}}{||g||_{k,1}} \geq \frac{1-\varepsilon_E-\varepsilon_W}{1+\varepsilon_W},$$

which implies that $\|P_E\|_{1,W} \geq \frac{1-\varepsilon_E-\varepsilon_W}{1+\varepsilon_W}$. Lemma 8 yields the desired inequality. ∎

Proposition 8. *Let $f \in L^2_k(\mathbb{R}^d) \cap L^1_k(\mathbb{R}^d)$ with $\|f\|_{k,2} = 1$. If f is ε_E-concentrated to E in L^1_k-norm and $\mathcal{F}_D(f)$ is ε_W-concentrated to W in L^2_k-norm, then*

$$mes_k(E) \geq (1-\varepsilon_E)^2 \|f\|^2_{k,1} \quad \text{and} \quad mes_k(W)\|f\|^2_{k,1} \geq c_k^2(1-\varepsilon_W)^2.$$

In particular,

$$mes_k(E)mes_k(W) \geq c_k^2(1-\varepsilon_E)^2(1-\varepsilon_W)^2.$$

Proof. Since $\|f\|_{k,2} = \|\mathcal{F}_D(f)\|_{k,2} = 1$ and $\mathcal{F}_D(f)$ is ε_W-concentrated to W in L^2_k-norm, it follows that $\|P_W(\mathcal{F}_D(f))\|_{k,2} \geq \|\mathcal{F}_D(f)\|_{k,2} - \|\mathcal{F}_D(f) - P_W(\mathcal{F}_D(f))\|_{k,2} \geq 1 - \varepsilon_W$ and thus,

$$\begin{aligned}(1-\varepsilon_W)^2 &\leq \int_W |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi \\ &\leq mes_k(W)||\mathcal{F}_D(f)||^2_{k,\infty} \leq \frac{mes_k(W)}{c_k^2}||f||^2_{k,1}\end{aligned}$$

by (2.10). Similarly, $\|f\|_{k,2} = 1$ and f is ε_E-concentrated to E in L^1_k-norm,

$$(1-\varepsilon_E)\|f\|_{k,1} \leq \int_E |f(x)|\omega_k(x)dx \leq \sqrt{mes_k(E)}$$

Here we used the Cauchy-Schwarz inequality and the fact that $\|f\|_{k,2} = 1$. ∎

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