# $H^1$-estimates of Littlewood-Paley and Lusin functions for Jacobi analysis

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**Abstract**

For $\alpha \geq \beta \geq -1/2$ let $\Delta(x) = (2\text{sh}x)^{2\alpha+1}(2\text{ch}x)^{2\beta+1}$ denote a weight function on $\mathbb{R}_+$ and $L^1(\Delta)$ the space of integrable functions on $\mathbb{R}_+$ with respect to $\Delta(x)dx$, equipped with a convolution structure. For a suitable $\phi \in L^1(\Delta)$, we put $\phi_t(x) = t^{-1}\Delta(x)^{-1}\Delta(x/t)\phi(x/t)$ for $t > 0$ and define the radial maximal operator $M_\phi$ as a usual manner. We introduce a real Hardy space $H^1(\Delta)$ as the set of all locally integrable functions $f$ on $\mathbb{R}_+$ whose radial maximal function $M_\phi(f)$ belongs to $L^1(\Delta)$. In this paper we shall obtain a relation between $H^1(\Delta)$ and $H^1(\mathbb{R})$. Indeed, we characterize $H^1(\Delta)$ in terms of weighted $H^1$ Hardy spaces on $\mathbb{R}$ via the Abel transform of $f$. As applications of $H^1(\Delta)$ and its characterization, we shall consider $(H^1(\Delta), L^1(\Delta))$-boundedness of some operators associated to the Poisson kernel for Jacobi analysis; the Poisson maximal operator $M_P$, the Littlewood-Paley $g$-function and the Lusin area function $S$. They are bounded on $L^p(\Delta)$ for $p > 1$, but not true for $p = 1$. Instead, $M_P$, $g$ and a modified $S_{a,\gamma}$ are bounded form $H^1(\Delta)$ to $L^1(\Delta)$.

## 1 Introduction

Let $\alpha \geq \beta \geq -1/2$ and $\Delta(x) = \Delta_{\alpha,\beta}(x) = (2\text{sh}x)^{2\alpha+1}(2\text{ch}x)^{2\beta+1}$ for $x \in \mathbb{R}_+ = [0, \infty)$. We define $L^1(\Delta)$ as the space of integrable functions on $\mathbb{R}_+$ with respect to $\Delta(x)dx$. Let $\phi_\lambda(x) = \phi_\lambda^{\alpha,\beta}(x)$ denote the Jacobi function of order $(\alpha, \beta)$, which satisfies a product formula:

$$\phi_\lambda(x)\phi_\lambda(y) = \int_0^\infty \phi_\lambda(z)K(x, y, z)\Delta(z)dx.$$

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Then by using the kernel $K(x, y, z)$, the generalized translation is defined as

$$T_x f(y) = \int_0^\infty f(z)K(x, y, z)\Delta(z)dz$$

and the convolution on $L^1(\Delta)$ is given by

$$f * g(x) = \int_0^\infty f(y)T_x g(y)\Delta(y)dy.$$

We call harmonic analysis associated to $(\mathbb{R}_+, \Delta, *)$ Jacobi analysis. It is not of homogeneous type, because $\Delta(x)$ has an exponential growth order $e^{2\rho x}$ when $x$ goes to $\infty$ whereas $\rho = \alpha + \beta + 1 > 0$. In this paper we treat some integral operators associated to the Poisson kernel $p_t$ (see §5): For a suitable function $f$ on $\mathbb{R}_+$ the Poisson maximal operator is given by

$$M_P(f) = \sup_{t>0} f * p_t(x)$$

and the Littlewood-Paley $g$-function $g(f)$ is defined by

$$g(f)(x) = \Big(\int_0^\infty \Big|t\frac{\partial}{\partial t} f * p_t(x)\Big|^2 \frac{dt}{t}\Big)^{1/2}.$$

Then $M_P$ and $g$ satisfy the maximal theorem; they are bounded on $L^p(\Delta)$ for $1 < p < \infty$ and satisfy a weak type $L^1$ estimate with respect to $\Delta(x)dx$ (see [1], [5], [6]). The aim of this paper is to find a subspace $H^1(\Delta) \subset L^1(\Delta)$, from which $M_P$ and $g$ are strongly bounded to $L^1(\Delta)$. Furthermore, we define a modified Lusin area function $S_{a,\gamma}(f)$, $a > 0$ and $\gamma \geq 0$, as

$$S_{a,\gamma}(f)(x) = \Big(\int_0^\infty \frac{1}{|B(t)|}\chi_{B(at)} * \Big|w_\gamma \cdot t\frac{\partial}{\partial t} p_t * f\Big|^2 (x)\frac{dt}{t}\Big)^{1/2},$$

where $w_\gamma(x) = (\mathrm{th}x)^\gamma$, $B(t) = [0, t]$, $\chi_{B(t)}$ is the characteristic function of $B(t)$ and $|B(t)|$ the volume of $B(t)$ with respect to $\Delta(x)dx$. For $p > 1$, $L^p$-boundedness of $S_{a,0}$ was investigated in [5]. Here we show that if $a < 1/3$, then $S_{a,\alpha+1/2}$ is bounded from $H^1(\Delta)$ to $L^1(\Delta)$.

This paper is organized as follows. Basic notations are given in §2 and the Abel transform $W_+^1(f)$ is defined by using fractional integral operators for Jacobi analysis. In §3 we shall obtain a key relation between the fractional derivatives for Jacobi analysis and the ones for the classical Euclidean analysis, especially, we can rewrite the inverse operator $W_-^1$ of $W_+^1$ in terms of the Euclidean fractional derivatives on $\mathbb{R}_+$ (see Theorem 3.5). We recall the definition of the real Hardy space $H^1(\Delta)$ in §4, which was introduced in [3].

We characterize $H^1(\Delta)$ by using a Euclidean maximal function of $W_+^1(f)$ via the key relation obtained in §3. Then it becomes clear that $H^1(\Delta)$ is related with Euclidean weighted Hardy spaces $H_w^1(\mathbb{R})$ (see (19)). In §5 we consider the $H^1$-estimate of the Poisson maximal operator $M_P$ (see Theorem 5.1) and in §6 the one of the $g$-function (see Theorem 6.5). In §7 we treat the modified area function $S_{a,\gamma}$ and obtain that $S_{a,\alpha+1/2}$, $0 < a < 1/3$, is bounded from $H^1(\Delta)$ to $L^1(\Delta)$ (see Theorem 7.2).

# 2 Notations

Let $\alpha \geq \beta \geq -1/2$ and $\Delta = \Delta_{\alpha,\beta}$ be as before. We put

$$\rho = \alpha + \beta + 1 \quad \text{and} \quad \gamma_\alpha = \alpha + 1/2.$$

Let $L^p(\Delta)$ denote the space of functions $f$ on $\mathbb{R}_+$ with finite $L^p$-norm:

$$\|f\|_{L^p(\Delta)}^p = \int_0^\infty |f(x)|^p \Delta(x) dx$$

and $L^1_{\mathrm{loc}}(\Delta)$ the space of locally integrable functions on $\mathbb{R}_+$. We regard often functions on $\mathbb{R}_+$ as even functions on $\mathbb{R}$, which are denoted by the same symbol. Let $C_c^\infty(\Delta)$ be the space of compactly supported $C^\infty$ even functions on $\mathbb{R}$. For $f \in C_c^\infty(\Delta)$ we define the Jacobi transform $\hat{f}(\lambda)$, $\lambda \in \mathbb{R}$, of $f$ by

$$\hat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x)\Delta(x)dx.$$

We refer to [4] for some basic properties of $\hat{f}$: The map $f \to \hat{f}$ is a bijection of $C_c^\infty(\Delta)$ onto the space of entire holomorphic even functions of exponential type, and the inverse transform is given as

$$f(x) = \int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda,$$

where $C(\lambda)$ is Harish-Chandra's $C$-function (cf. [4, (2.6)]). Furthermore, the map $f \to \hat{f}$ extends to an isometry of $L^2(\Delta)$ onto $L^2(\mathbb{R}_+, |C(\lambda)|^{-2}d\lambda)$. We recall that, as a function of $\lambda$, $\phi_\lambda(x)$ is the Fourier transform of a function $A(x, \cdot)$, which is compactly supported on $[0, x]$:

$$\Delta(x)\phi_\lambda(x) = \int_0^x \cos \lambda y A(x, y)dy.$$

Then the Abel transform $W_+^s(f)$, $s \in \mathbb{R}$, is defined by for $x \in \mathbb{R}_+$,

$$W_+^s(f)(x) = e^{\rho(1+s)x} \int_x^\infty f(y)A(x, y)dy.$$

By the integral formula of $A(x,y)$ (see [4, (2.18)], it follows that for $y \geq x$,

$$A(x,y) \leq ce^{\rho y}(\text{th}y)(\text{th}(y-x))^{\alpha-1/2}(\text{th}(x+y))^{\alpha-1/2} \leq ce^{\rho y}(\text{th}y)^{2\alpha} \tag{1}$$

and

$$\|W_+^1(f)\|_{L^1(\mathbb{R}_+)} \leq c\|f\|_{L^1(\Delta)}.$$

As shown in [4, §6], $W_+^s$ is explicitly given by a composition of the generalized Weyl type fractional operators on $\mathbb{R}_+$:

$$W_+^s(f)(x) = e^{s\rho x}W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(f)(x) = e^{s\rho x}W_+^0(f),$$

where for $\Re\mu > -n$, $n = 0, 1, 2, \cdots$,

$$W_\mu^\sigma(f)(s) = \frac{(-1)^n}{\Gamma(1+n)}\int_s^\infty\Big(\frac{d^n}{d(\text{ch}\sigma t)^n}f(t)\Big)(\text{ch}\sigma t - \text{ch}\sigma s)^{\mu+n-1}d(\text{ch}\sigma t). \tag{2}$$

Hence, the inverse operator $W_-^s$ of $W_+^s$ is given by

$$W_-^s(f) = W^2_{-(\beta+1/2)} \circ W^1_{-(\alpha-\beta)}(e^{-s\rho x}f) = W_-^0(e^{-s\rho x}f). \tag{3}$$

We recall (cf. [4, (3.7)]) that $W_+^s(f) \in C_c^\infty(\mathbb{R})$ if $f \in C_c^\infty(\Delta)$ and the Euclidean Fourier transform $W_+^s(f)^\sim$ on $\mathbb{C}$ of $W_+^s(f)$ coincides with the Jacobi transform $\hat{f}$ on $\mathbb{C} + is\rho$ of $f$: For $\lambda \in \mathbb{C}$,

$$\hat{f}(\lambda + is\rho) = W_+^s(f)^\sim(\lambda).$$

Hence, if we put $\check{f}(x) = f(-x)$, then

$$W_-^s(f) = W_-^0(e^{-s\rho x}f) = W_-^0(e^{s\rho x}\check{f}). \tag{4}$$

# 3 A key relation

We shall rewrite the fractional differential operator $W_-^1$ on $\mathbb{R}_+$ in terms of Euclidean fractional operators $W_\mu^{\mathbb{R}}$ on $\mathbb{R}_+$ (see (5) below).

First we shall obtain some basic properties of $W_\mu^\sigma$ on $\mathbb{R}_+$ (see (2)) and $W_\mu^{\mathbb{R}}$ on $\mathbb{R}_+$, which is defined by

$$W_\mu^{\mathbb{R}}(f)(s) = \frac{(-1)^n}{\Gamma(1+n)}\int_s^\infty f^{(n)}(t)(t-s)^{\mu+n-1}dt \tag{5}$$

for $\Re\mu > -n$, $n = 0, 1, 2, \cdots$. In what follows we denote for $x, \nu > 0$,

$$\Delta_\nu^\sigma(x) = (\text{sh}\sigma x)^{2\nu} \quad \text{and} \quad \nabla_\nu^\sigma(x) = \frac{(\sigma^{-1}\text{ch}\sigma x)^\nu}{\Delta_\nu^\sigma(x)}.$$

**Lemma 3.1.** *Let $\sigma > 0$ and $0 < \mu < 1$. For $F \in C_c^\infty(\mathbb{R}_+)$ and $x > 0$,*

$$W^\sigma_{-\mu}(F)(x) = \nabla^\sigma_\mu(x)\Big(W^{\mathbb{R}}_{-\mu}(F)(x)(\text{th}\sigma x)^\mu + \int_x^\infty F(s)A^\sigma_\mu(x,s)ds\Big),$$

*whete $A^\sigma_\mu(x,s)$ is of the form $A^\sigma_\mu(x,s) = Q_\mu(x,s)Z_\mu(s-x)$ and*

(i) $Z_\mu(x) = e^{-\sigma x}x^{-\mu}$,

(ii) $|Q^\sigma_\mu(x,s)| \le c\dfrac{(\text{th}\sigma x)^{2\mu}}{(\text{th}\sigma(s+x))^{\mu+1}}$ *for $s > x$,*

(iii) $\displaystyle\int_0^s A^\sigma_\mu(x,s)dx \le c$ *for all $s > 0$,*

(iv) $\displaystyle\int_x^\infty A^\sigma_\mu(x,s)ds \le c$ *for all $x > 0$.*

*Proof.* Let $K^\sigma_\mu(x,s) = \Big(\dfrac{\text{ch}\sigma s - \text{ch}\sigma x}{s-x}\Big)^{-\mu}$. We note that

$$W^\sigma_{-\mu}(F)(x) = \int_x^\infty F'(s)(\text{ch}\sigma s - \text{ch}\sigma x)^{-\mu}ds$$
$$= \int_x^\infty F'(s)(s-x)^{-\mu}ds \cdot K^\sigma_\mu(x,x) + \int_x^\infty F'(s)\frac{K^\sigma_\mu(x,s) - K^\sigma_\mu(x,x)}{(s-x)^\mu}ds$$
$$= W^{\mathbb{R}}_{-\mu}(F)(x)K^\sigma_\mu(x,x)$$
$$+ \int_x^\infty F(s)\Big(-\frac{dK^\sigma_\mu}{ds}(x,s) + \frac{\mu\Big(K^\sigma_\mu(x,s) - K^\sigma_\mu(x,x)\Big)}{s-x}\Big)\frac{1}{(s-x)^\mu}ds$$
$$= W^{\mathbb{R}}_{-\mu}(F)(x)K^\sigma_\mu(x,x) + \int_x^\infty F(s)a^\sigma_\mu(x,s)ds.$$

Since $K^\sigma_\mu(x,x) = (\sigma\text{sh}\sigma x)^{-\mu}$, it follows that $\Delta^\sigma_\mu(x)(\sigma^{-1}\text{ch}\sigma x)^{-\mu}K^\sigma_\mu(x,x) = (\text{th}\sigma x)^\mu$. Moreover, since $\dfrac{s-x}{\text{ch}\sigma s - \text{ch}\sigma x} \sim \text{th}(x+s)^{-1}\dfrac{s-x}{\text{th}(s-x)}e^{-\sigma s}$, it follows that $|a^\sigma_\mu(x,s)| \sim \text{th}\sigma(x+s)^{-(\mu+1)}\ e^{-\sigma\mu s}\ (\text{th}\sigma(s-x))^{-\mu}$. Therefore, if we put

$$A^\sigma_\mu(x,s) = \nabla^\sigma_\mu(x)^{-1}a^\sigma_\mu(x,s) = Q^\sigma_\mu(x,s)\cdot(\text{th}(s-x))^{-\mu}e^{-\sigma\mu(s-x)}$$
$$= Q^\sigma_\mu(x,s)Z^\sigma_\mu(s-x),$$

then we can easily deduce that

$$|Q^\sigma_\mu(x,s)| \le c\frac{(\text{th}\sigma x)^{2\mu}}{(\text{th}\sigma(x+s))^{\mu+1}}.$$

In particular, $|Q^\sigma_\mu(x,s)| \le c\dfrac{(\text{th}x)^\mu}{\text{th}s}$ for $x < s$. Hence (iii) and (iv) are obvious from the following lemma. □

**Lemma 3.2.** *We suppose that* $0 < \mu < 1$ *and for* $x < s$, $|g(x,s)| \le \frac{(\mathrm{th}x)^\mu}{\mathrm{th}s}\mathrm{th}(s-x)\cdot(s-x)^{-(1+\mu)}$. *Then it follows that*

$$\int_0^s g(x,s)dx \le c \quad \text{and} \quad \int_x^\infty g(x,s)ds \le c.$$

*Proof.* As for the integral over $x$, since $|g(x,s)| \le (\mathrm{th}x)^{\mu-1}$, it follows that, when $s$ is small, $\int_0^s |g(x,s)dx \le c\int_0^s x^{\mu-1}(s-x)^{-\mu}dx \le c$, and when $s$ is large, the integral is dominated by

$$\begin{aligned}&c\int_0^1 x^{\mu-1}(s-x)^{-\mu}dx + c\int_1^{s-1}(s-x)^{-(\mu+1)}dx + c\int_{s-1}^s (s-x)^{-\mu}dx\\ \le& c\int_0^1 x^{\mu-1}(1-x)^{-\mu}dx + c\int_1^\infty x^{-(\mu+1)}dx + c\int_0^1 x^{-\mu}dx \le c.\end{aligned}$$

On the other hand, as for the integral over $s$, when $x \ge 1$, $\int_x^\infty |g(x,s)|ds \le c\int_0^\infty \mathrm{th}s\cdot s^{-(1+\mu)}ds \le c$, and when $0 < x < 1$, the integral is dominated by

$$\begin{aligned}&cx^\mu\int_x^{2x+1} s^{-1}(s-x)^{-\mu}ds + cx^\mu\int_{2x+1}^\infty (s-x)^{-(1+\mu)}ds\\ \le& c\int_1^\infty s^{-1}(s-1)^{-\mu} + c\int_2^\infty (s-1)^{-(1+\mu)}ds \le c.\end{aligned}$$

This completes the proof. □

**Lemma 3.3.** *Let* $0 < \mu < 1$. *For* $f \in C_c^\infty(\mathbb{R}_+)$ *and* $g \in C^\infty(\mathbb{R}_+)$,

$$W_{-\mu}^{\mathbb{R}}(fg)(x) = W_{-\mu}^{\mathbb{R}}(f)(x)g(x) + \int_x^\infty f(s)B_{g,\mu}(x,s)ds,$$

*where* $B_{g,\mu}(x,s) = \mu(g(x)-g(s))(s-x)^{-(\mu+1)}$. *In particular, when* $g(x) = g_{l,m}(x) = (\mathrm{th}\sigma x)^{-l}(\sigma\mathrm{ch}x)^{-m}$ *for* $l, m > 0$, *it follows that*

$$B_{g_{l,m},\mu}(x,s) = g_{l-1,m}(x)A^1_{l,m,\mu}(x,s) + g_{l+1,m+2}(x)A^2_{l,m,\mu}(x,s),$$

*where* $A^i_{l,m,\mu}(x,s)$ *is of the form* $A^i_{l,m,\mu}(x,s) = Q^i_{l,m,\mu}(x,s)Z^1_{l,m,\mu}(s-x)$ *and*

(*i*) $Z^1_{l,m,\mu}(x) = (\mathrm{th}\sigma x)x^{-(\mu+1)}$, $|Q^1_{l,m,\mu}(x,s)| \le c$,

(*ii*) $Z^2_{l,m,\mu}(x) = e^{-m\sigma x}(\mathrm{th}\sigma x)x^{-(\mu+1)}$, $|Q^2_{l,m,\mu}(x,s)| \le c\frac{\mathrm{th}\sigma x}{\mathrm{th}\sigma s}$.

*In particular, each* $A^i_{l,m,\mu}(x,s)$ *satisfies (iii) and (iv) in Lemma 3.1. Moreover,* $(\mathrm{th}\sigma x)^{\mu-1}A^2_{l,m,\mu}(x,s)$ *also satisfies these properties.*

*Proof.* Since $(fg)'(s) = f'(s)g(x) + \Big(f(s)(g(s) - g(x))\Big)'$, the formula follows by integrating by parts. Let $g = g_{l,m}$ and $\sigma = 1$ without loss of generality. Then it follows that

$$\begin{aligned}
B_{\tilde{g}_{l,m},\mu}(x,s) = \mu\Big(&(\mathrm{th}x)^{-l}(\mathrm{ch}x)^{-m} - (\mathrm{th}x)^{-l}(\mathrm{ch}s)^{-m} \\
&+ (\mathrm{th}x)^{-l}(\mathrm{ch}s)^{-m} - (\mathrm{th}s)^{-l}(\mathrm{ch}s)^{-m}\Big)(s-x)^{-(\mu+1)} \\
=&\mu(\mathrm{th}x)^{-l+1}(\mathrm{ch}x)^{-m} \cdot (\mathrm{th}x)^{-1}\frac{1 - (\mathrm{ch}x)^m/(\mathrm{ch}s)^m}{s-x}(s-x)^{-\mu} \\
&+ \mu(\mathrm{th}x)^{-l-1}(\mathrm{ch}x)^{-m-2} \cdot (\mathrm{th}x)^{l+1}(\mathrm{ch}x)^2\frac{(\mathrm{th}x)^{-l} - (\mathrm{th}s)^{-l}}{s-x}\frac{(\mathrm{ch}x)^m}{(\mathrm{ch}s)^m}(s-x)^{-\mu} \\
=&g_{l-1,m}(x)A^1_{l,m,\mu}(x,s) + g_{l+1,m+2}(x)A^2_{l,m,\mu}(x,s) \\
=&g_{l-1,m}(x)\Big(\frac{1 - (\mathrm{ch}x)^m/(\mathrm{ch}s)^m}{s-x}\frac{s-x}{\mathrm{th}(s-x)}\Big) \cdot \frac{\mathrm{th}(s-x)}{s-x}(s-x)^{-\mu} \\
&+ g_{l+1,m+2}(x)\Big((\mathrm{th}x)^{l+1}(\mathrm{ch}x)^2\frac{(\mathrm{th}x)^{-l} - (\mathrm{th}s)^{-l}}{s-x}\frac{(\mathrm{ch}x)^m}{(\mathrm{ch}s)^m}e^{m(s-x)}\frac{s-x}{\mathrm{th}(s-x)}\Big) \\
&\qquad \times e^{-m(s-x)}\frac{\mathrm{th}(s-x)}{s-x}(s-x)^{-\mu} \\
=&g_{l-1,m}(x) \cdot Q^1_{l,m,\mu}(x,s)Z^1_{l,m,\mu}(s-x) + g_{l+1,m+2}(x) \cdot Q^2_{l,m,\mu}(x,s)Z^2_{l,m,\mu}(s-x).
\end{aligned}$$

Clearly, $Q^i_{l,m,\mu}(x)$ and $Z^i_{l,m,\mu}(x,s)$ satisfy the desired properties. Hence, it follows from Lemma 3.2 that $A^1_{l,m,\mu}(x,s)$ satisfies (iii) and (iv) in Lemma 3.1. Moreover, since $(\mathrm{th}x)^{\mu-1}\frac{\mathrm{th}x}{\mathrm{th}s} \leq \frac{(\mathrm{th}x)^\mu}{\mathrm{th}s}$ for $s \geq x$, $(\mathrm{th}x)^{\mu-1}A^2_{l,m,\mu}(x,s)$ also satisfies (iii) and (iv). □

**Lemma 3.4.** *Let $\sigma > 0$, $0 < \mu < 1$. For $F \in C_c^\infty(\mathbb{R}_+)$ and $G \in C^\infty(\mathbb{R}_+)$*

$$W^\sigma_{-\mu}(FG)(x) = W^{\mathbb{R}}_{-\mu}(F)(x)S_{G,\mu}(x) + \int_x^\infty F(s)T_{G,\mu}(x,s)ds,$$

*where*

$$\begin{aligned}
S_{G,\mu}(x) &= \nabla^\sigma_\mu(x)G(x)(\mathrm{th}\sigma x)^\mu, \\
T_{G,\mu}(x,s) &= \nabla^\sigma_\mu(x)\Big(B_{G,\mu}(x,s)(\mathrm{th}\sigma x)^\mu + G(s)A^\sigma_\mu(x,s)\Big).
\end{aligned}$$

*Proof.* It follows from Lemmas 3.1 and 3.3 that $W^{\sigma}_{-\mu}(FG)(x)$ equals to

$$\begin{aligned}
&\nabla^{\sigma}_{\mu}(x)\Big(W^{\mathbb{R}}_{-\mu}(FG)(x)(\mathrm{th}\sigma x)^{\mu} + \int_x^{\infty}(FG)(s)A^{\sigma}_{\mu}(x,s)ds\Big)\\
=&\nabla^{\sigma}_{\mu}(x)\Big(W^{\mathbb{R}}_{-\mu}(F)(x)G(x)(\mathrm{th}\sigma x)^{\mu} + \int_x^{\infty}F(s)B_{G,\mu}(x,s)ds\cdot(\mathrm{th}\sigma x)^{\mu}\\
&\qquad + \int_x^{\infty}F(s)G(s)A^{\sigma}_{\mu}(x,s)ds\Big).
\end{aligned}$$

□

Now we suppose that $\nu = n+\mu$, $n = 0,1,2,\cdots$ and $0\le\mu<1$. We shall rewrite the fractional derivative $W^{\sigma}_{-\nu}$ in terms of Euclidean fractional derivatives $W^{\mathbb{R}}_{-\gamma}$, $\gamma\in\Gamma$. When $\mu=0$ and $\nu=n=1,2,\cdots$, it easily follows that

$$W^{\sigma}_{-n}(F)(x) = \sum_{k=1}^{n}\sum_{p=n}^{2n-k} c^n_{k,p}g_{2n-k,p}(x)W^{\mathbb{R}}_{-k}(F)(x),$$

where $g_{2n-k,p} = (\mathrm{th}\sigma x)^{-(2n-k)}(\mathrm{ch}\sigma x)^{-p}$. Hence, when $0<\mu<1$ and $\nu = n+\mu$, it follows that

$$W^{\sigma}_{-\nu}(F)(x) = \sum_{k=1_n}^{n}\sum_{p=n}^{2n-k} c^n_{k,p}W^{\sigma}_{-\mu}(g_{2n-k,p}\cdot W^{\mathbb{R}}_{-k}(F))(x),$$

where $1_n = 1$ if $n\ge 1$ and $1_n = 0$ if $n=0$. Here we apply Lemma 3.4 by substituting $F$ and $G$ with $W^{\mathbb{R}}_{-k}(F)$ and $g_{2n-k,p}$ respectively. Then $W^{\mathbb{R}}_{-\mu}(F)$ in Lemma 3.4 corresponds to $W^{\mathbb{R}}_{-(k+\mu)}(F)$ and

$$\begin{aligned}
S_{g_{2n-k,p},\mu}(x) &= \nabla^{\sigma}_{\mu}(x)g_{2n-k,p}(x)(\mathrm{th}\sigma x)^{\mu}\\
&= c_{\sigma}\nabla^{\sigma}_{\nu}(x)(\mathrm{th}\sigma x)^{k+\mu}(\mathrm{ch}\sigma x)^{-(p-n)}.
\end{aligned}$$

As for $T_{g_{2n-k,p},\mu}(x,s)$, $s>x$, since $B_{g_{2n-k,p},\mu}(x,s) = cg_{2n-k-1,p}A^1_{2n-k,p,\mu}(x,s) + g_{2n-k+1,p+2}A^2_{2n-k,p,\mu}(x,s)$ by Lemma 3.3, it follows that the first term of $T_{g_{2n-k,p},\mu}(x,s)$ is equal to

$$\begin{aligned}
\nabla^{\sigma}_{\mu}(x)B_{g_{2n-k,p}\mu}(x,s)(\mathrm{th}\sigma x)^{\mu} =&\nabla^{\sigma}_{\nu}(x)(\mathrm{th}x)^{k}(\mathrm{ch}x)^{-(p-n)}\\
&\times\Big((\mathrm{th}x)^{\mu+1}A^1_{2n-k,p,\mu}(x,s) + (\mathrm{ch}x)^{-2}(\mathrm{th}x)^{\mu-1}A^2_{2n-k,p,\mu}(x,s)\Big)
\end{aligned}$$

and the second term of $T_{g_{2n-k,p},\mu}(x,s)$ is equal to

$$\begin{aligned}
&\nabla^{\sigma}_{\mu}(x)g_{2n-k,p}(s)A^{\sigma}_{\mu}(x,s)\\
=&\nabla^{\sigma}_{\nu}(x)(\mathrm{th}\sigma x)^{k}\cdot(\mathrm{ch}x)^{-(p-n)}\frac{g_{2n-k,p}(s)}{g_{2n-k,p}(x)}A^{\sigma}_{\mu}(s,x).
\end{aligned}$$

Here we define $\tilde{A}^i_{2n-k,p,\mu}$, $\tilde{Q}^i_{2n-k,p,\mu}$ and $\tilde{Z}^i_{2n-k,p,\mu}$ for $i = 0, 1, 2$ as

$$\begin{aligned}
\tilde{A}^0_{2n-k,p,\mu}(x, s) &= \frac{g_{2n-k,p}(s)}{g_{2n-k,p}(x)} A^\sigma_\mu(s, x) \\
&= \frac{g_{2n-k,p}(s)}{g_{2n-k,p}(x)} Q^\sigma_\mu(x, s) \cdot Z^\sigma_\mu(s - x) \\
&= \tilde{Q}^0_{2n-k,p,\mu}(x, s)\tilde{Z}^0_{2n-k,p\mu}(s - x), \\
\tilde{A}^1_{2n-k,p,\mu}(x, s) &= (\mathrm{th}x)^{\mu+1} A^1_{2n-k,p,\mu}(x, s) \\
&= (\mathrm{th}x)^{\mu+1} Q^1_{2n-k,p,\mu}(x, s) \cdot Z^1_{2n-k,p,\mu}(s - x) \\
&= \tilde{Q}^1_{2n-k,p,\mu}(x, s)\tilde{Z}^1_{2n-k,p,\mu}(s - x), \\
\tilde{A}^2_{2n-k,p,\mu}(x, s) &= (\mathrm{ch}x)^{-2}(\mathrm{th}x)^{\mu-1} A^2_{2n-k,p,\mu}(x, s) \\
&= (\mathrm{ch}x)^{-2}(\mathrm{th}x)^{\mu-1} Q^2_{2n-k,p,\mu}(x, s) \cdot Z^2_{2n-k,p,\mu}(s - x) \\
&= \tilde{Q}^2_{2n-k,p,\mu}(x, s)\tilde{Z}^2_{2n-k,p,\mu}(s - x).
\end{aligned}$$

Then, by noting Lemmas 3.1, Lemma 3.2 and 3.4, we can easily deduce that each $\tilde{A}^i_{2n-k,p,\mu}$ satisfy the properties (iii) and (iv) in Lemma 3.1 and moreover, each $\tilde{Z}^i_{2n-k,p,\mu}$ is dominated by $(\mathrm{th}\sigma x)x^{-(\mu+1)}$ and

$$\begin{aligned}
|\tilde{Q}^0_{2n-k,p,\mu}(x, s)| &\le c\frac{(\mathrm{th}\sigma x)^{2\mu}}{\mathrm{th}\sigma(s + x)^{\mu+1}}, \\
|\tilde{Q}^1_{2n-k,p,\mu}(x, s)| &\le c(\mathrm{th}\sigma x)^{\mu-1}, \\
|\tilde{Q}^2_{2n-k,p,\mu}(x, s)| &\le c\frac{(\mathrm{th}\sigma x)^{\mu}}{\mathrm{th}\sigma s}.
\end{aligned} \tag{6}$$

In particular, $|\tilde{Q}^i_{2n-k,p,\mu}(x, s)| \le c\frac{(\mathrm{th}\sigma x)^{\mu}}{\mathrm{th}\sigma s}$ holds for all $i = 0, 1, 2$. Hence, changing the notations by removing '~', we can obtain the following.

**Proposition 3.5.** *Let $\nu = n + \mu > 0$, $n = 0, 1, 2, \cdots$ and $0 \le \mu < 1$. Then for $F \in C^\infty_c(\mathbb{R}_+)$ and $x > 0$,*

$$\begin{aligned}
W^\sigma_\nu(F)(x) =& \nabla^\sigma_\nu(x) \sum_{k=1_n}^{n} \sum_{p=n}^{2n-k} c^n_{k,p}\Big(c_\sigma(\mathrm{th}\sigma x)^{k+\mu}(\mathrm{ch}\sigma x)^{-(p-n)} W^{\mathbb{R}}_{-(k+\mu)}(F)(x) \\
&+ (\mathrm{th}\sigma x)^k(\mathrm{ch}x)^{-(p-n)} \int_x^\infty W^{\mathbb{R}}_{-k}(F)(s) A_{2n-k,p,\mu}(x, s) ds
\end{aligned} \tag{7}$$

*where each $A_{2n-k,p,\mu}(x, s)$ is of the form*

$$A_{2n-k,p,\mu}(x, s) = Q_{2n-k,p\mu}(x, s) Z_{2n-k,p,\mu}(s - x)$$

*and*

$$(i)\quad Z_{2n-k,p,\mu}(x) \le c(\mathrm{th}\sigma x)x^{-(\mu+1)},$$
$$(ii)\quad |Q_{2n-k,p,\mu}(x,s)| \le c\frac{(\mathrm{th}\sigma x)^{\mu}}{(\mathrm{th}\sigma s)}\quad \textit{for } s > x,$$
$$(iii)\quad \int_0^s A_{2n-k,p,\mu}(x,s)dx \le c \textit{ for all } s > 0,$$
$$(iv)\quad \int_x^{\infty} A_{2n-k,p,\mu}(x,s)ds \le c\quad \textit{for all } x > 0.$$

Next we shall consider a composition of $W^2_{-\nu}$ and $W^1_{-\nu'}$. We suppose that $\nu = n+\mu$ and $\nu' = \mu' + n'$, where $n, n' = 0, 1, 2, \cdots$ and $0 \le \mu, \mu' < 1$. When one of $\mu$ and $\mu'$ is equal to 0, we can easily deduce the final theorem form Proposition 3.5. Hence we may assume that $\mu, \mu' > 0$ in the following. We note that $W^2_{-\nu} \circ W^1_{-\nu'} = W^2_{-\mu} \circ (W^2_{-n} \circ W^1_{-\nu'})$ and $W^2_{-1} = \dfrac{1}{\mathrm{ch}x} W^1_{-1}$. Thereby, it follows from Proposition 3.5 that

$$W^2_{-n} \circ W^1_{-\nu'}(F)(x) = \sum_{l=1_n}^{n} \frac{{}_nC_l}{(\mathrm{ch}x)^{2n-l}} W^1_{-(\nu'+l)}(F)(x)$$
$$= \sum_{l=1_n}^{n} \frac{{}_nC_l}{(\mathrm{ch}x)^{2n-l}} \nabla^1_{\nu'+l}(x) \sum_{k=1_{n'+l}}^{n'+l} \sum_{p=n'+l}^{2(n'+l)-k} \begin{pmatrix} \text{the right hand side of (7)} \\ \text{changed as } \sigma, \mu, n \to 1, \mu', n'+l \end{pmatrix}.$$

Hence in order to calculate $W^2_{-\nu} \circ W^1_{-\nu'}$, we first apply $W^2_{-\mu}$ to each term in the right hand side and then use Lemma 3.4 to rewrite it in terms of Euclidean fractional derivatives. Therefore, it is enough to estimate the following terms $I_{ij}$, $i, j = 1, 2$: For $\gamma = k + \mu'$ and $v = p - n$,

$$\begin{aligned} I_1(x) &= W^2_{-\mu}\Big(\frac{1}{(\mathrm{ch}x)^{2n-l}} \nabla^1_{\nu'+l}(x)(\mathrm{ch}x)^{-v}(\mathrm{th}x)^{\gamma} W^{\mathbb{R}}_{-\gamma}(F)(x)\Big) \\ &= W^2_{-\mu}\Big(g_{2\nu'+2l-\gamma,\nu'+2n+v} W^{\mathbb{R}}_{-\gamma}(F)(x)\Big) \\ &= W^{\mathbb{R}}_{-(\gamma+\mu)}(F)(x)S(x) + \int_x^{\infty} W^{\mathbb{R}}_{-\gamma}(F)(s)T(x,s)ds \\ &= I_{11}(x) + I_{12}(x), \end{aligned}$$

where

$$\begin{aligned} S(x) &= \nabla^2_{\mu}(x) g_{2\nu'+2l-\gamma,\nu'+2n+v}(x)(\mathrm{th}x)^{\mu}, \\ T(x,s) &= \nabla^2_{\mu}(x)\Big(B_{g_{2\nu'+2l-\gamma,\nu'+2n+v},\mu}(x,s)(\mathrm{th}x)^{\mu} \\ &\qquad + (\mathrm{th}x)^{\mu-1} g_{2\nu'+2l-\gamma,\nu'+2n+v}(x) A^2_{\mu}(x,s)\Big) \end{aligned}$$

and for $\gamma = k$ and $v = p - n$, by substituting $A_{2n-k,p,\mu'}(x,s)$ by $A_{\gamma,\mu'}(x,s)$,

$$\begin{aligned}
I_2(x) &= W^2_{-\mu}\Big(\frac{1}{(\mathrm{ch}x)^{2n-l}}\nabla^1_{\nu'+l}(x) \\
&\qquad\qquad \times (\mathrm{th}x)^{\gamma}(\mathrm{ch}x)^{-v}\int_x^\infty W^{\mathbb{R}}_{-\gamma}(F)(s)A_{\gamma,\mu'}(x,s)ds\Big) \\
&= W^2_{-\mu}\Big(g_{2\nu'+2l-\gamma,\nu'+2n+v}\int_x^\infty W^{\mathbb{R}}_{-\gamma}(F)(s)A_{\gamma,\mu'}(x,s)ds\Big) \\
&= W^{\mathbb{R}}_{-\mu}\Big(\int_x^\infty W^{\mathbb{R}}_{-\gamma}(F)(s)A_{\gamma,\mu'}(x,s)ds\Big)(x)S(x) \\
&\qquad\qquad + \int_x^\infty\Big(\int_s^\infty W^{\mathbb{R}}_{-\gamma}(F)(t)A_{\gamma,\mu'}(s,t)dt\Big)T(x,s)ds \\
&= I_{21}(x) + I_{22}(x).
\end{aligned}$$

<u>$I_{11}$ and $I_{12}$</u>: By the process which yields Proposition 3.5 from Lemma 3.3, it follows that

$$S(x) = \nabla^2_\nu(x)\nabla^1_{\nu'}(x)(\mathrm{th}x)^{2(n-l)+\gamma+\mu}(\mathrm{ch}x)^{-v}\frac{(\mathrm{ch}2x)^n}{(\mathrm{ch}x)^{2n}}, \tag{8}$$

$$\begin{aligned}
T(x,s) = \nabla^2_\nu(x)\nabla^1_{\nu'}(x)(\mathrm{th}x)^{2(n-l)+\gamma}(\mathrm{ch}x)^{-v} \\
\times\Big(A^0_{\gamma,\mu}(x,s) + A^1_{\gamma,\mu}(x,s) + A^2_{\gamma,\mu}(x,s)\Big),
\end{aligned} \tag{9}$$

where each $A^i_{\gamma,\mu}$ satisfies the corresponding properties (i)$\sim$ (iv) in Proposition 3.5. Hence, $I_{11}$ and $I_{12}$ can be written as

$$I_{11}(x) = \nabla^2_\nu(x)\nabla^1_{\nu'}(x)(\mathrm{th}x)^{2(n-l)}(\mathrm{ch}x)^{-v}\frac{(\mathrm{ch}2x)^n}{(\mathrm{ch}x)^{2n}}\cdot W^{\mathbb{R}}_{-(\gamma+\mu)}(F)(x)(\mathrm{th}x)^{\gamma+\mu},$$

$$I_{12}(x) = \nabla^2_\nu(x)\nabla^1_{\nu'}(x)(\mathrm{th}x)^{\gamma}\cdot(\mathrm{th}x)^{2(n-l)}(\mathrm{ch}x)^{-v}\int_x^\infty W^{\mathbb{R}}_{-\gamma}(F)(s)A_{\gamma,\mu}(x,s)ds,$$

where each $A_{\gamma,\mu}(x,s)$ satisfies the properties (i)$\sim$ (iv).
<u>$I_{21}$</u>: We define $q_{\gamma,\mu'}(x) = Q_{\gamma,\mu'}(x,x)(\mathrm{th}x)^{1-\mu'}$, which is bounded because of (6), and we put

$$\begin{aligned}
R_{\gamma,\mu'}(x,s) &= \Big(Q_{\gamma,\mu'}(x,s) - \frac{(\mathrm{th}x)^{\mu'}}{\mathrm{th}s}q_{\gamma,\mu'}(x)\Big)Z_{\gamma,\mu'}(s-x) \\
&= \Big(Q_{\gamma,\mu'}(x,s) - \frac{Q_{\gamma,\mu'}(x)}{\mathrm{th}s}\Big)Z_{\gamma,\mu'}(s-x).
\end{aligned}$$

Then

$$A_{\gamma,\mu'}(x,s) = \frac{Q_{\gamma,\mu'}(x)}{\mathrm{th}s}Z_{\gamma,\mu'}(s-x) + R_{\gamma,\mu'}(x,s). \tag{10}$$

Since $|\dfrac{Q_{\gamma,\mu'}(x)}{\text{th}s}| \leq c\dfrac{(\text{th}x)^{\mu'}}{\text{th}s}$ and $R_\gamma(x,x) = 0$, it follows that $|R_{\gamma,\mu'}(x,s)| \leq c\dfrac{(\text{th}x)^{\mu'}}{\text{th}s}(\text{th}(s-x))^2(s-x)^{-(1+\mu')}$ and moreover,

$$\begin{aligned}\frac{d}{dx}R_{\gamma,\mu'}(x,s) \sim &\frac{(\text{th}x)^{\mu'}}{\text{th}s}\big(\text{th}(s-x)^2(s-x)^{-(\mu'+2)} \\ &\qquad + (\text{th}x)^{-1}\text{th}(s-x)^2(s-x)^{-(\mu'+1)}\big), \qquad (11)\end{aligned}$$

where the second term appears when $x$ is small.

We here define $I_{21}^{QZ}$ and $I_{21}^{R}$ by replacing $A_{\gamma,\mu'}(x,s)$ by $(\text{th}s)^{-1}Q_{\gamma,\mu'}(x)$ $Z_{\gamma,\mu}(s-x)$ and $R_{\gamma,\mu'}(x,s)$ respectively (see (10)). Then $I_{21} = I_{21}^{QZ} + I_{21}^{R}$.

As for $I_{21}^{QZ}$, tt follows from Lemma 3.3 that

$$\begin{aligned}I_{21}^{QZ}(x) = &W_{-\mu}^{\mathbb{R}}\Big(\int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(s)(\text{th}s)^{-1}\cdot Z_{\gamma,\mu'}(s-x)ds\cdot Q_{\gamma,\mu'}(x)\Big)(x)S(x) \\ =&W_{-\mu}^{\mathbb{R}}\Big(\int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(s)(\text{th}s)^{-1}\cdot Z_{\gamma,\mu'}(s-x)ds\Big)(x)Q_{\gamma,\mu'}(x)S(x) \\ &+\int_x^\infty\int_s^\infty W_{-\gamma}^{\mathbb{R}}(F)(t)(\text{th}t)^{-1}\cdot Z_{\gamma,\mu'}(t-s)dt\cdot B_{Q_{\gamma,\mu'},\mu}(x,s)ds\cdot S(x) \\ =&W_{-\mu}^{\mathbb{R}}\Big(\int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(s)(\text{th}s)^{-1}\cdot Z_{\gamma,\mu'}(s-x)ds\Big)(x)Q_{\gamma,\mu'}(x)S(x) \\ &+\int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(t)(\text{th}t)^{-1}\Big(\int_x^t Z_{\gamma,\mu'}(t-s)B_{Q_{\gamma,\mu'},\mu}(x,s)ds\Big)dt\cdot S(x) \\ &= J_1 + J_2.\end{aligned}$$

To calculate $J_1$, we take $\delta > 0$ such that $\tilde{W}_\delta^{\mathbb{R}}Z_\gamma(s-x)\Big|_{s=x} = 0$, where $\tilde{W}_\delta^{\mathbb{R}}$ is the Riemann type fractional operator which is defined as

$$\tilde{W}_\delta^{\mathbb{R}}(f)(x) = \int_0^x f(s)(s-x)^{\delta-1}ds. \qquad (12)$$

Then, since $\dfrac{d}{dx}\tilde{W}_\delta^{\mathbb{R}}Z_{\gamma,\mu(}(s-x) = -\dfrac{d}{ds}\tilde{W}_\delta^{\mathbb{R}}Z_{\gamma,\mu'}(s-x)$, it follows that

$$\begin{aligned}&W_{-\mu}^{\mathbb{R}}\Big(\int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(s)(\text{th}s)^{-1}\cdot Z_{\gamma,\mu'}(s-x)ds\Big)(x) \\ =&W_{-\mu}^{\mathbb{R}}\Big(\int_x^\infty W_{-\delta}^{\mathbb{R}}\Big(W_{-\gamma}^{\mathbb{R}}(F)(t)(\text{th}t)^{-1}\Big)(s)\tilde{W}_\delta^{\mathbb{R}}Z_{\gamma,\mu'}(s-x)ds\Big) \\ =&\int_x^\infty W_{-(\mu+\delta)}^{\mathbb{R}}\Big(W_{-\gamma}^{\mathbb{R}}(F)(t)(\text{th}t)^{-1}\Big)(s)\tilde{W}_\delta^{\mathbb{R}}Z_{\gamma,\mu'}(s-x)ds \\ =&\int_x^\infty W_{-\mu}^{\mathbb{R}}\Big(W_{-\gamma}^{\mathbb{R}}(F)(t)(\text{th}t)^{-1}\Big)(s)Z_{\gamma,\mu'}(s-x)ds.\end{aligned}$$

Hence, applying Lemma 3.3, the first term $J_1$ of $I_{21}^{QZ}$ becomes

$$\begin{aligned}
&\int_x^\infty W^{\mathbb{R}}_{-(\gamma+\mu)}(F)(s)(\mathrm{th}s)^{-1}\cdot Z_{\gamma,\mu'}(s-x)ds\cdot Q_{\gamma,\mu'}(x)S(x)\\
&+\int_x^\infty\int_s^\infty W^{\mathbb{R}}_{-\gamma}(F)(t)B_{(\mathrm{th}t)^{-1},\mu}(s,t)dt\cdot Z_{\gamma,\mu'}(s-x)ds\cdot Q_{\gamma,\mu'}(x)S(x)\\
=&\int_x^\infty W^{\mathbb{R}}_{-(\gamma+\mu)}(F)(s)(\mathrm{th}s)^{-1}\cdot Z_{\gamma,\mu'}(s-x)ds\cdot Q_{\gamma,\mu'}(x)S(x)\\
&+\int_x^\infty W^{\mathbb{R}}_{-\gamma}(F)(t)\Big(\int_x^t B_{(\mathrm{th}t)^{-1},\mu}(s,t)Z_{\gamma,\mu'}(s-x)ds\Big)dt\cdot Q_{\gamma,\mu'}(x)S(x)\\
&=J_{11}+J_{12}.
\end{aligned}$$

For $J_{11}$, substituting (8) and $|Q_{\gamma,\mu'}(x)|\le c(\mathrm{th}x)^{\mu'}$, we see that

$$\begin{aligned}
&(\mathrm{th}s)^{-1}Z_{\gamma,\mu'}(s-x)\cdot Q_{\gamma,\mu'}(x)S(x)\\
\le& c\nabla^2_\nu(x)\nabla^1_{\nu'}(x)(\mathrm{th}x)^{2(n-l)}(\mathrm{ch}x)^{-v}\frac{(\mathrm{th}x)^{\mu+\mu'}}{s}Z_{\gamma,\mu'}(s-x).
\end{aligned}$$

For $J_{12}$, we note from Lemma 3.3 with $l=1,m=0$ that $|B_{(\mathrm{th}t)^{-1},\mu}(s,t)|\le \dfrac{1}{\mathrm{th}t\cdot\mathrm{th}s}\mathrm{th}(t-s)(t-s)^{-(\mu+1)}$. When $t$ is small, we here take $0<\epsilon<\min\{1-\mu',\mu\}$ and let $s^{-1}\le x^{-(\mu'+\epsilon)}(s-x)^{-1+\mu'+\epsilon}$. Then it follows that

$$\begin{aligned}
L=&\int_x^t B_{(\mathrm{th}t)^{-1},\mu}(s,t)Z_{\gamma,\mu'}(s-x)ds\\
\sim&\frac{1}{t}\int_x^t\frac{1}{s}(t-s)^{-\mu}(s-x)^{-\mu'}ds\\
\le&\frac{1}{tx^{\mu'+\epsilon}}\int_0^{t-x}((t-x)-s)^{-\mu}s^{-1+\epsilon}ds=c\frac{1}{tx^{\mu'+\epsilon}}(t-x)^{-(\mu-\epsilon)}.
\end{aligned}$$

When $t$ is large and $t-x$ is small, since $x$ is large, $L$ is dominated by

$$\int_x^t(t-s)^{\mu}(s-x)^{-\mu'}ds=(t-x)^{-1+(2-\mu'-\mu)}\ \text{ and }\ 2-\mu'-\mu>0,$$

and when $t$ is large and $t-x$ is large, $L$ is dominated by

$$\begin{aligned}
&\int_x^t\mathrm{th}(t-s)\cdot(t-s)^{-(1+\mu)}\cdot\mathrm{th}(s-x)\cdot(s-x)^{-(1+\mu')}ds \qquad (13)\\
\sim&\int_x^t\Big(\frac{1}{1+(t-s)}\Big)^{\mu+1}\Big(\frac{1+(t-s)}{t-s}\Big)^{\mu}\Big(\frac{1}{1+(s-x)}\Big)^{\mu'+1}\Big(\frac{1+(s-x)}{s-x}\Big)^{\mu'}ds\\
=&(t-x)^{-\mu-\mu'+1}\int_0^1\frac{(1-s)^{-\mu}}{1+(t-x)(1-s)}\cdot\frac{s^{-\mu'}}{1+(t-x)s}ds.
\end{aligned}$$

Then, by dividing the last integral as $\int_0^1 ds = \int_0^{1/2} ds + \int_{1/2}^1 ds$, we see that $L$ is dominated by $(t-x)^{-(1+\mu)} + (t-x)^{-(1+\mu')}$. $(t-x)^{-1-(\mu-\epsilon)}$. Therefore, substituting (8) and $|Q_{\gamma,\mu'}(x)| \leq c(\mathrm{th}x)^{\mu'}$, we can find a $\xi > 0$ such that

$$\begin{aligned}&\Big(\int_x^t B_{(\mathrm{th}t)^{-1},\mu}(s,t)Z_{\gamma,\mu'}(s-x)ds\Big)\cdot Q_{\gamma,\mu'}(x)S(x)\\ \leq& c\nabla_\nu^2(x)\nabla_{\nu'}^1(x)(\mathrm{th}x)^{2(n-l)}(\mathrm{ch}x)^{-v}\frac{(\mathrm{th}x)^\xi}{t}\mathrm{th}(t-x)(t-x)^{-(1+\xi)}.\end{aligned}$$

We recall that the properties (iii) and (iv) follows from (i) and (ii). Hence we can conclude that

$$\begin{aligned}J_1 =& \nabla_\nu^2(x)\nabla_{\nu'}^1(x)(\mathrm{th}x)^{2(n-l)}(\mathrm{ch}x)^{-v}\\ &\times\Big((\mathrm{th}x)^{\gamma+\mu}\int_x^\infty W^{\mathbb{R}}_{-(\gamma+\mu)}(F)(s)A_{\gamma,\mu}(x,s)ds\\ &\qquad + (\mathrm{th}x)^\gamma\int_x^\infty W^{\mathbb{R}}_{-\gamma}(F)(s)A_\gamma(x,s)ds\Big),\end{aligned} \tag{14}$$

where $A_{\gamma,\mu}(x,s)$ and $A_\gamma(x,s)$ satisfy the corresponding properties (i)$\sim$(iv) in Proposition 3.5. To calculate $J_2$ we recall that $|Q_{\gamma,\mu'}(x)| \leq c(\mathrm{th}x)^{\mu'}$. Since $\mu' < 1$, it follows that $|B_{Q_{\gamma,\mu'},\mu}(x,s)| \leq cs^{\mu'-1}(s-x)^{-\mu}$ for all small $s$, and since $\gamma \geq 0$, it follows that $|B_{Q_{\gamma,\mu'},\mu}(x,s)| \leq c(s-x)^{-(\mu+1)}$ for all $x < s$. We shall estimate the inside integral of $J_2$. When $t$ is small, similarly as above, we take $0 < \epsilon < \min\{1-\mu', \mu\}$ and let $s^{\mu'-1} \leq x^{-\epsilon}(s-x)^{\mu'-1+\epsilon}$. Then the inside integral of $J_2$ is dominated by $x^{-\epsilon}(t-x)^{-(\mu-\epsilon)}$. When $t$ is large, also similarly as above, it is dominated by $(t-x)^{-(1+\mu)} + (t-x)^{-(1+\mu')}$. Hence, substituting (8), we can find a $\xi$ such that

$$\begin{aligned}&(\mathrm{th}t)^{-1}\Big(\int_x^t Z_{\gamma,\mu'}(t-s)B_{Q_{\gamma,\mu'},\mu}(x,s)ds\Big)dt\cdot S(x)\\ \sim& \nabla_\nu^2(x)\nabla_{\nu'}^1(x)(\mathrm{th}x)^{2(n-l)}(\mathrm{ch}x)^{-v}\frac{(\mathrm{th}x)^\xi}{\mathrm{th}t}\mathrm{th}(t-x)(t-x)^{-(1+\xi)}.\end{aligned}$$

Therefore, $J_2$ can be rewritten as the last term in (14).

As for $I_{21}^R$, we recall that $R_\gamma(x,x)=0$. Then it follows that

$$\begin{aligned}
I_{21}^R(x) &= W_{-\mu}^{\mathbb{R}}\Big(\int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(s)R_\gamma(x,s)ds\Big)(x)S(x)\\
&= \int_x^\infty\Big(\int_s^\infty W_{-\gamma}^{\mathbb{R}}(F)(t)\frac{d}{dx}R_\gamma(s,t)dt\Big)(s-x)^{-\mu}ds\cdot S(x)\\
&= \int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(t)\Big(\int_x^t \frac{d}{dx}R_\gamma(s,t)(s-x)^{-\mu}ds\Big)dt\cdot S(x)\\
&= \nabla_\nu^2(x)\nabla_{\nu'}^1(x)(\mathrm{th}x)^{2(n-l)+u}(\mathrm{ch}x)^{-v}c_n(x)\cdot(\mathrm{th}x)^{\gamma+\mu}\\
&\qquad\times\int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(t)\Big(\int_x^t\frac{d}{dx}R_\gamma(s,t)(s-x)^{-\mu}ds\Big)dt.
\end{aligned}$$

We shall prove that $M=(\mathrm{th}x)^\mu\displaystyle\int_x^t\frac{d}{dx}R_\gamma(s,t)(s-x)^{-\mu}ds$ satisfies the properties (i) and (ii). We recall (11) and we consider separately the cases where (a) $t$ is small, (b) $t$ is large and $t-x$ is small, (c) $t$, $t-x$ are large and $x$ is large, (d) $t$, $t-x$ are large and $x$ is small.

(a) When $t$ is small, $M$ is dominated by

$$\frac{x^\mu}{t}\int_x^t\Big(s^{\mu'}(t-s)^{-\mu'}(s-x)^{-\mu}+s^{\mu'-1}(t-s)^{1-\mu'}(s-x)^{-\mu}\Big)ds=M_1+M_2.$$

Then $M_1\le c\dfrac{x^\mu}{t}\displaystyle\int_x^t(t-s)^{-\mu'}(s-x)^{-\mu}ds\le c\dfrac{x^\mu}{t}(t-s)^{-1+(2-\mu-\mu')}$. As for $M_2$, if $\mu+\mu'>1$, then $M_2\le c\dfrac{x^{\mu+\mu'-1}}{t}(t-x)^{2-\mu-\mu'}$ and $\mu+\mu'-1,\ 2-\mu-\mu'>0$. If $\mu+\mu'\le 1$, we take $0<\epsilon<\min\{\mu,\mu'\}$ and let $s^{\mu'-1}\le x^{-\mu+\epsilon}(s-x)^{-1+\mu+\mu'-\epsilon}$. Then $M_2\le c\dfrac{x^\epsilon}{t}(t-x)^{1-\epsilon}$.

(b) When $t$ is large and $t-x$ is small, since $x$ is large, $M\le c\displaystyle\int_x^t(t-s)^{-\mu'}(x-s)^{-\mu}ds=c(t-x)^{-1+(2-\mu-\mu')}$.

(c) When $t,t-x,x$ are large, we divide the integral $M$ as $\displaystyle\int_x^{(t+x)/2}dx+\int_{(t+x)/2}^t ds=M_-+M_+$. Similarly as (13), $M_-$ is dominated by

$$(t-x)^{-\mu-\mu'+1}\int_0^{1/2}\frac{(1-s)^{-\mu'}s^{-\mu}}{(1+(t-x)(1-s))^2}ds\le c(t-x)^{-1-\mu-\mu'}.$$

As for $M_+$, by integration by part, it follows that $M_+$ is dominated by

$$R_{\gamma,\mu'}\Big(\frac{t+x}{2},t\Big)\Big(\frac{t-x}{2}\Big)^{-\mu}+\mu\int_{(t+x)/2}^t R_{\gamma,\mu'}(s,t)(s-x)^{-\mu-1}ds.$$

Then the last integral is dominated as

$$\int_{(t+x)/2}^{t}(1+(t-s))^{-(1+\mu')}(s-x)^{-\mu-1}ds \le \Big(\frac{t-x}{2}\Big)^{-(1+\mu)}\int_0^{\infty}(1+s)^{-(1+\mu')}ds.$$

Hence, it is easy to see that $M_+$ is dominated by $c(t-x)^{-(1+\mu)}$.

(d) When $t, t-x$ are large and $x$ is small, we divide the integral as $\int_x^1 ds + \int_1^t ds$. Then the last integral satisfies the same estimate in (c) because $\int_1^t ds \le \int_x^t ds$. On the other hand, the first one is dominated by

$$\begin{aligned} &x^{\mu}\int_x^1 \Big(s^{\mu'-1}(t-s)^{-(1+\mu')}(s-x)^{-\mu}+s^{\mu'}(t-s)^{-(2+\mu')}(s-x)^{-\mu}\Big)ds\\ =&m_1+m_2. \end{aligned}$$

As for $m_1$, we replace $\int_x^1 ds \le \int_x^t ds$ and apply the same argument as used for $M_1$ in (a). Then we can deduce that, if $\mu+\mu'>1$, then $m_1 \le cx^{\mu+\mu'-1}(t-x)^{-(\mu+\mu')}$ and, if $\mu+\mu'\le 1$, then $m_1\le cx^{\epsilon}(t-x)^{-(1+\epsilon)}$, where $0<\epsilon<\min\{\mu,\mu'\}$. As for $m_2$, by replacing $s^{\mu}$ by 1, $m_2$ is dominated by

$$x^{\mu}(t-x)^{-1-\mu-\mu'}\int_0^{(1-x)/(t-x)}(1-s)^{-(2+\mu')}s^{-\mu}ds \le cx^{\mu}(t-x)^{-1-\mu-\mu'},$$

because we may suppose that $(1-x)/(t-x)\le 1/2$.

Hence, in all cases $M$ satisfies the desired properties (i) and (ii). Therefore, $I_{21}^R$ can be rewritten as the last term in (14).

$\underline{I_{22}}$: Last we shall estimate $I_{22}$. Substituting (8) and changing the order of integrations, it follows that

$$\begin{aligned} I_{22}(x) &= \int_x^{\infty} W_{-\gamma}^{\mathbb{R}}(F)(t)\Big(\int_x^t A_{\gamma}(s,t)T(x,s)ds\Big)dt\\ &= \nabla_{\nu}^2(x)\nabla_{\nu'}^1(x)(\mathrm{th}x)^{\gamma+\mu+\mu'-2}\\ &\qquad \times(\mathrm{th}x)^{2(n-l)+u}(\mathrm{ch}x)^{-v}\int_x^{\infty}W_{-\gamma}^{\mathbb{R}}(F)(t)A_{\gamma,\mu}(x,t)dt, \end{aligned}$$

where we denote each $\int_x^t A_{\gamma,\mu'}(s,t)A_{\gamma,\mu}^i(x,s)ds$ by $A_{\gamma}(x,s)$. Since $A_{\gamma,\mu'}(s,t)$ and $A_{\gamma,\mu}^i(x,s)$ satisfy the desired properties (i) and (ii), it is easy to see that $A_{\gamma}(x,s)$ also satisfies the same properties. Therefore, $I_{22}$ also can be rewritten as the last term in (14).

Finally, we can obtain the following.

**Theorem 3.6.** *Let $\nu = n + \mu$ and $\nu' = n' + \mu'$, where $n, n' = 0, 1, 2, \cdots$ and $0 \le \mu, \mu' < 1$. Then for $F \in C_c^\infty(\mathbb{R}_+)$,*

$$W_\nu^2 \circ W_{\nu'}^1(F)(x) \sim \frac{(2^{-1}\mathrm{ch}2x)^\nu (\mathrm{ch}x)^{\nu'}}{(\mathrm{sh}2x)^{2\nu}(\mathrm{sh}x)^{2\nu'}} \Big( \sum_{\gamma\in\Gamma_0} (\mathrm{th}x)^\gamma W_{-\gamma}^{\mathbb{R}}(F)(x) + \sum_{\gamma\in\Gamma_1} (\mathrm{th}x)^\gamma \int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(s) A_\gamma(x,s)ds \Big),$$

*where $\Gamma_0 = \{k + \mu + \mu' | 1_{n+n'} \le k \le n + n'\}$ and $\Gamma_1 = \{k, k+\mu, k+\mu' | 1_{n+n'} \le k \le n + n'\}$. For each $\gamma \in \Gamma_1$, $A_\gamma(x,s)$ is of the form $A_\gamma(x,s) = Q_\gamma(x,s) Z_\gamma(s-x)$ and there exists $0 < \xi < 1$ such that*

$$\begin{aligned}
&(i) \quad Z_\gamma(x) \le c(\mathrm{th}x) x^{-(\xi+1)}, \\
&(ii) \quad |Q_\gamma(x,s)| \le c\frac{(\mathrm{th}x)^\xi}{(\mathrm{th}s)} \quad \text{for } s > x, \\
&(iii) \quad \int_0^s A_\gamma(x,s)dx \le c \text{ for all } s > 0, \\
&(iv) \quad \int_x^\infty A_\gamma(x,s)ds \le c \quad \text{for all } x > 0.
\end{aligned}$$

Let $\nu = \beta + 1/2$ and $\nu' = \alpha - \beta$ in Theorem 3.6. If we replace $F$ by $e^{-\rho x}F(x)$ and $e^{\rho x}\check{F}(x)$ respectively (see (4)) , then we can obtain the following.

**Corollary 3.7.** *For $F \in C_c^\infty(\mathbb{R}_+)$,*

$$W_-^1(F)(x) \sim \frac{1}{\Delta(x)} \Big( \sum_{\gamma\in\Gamma_0} (\mathrm{th}x)^\gamma W_{-\gamma}^{\mathbb{R}}(F)(x) + \sum_{\gamma\in\Gamma_1} (\mathrm{th}x)^\gamma \int_x^\infty W_{-\gamma}^{\mathbb{R}}(F)(s) A_\gamma(x,s)ds \Big) \tag{15}$$

$$\sim \frac{e^{\rho x}}{\Delta(x)} \Big( \sum_{\gamma\in\Gamma_0} (\mathrm{th}x)^\gamma e^{\rho x} W_{-\gamma}^{\mathbb{R}}(\check{F})(x) + \sum_{\gamma\in\Gamma_1} (\mathrm{th}x)^\gamma \int_x^\infty e^{\rho s} W_{-\gamma}^{\mathbb{R}}(\check{F})(s) A_\gamma(x,s)ds \Big) \tag{16}$$

*where $\Gamma_i$ and $A_\gamma(x,s)$ are same as in Theorem 3.6.*

# 4 Real Hardy spaces

We keep the notations in the previous section. We put $\Gamma = \Gamma_0 \cup \Gamma_1$ and for each $\gamma \in \Gamma$ we define

$$w_\gamma(x) = (\mathrm{th}x)^\gamma, \; x \in \mathbb{R}_+.$$

We regard often $W^{\mathbb{R}}_{-\gamma}(F)$ and $w_\gamma$ on $\mathbb{R}_+$ as even functions on $\mathbb{R}$. We suppose that $f \in L^1(\Delta)$ and put $F = W^1_+(f)$. Since $f = W^1_- \circ W^1_+(f) = W^1_-(F)$, it follows from (15), (16) and the property (iii) of Theorem 3.6 that

$$\|f\|_{L^1(\Delta)} \le c \sum_{\gamma\in\Gamma} \|W^{\mathbb{R}}_{-\gamma}(F)\|_{L^1_{w_\gamma}(\mathbb{R}_+)} \sim \sum_{\gamma\in\Gamma} \|e^{2\rho x} W^{\mathbb{R}}_{-\gamma}(\check{F})\|_{L^1_{w_\gamma}(\mathbb{R}_+)},$$

where $L^1_{w_\gamma}(\mathbb{R}_\pm)$ is the $w_\gamma$-weighted $L^1$-space on $\mathbb{R}_\pm$. Here we recall that $W^{\mathbb{R}}_{-\gamma}$ is a Fourier multiplier of an even or an odd function on $\mathbb{R}$. Therefore, $W^{\mathbb{R}}_{-\gamma}(\check{F})(x) = \pm W^{\mathbb{R}}_{-\gamma}(F)(-x)$ and thus,

$$\|e^{2\rho x} W^{\mathbb{R}}_{-\gamma}(\check{F})\|_{L^1_{w_\gamma}(\mathbb{R}_+)} = \|e^{-2\rho x} W^{\mathbb{R}}_{-\gamma}(F)\|_{L^1_{w_\gamma}(\mathbb{R}_-)}.$$

Hence, it follows that

$$\|W^{\mathbb{R}}_{-\gamma}(F)\|_{L^1_{w_\gamma}(\mathbb{R}_-)} \le \|e^{-2\rho x} W^{\mathbb{R}}_{-\gamma}(F)\|_{L^1_{w_\gamma}(\mathbb{R}_-)} \sim \|W^{\mathbb{R}}_{-\gamma}(F)\|_{L^1_{w_\gamma}(\mathbb{R}_+)}.$$

Then we obtain that

$$\|f\|_{L^1(\Delta)} \le c \sum_{\gamma\in\Gamma} \|W^{\mathbb{R}}_{-\gamma}(F)\|_{L^1_{w_\gamma}(\mathbb{R})}.$$

For the converse, first we note that for $0 \le \gamma \le \gamma_\alpha$, if $-n < \gamma \le -n+1$, then

$$F^{(n)}(x) = \int_x^\infty \frac{d^n}{dx^n}\big(e^{\rho x} A(x,y)\big) dy,$$

because $\frac{d^k}{dx^k}\big(e^{\rho x} A(x,y)\big)\Big|_{y=x} = 0$ for $0 \le k \le n-1$ (see (1)) and thereby, $|W^{\mathbb{R}}_{-\gamma}\big(e^{\rho x} A(x,y)\big)| \le e^{2\rho y}(\mathrm{th}y)^{2\alpha-\gamma}$. Since $e^{2\rho s}(\mathrm{th}s)^{2\alpha+1} \sim \Delta(s)$, we see that

$$\begin{aligned} \int_0^\infty |W^{\mathbb{R}}_{-\gamma}(F)(x)|(\mathrm{th}x)^\gamma dx &\le c \int_0^\infty |f(s)| \Big( \int_0^s |W^{\mathbb{R}}_{-\gamma}(e^{\rho x} A(x,s))|(\mathrm{th}x)^\gamma dx \Big) ds \\ &\le c \int_0^\infty |f(s)|\Delta(s) ds = \|f\|_{L^1(\Delta)}. \end{aligned} \tag{17}$$

Since $\|W^{\mathbb{R}}_{-\gamma}(F)\|_{L^1_{w_\gamma}(\mathbb{R}_-)} \le c\|W^{\mathbb{R}}_{-\gamma}(F)\|_{L^1_{w_\gamma}(\mathbb{R}_+)}$, the converse follows. Hence we can obtain the following.

**Theorem 4.1.** *For $f \in L^1(\Delta)$, it follows that*

$$\|f\|_{L^1(\Delta)} \sim \sum_{\gamma\in\Gamma} \|W^{\mathbb{R}}_{-\gamma}(F)\|_{L^1_{w_\gamma}(\mathbb{R})}. \tag{18}$$

We recall that $\gamma_\alpha = \alpha + 1/2$ is the maximum in $\Gamma$ and rewrite $W^{\mathbb{R}}_{-\gamma}(F)$ as $W^{\mathbb{R}}_{\gamma_\alpha-\gamma} \circ W^{\mathbb{R}}_{-\gamma_\alpha}(F)$. Since $|W^{\mathbb{R}}_{\gamma_\alpha-\gamma} \circ W^{\mathbb{R}}_{-\gamma_\alpha}(F)| \le W^{\mathbb{R}}_{\gamma_\alpha-\gamma}(|W^{\mathbb{R}}_{-\gamma_\alpha}(F)|)$, we have

$$\begin{aligned}\int_0^\infty |W^{\mathbb{R}}_{-\gamma}(F)(x)|w_\gamma(x)dx &\le \int_0^\infty W^{\mathbb{R}}_{\gamma_\alpha-\gamma}(|W_{\gamma_\alpha}(F)|)(x)w_\gamma(x)dx\\ &\le \int_0^\infty |W_{\gamma_\alpha}(F)|(x)\tilde{W}_{\gamma_\alpha-\gamma}(w_\gamma)(x)dx,\end{aligned}$$

where $\tilde{W}_{\gamma_\alpha-\gamma}$ is a fractional integral defined by (12). Since $\tilde{W}^{\mathbb{R}}_{\gamma_\alpha-\gamma}((\mathrm{th}x)^\gamma) \sim (\mathrm{th}x)^{\gamma_\alpha}$ when $x$ is small, it follows that

$$\|f\|_{L^1(\Delta)} \sim \|W^{\mathbb{R}}_{-\gamma_\alpha}(F)\chi_1\|_{L^1_{w_{\gamma_\alpha}}(\mathbb{R})} + \sum_{\gamma\in\Gamma} \|W^{\mathbb{R}}_{-\gamma}(F)(1-\chi_1)\|_{L^1(\mathbb{R})},$$

where $\chi_1(x)$ is the characteristic function of $[0,1]$. In particular, if $f(x)$ is supported on $[0,R]$, then $F = W^1_+(f)$ is also supported on $[0,R]$ and

$$\|f\|_{L^1(\Delta)} \sim c_R \|W^{\mathbb{R}}_{-\gamma_\alpha}(F)\|_{L^1_{w_{\gamma_\alpha}}(\mathbb{R})}.$$

We shall introduce a real Hardy spaces $H^p(\Delta)$, $p > 0$. For $\phi \in C^\infty_c(\Delta)$ with $\int_{-\infty}^\infty \phi(x)\Delta(x) = 1$, we define a dilation $\phi_t$, $t > 0$, of $\phi$ as

$$\phi_t(x) = \frac{1}{t\Delta(x)}\Delta\Big(\frac{x}{t}\Big)\phi\Big(\frac{x}{t}\Big),$$

which keeps the $L^1(\Delta)$-norm of $\phi$, and by using this dilation, we define the radial maximal operator $M_\phi$ by $M_\phi(f)(x) = \sup_{t>0} |f * \phi_t(x)|$. We set

$$H^p(\Delta) = \{f \in L^1_{\mathrm{loc}}(\Delta) \; ; \; M_\phi(f) \in L^p(\Delta)\}$$

and $\|f\|_{H^1(\Delta)} = \|M_\phi(f)\|_{L^p(\Delta)}$. Then it follows from [3, §4] that $H^p(\Delta) = L^p(\Delta)$ for $1 < p < \infty$ and $H^1(\Delta) \subset L^1(\Delta)$. We now apply the formula (15) to $f * \phi_t = W^1_-(W^1_+(f * \phi_t)) = W^1_-(F \circledast W^1_+(\phi_t))$, where $\circledast$ denotes the convolution on $\mathbb{R}$. Then, since $W^{\mathbb{R}}_{-\gamma}(F \circledast W^1_+(\phi_t)) = W^{\mathbb{R}}_{-\gamma}(F) \circledast W^1_+(\phi_t)$ and the Euclidean Fourier transform $\tilde{W}^1_+(\phi_t)$ of $W^1_+(\phi_t)$ has the same properties of the Fourier transform $\tilde{\psi}(t\lambda)$ of a Euclidean dilation $\psi_t$ with non-vanishing moment (see [3, §3]). Therefore, by taking the supremum over $t > 0$ and integrating with respect to $\Delta(x)dx$, we see that $\|M_\phi(f)\|_{L^1(\Delta)}$ is bounded by $\sum_{\gamma\in\Gamma}\|M^{\mathbb{R}}(W_{-\gamma}^{\mathbb{R}}(F))\|_{L^1_{w_\gamma}(\mathbb{R})}$, where $M^{\mathbb{R}}$ is a Euclidean radial maximal operator. Then it follow from (18) that

$$\|f\|_{H^1(\Delta)} \leq c\sum_{\gamma\in\Gamma}\|W_{-\gamma}^{\mathbb{R}}(F)\|_{H^1_{w_\gamma}(\mathbb{R})},$$

where $H^1_{w_\gamma}(\mathbb{R})$ is the $w_\gamma$-weighted $H^1$-space on $\mathbb{R}$ (see [8, Chap.6]). For the converse, we note that $\|W_{-\gamma}^{\mathbb{R}}(F)\|_{H^1_{w_\gamma}(\mathbb{R})} \sim \|\sup_{t>0} W_{-\gamma}^{\mathbb{R}}(F) \circledast W_+^1(\phi_t)\|_{L^1_{w_\gamma}(\mathbb{R})}$ $= \|\sup_{t>0} W_{-\gamma}^{\mathbb{R}}(F \circledast W_+^1(\phi_t))\|_{L^1_{w_\gamma}(\mathbb{R})} \leq c\|\sup_{t>0}|f*\phi_t|\|_{L^1(\Delta)} = \|f\|_{H^1(\Delta)}$ similarly as in (17). Therefore, we can deduce that

**Theorem 4.2.** *For $f \in H^1(\Delta)$, it follows that*

$$\|f\|_{H^1(\Delta)} \sim \sum_{\gamma\in\Gamma}\|W_{-\gamma}^{\mathbb{R}}(F)\|_{H^1_{w_\gamma}(\mathbb{R})}. \tag{19}$$

We here define a norm $\|f\|_{H^1_0(\Delta)}$ as

$$\|f\|_{H^1_0(\Delta)} = \sum_{\gamma\in\Gamma}\|W_{-\gamma}^{\mathbb{R}}(F)\|_{H^1(\mathbb{R})}$$

and denote by $H^1_0(\Delta)$ the set of all $f \in L^1_{\mathrm{loc}}(\Delta)$ satisfying $|f\|_{H^1_0(\Delta)} < \infty$. Clearly, it follows that

$$H^1_0(\Delta) \subset H^1(\Delta)$$

and moreover, since $C(\lambda+i\rho)^{-1} = O(|\lambda|^{\alpha+1/2})$,

$$\|f\|_{H^1_0(\Delta)} \sim \|M_{C(\lambda+i\rho)^{-1}}F\|_{H^1(\mathbb{R})}, \tag{20}$$

where $M_{C(\lambda+i\rho)^{-1}}$ is the Fourier multiplier corresponding to $C(\lambda+i\rho)^{-1}$.

**Remark 4.3.** The right hand sides of (19) and (20) can be characterized in terms of Triebel-Lizorkin spaces $F^s_{1,2}$ on $\mathbb{R}$. Therefore, by taking the inverse $W^1_-$, we can pull back some properties of $F^s_{1,2}$ to Jacobi analysis, such as atomic decompositions and interpolations, which will be investigated in the forthcoming papers.

# 5 Poisson maximal operator

The Poisson kernel $p_t$, $t>0$, is a function on $\mathbb{R}_+$ whose Jacobi transform is given as $\hat{p}_t(\lambda) = e^{-t\sqrt{\lambda^2+\rho^2}}$, $\lambda \in \mathbb{C}$. We define the Poisson maximal operator $M_P$ by

$$M_P(f)(x) = \sup_{t>0}|f*p_t(x)|.$$

Then $M_P$ is bounded on $L^p(\Delta)$ for $1 < p < \infty$ and satisfies the weak type $L^1$ estimate with respect to $\Delta(x)dx$ (see [1], [5], [6]). When $p = 1$, we can prove that $M_P$ is bounded from $H^1(\Delta)$ to $L^1(\Delta)$. Since the proof is essentially same as in [3, Theorem 7.7], we shall give a sketch of the proof.

For $f \in H^1(\Delta)$ we put $F = W_+^1(f)$. In what follows we regard functions on $\mathbb{R}_+$ as even functions on $\mathbb{R}$ denoted by the same symbol. Then for each $\gamma \in \Gamma$, $W_{-\gamma}(F)$ belongs to $H^1_{w_\gamma}(\mathbb{R})$. Since $f * p_t = W_-^1 \circ W_+^1(f * p_t) = W_-^1(F \circledast W_+^1(p_t))$, applying (15) and taking the supremum over $t > 0$, we can deduce that

$$\|M_P(f)\|_{L^1(\Delta)} \le c \sum_{\gamma\in\Gamma} \|M_{\mathbb{R}}(W_{-\gamma}(F))\|_{L^1_{w_\gamma}(\mathbb{R})},$$

where $M_{\mathbb{R}}$ is a maximal operator on $\mathbb{R}$ defined by

$$M_{\mathbb{R}}(H)(x) = \sup_{t>0} |H \circledast W_+(p_t)(x)|.$$

Therefore, to prove the $(H^1(\Delta), L^1(\Delta))$-boundedness of $M_P$, it is enough to show the $(H^1_{w_\gamma}(\mathbb{R}), L^1_{w_\gamma}(\mathbb{R}))$-boundedness of $M_{\mathbb{R}}$ for each $\gamma$. Let $H \in H^1_{w_\gamma}(\mathbb{R})$. We denote a $(1, \infty, 2)$-atomic decomposition of $H$ as

$$H = \sum_i \lambda_{\gamma,i} A_{\gamma,i} \tag{21}$$

where $\lambda_{\gamma,i} \ge 0$, $A_{\gamma,i}$ is a $(1, \infty, 2)$-atom on $\mathbb{R}$ supported on $B_{\gamma,i} = B(x_{\gamma,i}, r_{\gamma,i})$ and

$$\|\sum_i \lambda_{\gamma,i} r_{\gamma,i}^{-1} \chi_{B_{\gamma,i}}\|_{L^1_{w_\gamma}(\mathbb{R})} \le \|H\|_{H^1_{w_\gamma}(\mathbb{R})} \tag{22}$$

(see [8, Chap. 8]). To prove the $(H^1_{w_\gamma}(\mathbb{R}), L^1_{w_\gamma}(\mathbb{R}))$-boundedness of $M_{\mathbb{R}}$, we shall determine a shape of $M_{\mathbb{R}}(A)(x)$ for each $(1, \infty, 2)$-atom $A$ on $\mathbb{R}$. We may suppose that $A$ is centered, that is, $A$ is supported on $[-r, r]$. As in [3, Lemma 7.8], we see that $M_{\mathbb{R}}$ is dominated by the Hardy-Littlewood maximal operator. Thereby, $M_{\mathbb{R}}$ is bounded on $L^2(\mathbb{R})$ and $\int_{|x|<2r} |M_{\mathbb{R}}(A)(x)|^2 dx \le \|M_{\mathbb{R}}(A)\|_2^2 \le c\|A\|_2^2 \le cr^{-1}$. If $x \ge 2r$, as in [3, Lemma 7.9], we see that $M_{\mathbb{R}}(A) \le c\dfrac{r^{1/2}}{|x-r|^{3/2}}$. Then, combining these results, we obtain that

$$\begin{aligned} M_{\mathbb{R}}(A)(x) &\le M_{\mathbb{R}}(A)(x)\chi_{B(0,2r)}(x) + cr^{1/2}|x|^{-3/2}\chi_{B(0,2r)^c}(x) \\ &\le ca(x) + c\sum_{k=2}^{\infty} r^{-1}2^{-3k/2}\chi_{B(0,2^k r)}(x), \end{aligned} \tag{23}$$

where $a \geq 0$, $a$ is supported on $B(0, 2r)$ and $\|a\|_{L^2(\mathbb{R})} \leq r^{-1/2}$. Especially, it follows from (21) that

$$M_{\mathbb{R}}(H)(x) \quad \leq \quad c \sum_i \lambda_{\gamma,i} \Big( a_{\gamma,i}(x) + \sum_{k=2}^{\infty} r_{\gamma,i}^{-1} 2^{-3k/2} \chi_{B(x_{\gamma,i}, 2^k r_{\gamma,i})}(x) \Big), \quad (24)$$

where $a_{\gamma,i} \geq 0$ is supported on $B(x_{\gamma,i}, 2r_{\gamma,i})$ and $\|a_{\gamma,i}\|_{L^2(\mathbb{R})} \leq r_{\gamma,i}^{-1/2}$. Therefore, it follows from [8, Lemmas 4 and 5 in Chap. 8] and (22) that

$$\begin{aligned} \|M_{\mathbb{R}}(H)\|_{L^1_{w_\gamma}(\mathbb{R})} &\leq \| \sum_i \sum_{k=1}^{\infty} \lambda_{\gamma,i} r_{\gamma,i}^{-1} 2^{-3k/2} \chi_{B(x_{\gamma,i}, 2^k r_{\gamma,i})} \|_{L^1_{w_\gamma}(\mathbb{R})} \\ &\leq c \| \sum_i \sum_{k=1}^{\infty} \lambda_{\gamma,i} r_{\gamma,i}^{-1} 2^{-k/2} \chi_{B(x_{\gamma,i}, r_{\gamma,i})} \|_{L^1_{w_\gamma}(\mathbb{R})} \qquad (25) \\ &\leq c \| \sum_i \lambda_{\gamma,i} r_{\gamma,i}^{-1} \chi_{B(x_{\gamma,i}, r_{\gamma,i})} \|_{L^1_{w_\gamma}(\mathbb{R})} \\ &\leq c \|H\|_{H^1_{w_\gamma}(\mathbb{R})}. \end{aligned}$$

Hence $M_{\mathbb{R}}$ is bounded from $H^1_{w_\gamma}(\mathbb{R})$ to $L^1_{w_\gamma}(\mathbb{R})$.

**Theorem 5.1.** *$M_P$ is bounded from $H^1(\Delta)$ to $L^1(\Delta)$.*

# 6 Littlewood-Paley $g$-function

The Littelwood-Paley $g$-function $g(f)$ is defined as

$$g(f)(x) = \Big( \int_0^{\infty} \Big| t \frac{\partial}{\partial t} f * p_t(x) \Big|^2 \frac{dt}{t} \Big)^{1/2}.$$

Then $g$ is bounded on $L^p(\Delta)$ for $1 < p < \infty$ and satisfies the weak type $L^1$ estimate with respect to $\Delta(x)dx$ (see [1], [5], [6]). We put $F = W_+^1(f)$ and $K_t = t(\partial/\partial t)p_t$. Since $t(\partial/\partial t) f * p_t = W_-^1 \circ W_+^1(f * K_t) = W_-^1(W_+^1(f) \circledast W_+^1(K_t)) = W_-^1(F \circledast W_+(K_t))$, it follows that

$$g(f)(x) = \Big( \int_0^{\infty} \Big| W_-^1(F \circledast W_+^1(K_t))(x) \Big|^2 \frac{dt}{t} \Big)^{1/2}. \qquad (26)$$

We here define

$$g_{\mathbb{R}}(H)(x) = \Big( \int_0^{\infty} |H \circledast W_+(K_t)(x)|^2 \frac{dt}{t} \Big)^{1/2}. \qquad (27)$$

**Proposition 6.1.** *Let notation be as above. Then*

$$\|g(f)\|_{L^1(\Delta)} \leq c \sum_{\gamma \in \Gamma} \|g_{\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))\|_{L^1_{w_\gamma}(\mathbb{R})}.$$

*Proof.* We apply the formula (15) to (26). Since $W^{\mathbb{R}}_{-\gamma}(F \circledast W^1_+(K_t)) = W^{\mathbb{R}}_{-\gamma}(F) \circledast W^1_+(K_t)$, we see that

$$\begin{aligned} g(f)(x) \;&\leq\; c\Delta(x)^{-1}\Big( \sum_{\gamma\in\Gamma_0} (\mathrm{th}x)^\gamma g_{\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))(x) \\ &\quad + \sum_{\gamma\in\Gamma_1} (\mathrm{th}x)^\gamma \int_x^\infty g_{\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))(s) A_\gamma(x,s)ds \Big). \end{aligned} \tag{28}$$

We take the integration of the right hand side with respect to $\Delta(x)dx$. Since $A_\gamma(x,s)$ satisfies the property (iii) of Theorem 3.6, it follows that the $\Delta$-integral is dominated by $\|g_{\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))\|_{L^1_{w_\gamma}(\mathbb{R})}$. □

Now we shall consider the $(H^1(\Delta), L^1(\Delta))$-boundedness of $g$. Let $f \in H^1(\Delta)$ and put $F = W_+(f)$. For each $\gamma \in \Gamma$, $W^{\mathbb{R}}_{-\gamma}(F)$ belongs to $H^1_{w_\gamma}(\mathbb{R})$. Hence, Proposition 6.1 and (18) imply that $g$ is of type $(H^1(\Delta), L^1(\Delta))$ provided that $g_{\mathbb{R}}$ is of type $(H^1_{w_\gamma}(\mathbb{R}), L^1_{w_\gamma}(\mathbb{R}))$ for each $\gamma \in \Gamma$. In what follows we shall prove that $g_{\mathbb{R}}$ is bounded from $H^1_{w_\gamma}(\mathbb{R})$ to $L^1_{w_\gamma}(\mathbb{R})$.

Let $H \in H^1_{w_\gamma}(\mathbb{R})$. Then it has a $(1,\infty,1)$-atomic decomposition: $H = \sum_i \lambda_{\gamma,i} A_{\gamma,i}$, which satisfies (22). Similarly as in the case of $M_P$ in §5, we shall determine a shape of $g_{\mathbb{R}}(A)(x)$ for each $(1,\infty,1)$-atom $A$ on $\mathbb{R}$. We may suppose that $A$ is centered, that is, $A$ is supported on $[-r,r]$, $\|A\|_\infty \leq (2r)^{-1}$ and $\displaystyle\int_{-\infty}^\infty A(x)x^k dx = 0$ for $k = 0, 1$.

**Proposition 6.2.** *$g_{\mathbb{R}}$ is $L^2$ bounded on $\mathbb{R}$.*

*Proof.* For $H \in L^2(\mathbb{R})$,

$$\begin{aligned} \|g_{\mathbb{R}}(H)\|^2_{L^2(\mathbb{R})} &= \int_0^\infty \|H \circledast W_+(K_t)\|^2_{L^2(\mathbb{R})} \frac{dt}{t} = \int_0^\infty \|\tilde{H} \cdot W_+(K_t)^\sim\|^2_{L^2(\mathbb{R})} \frac{dt}{t} \\ &= \int_0^\infty \|\tilde{H}(\lambda) \cdot t\sqrt{\lambda(\lambda+2i\rho)} e^{-t\sqrt{\lambda(\lambda+2i\rho)}}\|^2_{L^2(\mathbb{R})} \frac{dt}{t} \\ &= \int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 \left( \int_0^\infty t|\lambda(\lambda+2i\rho)| e^{-2t\Re\sqrt{\lambda(\lambda+2i\rho)}} dt \right) d\lambda \\ &= \int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 \left( \int_0^\infty tre^{-2t\sqrt{r}\cos(\theta/2)} dt \right) d\lambda \\ &\leq c\int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 d\lambda = c\|H\|^2_{L^2(\mathbb{R})}, \end{aligned}$$

where we set $\lambda(\lambda+2i\rho)=re^{i\theta}$ and we used the fact that $\Re(\lambda(\lambda+2i\rho)) = r\cos\theta \geq 0$ and $\cos(\theta/2)=\sqrt{(\cos\theta+1)/2}\geq 1/\sqrt{2}$. $\square$

In particular, we have

$$\int_{|x|\leq 2r} g_{\mathbb{R}}(A)^2(x)dx \leq \|g_{\mathbb{R}}(A)\|^2_{L^2(\mathbb{R})} \leq c\|A\|^2_{L^2(\mathbb{R})} \leq cr^{-1}. \tag{29}$$

Next we shall obtain an estimate of $g_{\mathbb{R}}(A)$ for $|x|>2r$. We recall that

$$\begin{aligned} & W_+(K_t)(x) = te^{\rho x}(\partial/\partial t)W_+(p_t)(x) = te^{\rho x}(\partial/\partial t)F^0_{p_t}(x) \\ = \; & Cte^{\rho x}(\partial/\partial t)\Big(t(t^2+x^2)^{-1/2}\mathsf{K}_1(\rho(t^2+x^2)^{1/2})e^{\rho(t^2+x^2)^{1/2}}\cdot e^{-\rho(t^2+x^2)^{1/2}}\Big), \end{aligned}$$

where $\mathsf{K}_\nu$ is the modified Bessel function (see [1, p. 289]), which satisfies $(d/dx)^k\mathsf{K}_\nu(x) = O(x^{-1/2-k}e^{-x})$ if $x\to\infty$, and $O(x^{-\nu-k})$ if $x\to 0$.

**Lemma 6.3.** *Let notation be as above.*

$$\begin{aligned} (g_1): &\quad W_+(K_t)(x) \leq ct(t^2+x^2)^{-3/4} \text{ if } t^2+x^2\geq 1, \\ (l_1): &\quad W_+(K_t)(x) \leq ct(t^2+x^2)^{-1} \text{ if } t^2+x^2\leq 1, \\ (g_2): &\quad (d/dx)^2(W_+(K_t))(x) \leq ct(t^2+x^2)^{-7/4} \text{ if } t^2+x^2\geq 1, \\ (l_2): &\quad (d/dx)^2(W_+(K_t))(x) \leq ct(t^2+x^2)^{-2} \text{ if } t^2+x^2\leq 1, \\ (g_3): &\quad (d/dx)^2(W_+(K_t))(x) \leq ct^{-2}(t^2+x^2)^{-1} \text{ if } t^2+x^2\geq 1. \end{aligned}$$

*Proof.* Except $(g_3)$, all estimates follow from the asymptotic behavior of $W_+(K_t)(x)$. As for $(g_3)$, as a function $t\in\mathbb{R}_+$, $t^{2l}e^{-\rho(t^2+x^2)^{1/2}}$, $l\in\mathbb{R}$, has the maximum $O(|x|^l e^{-\rho x})$ at $t\sim|x|^{1/2}$. Then $(g_3)$ follows from $(g_2)$. $\square$

**Lemma 6.4.** *Let notation be as above and suppose $|x|\geq 2r$. Then $|A\circledast W_+(K_t)(x)|$ is dominated by*

$$G_r(t,x) = c\begin{cases} t(t+|x|)^{-3/2} & \text{if } t+|x|\geq 1 \qquad (G_1) \\ t(t+|x|)^{-2} & \text{if } t+|x|\leq 1 \qquad (L_1) \\ r^2t^{-2}(t+|x|)^{-2} & \text{if } t+|x|\geq 1 \qquad (G_3) \\ r^2t^{-1}(t+|x|)^{-2} & \text{if } t+|x|\leq 1. \qquad (L_2) \end{cases}$$

*Proof.* Let $|y|\leq r$. Since $|x|\geq 2r$, $|x-y|\leq|x|+r\leq 3|x|/2$ and $|x-y|\geq |x|-r\geq|x|/2$, that is, $|x-y|\sim|x|$ and $t+|x-y|\sim t+|x|$. Therefore, since $A\circledast W_+(K_t)(x) = \int_{-\infty}^{\infty} A(y)W_+(K_t)(x-y)dy$ and $\|A\|_{L^1(\mathbb{R})}=1$, $(G_1)$ and $(L_1)$ follow from $(g_1)$ and $(l_1)$ in Lemma 6.3 respectively. Since $A$ satisfies the moment conditions, it follows that $B(x) = \int_{-\infty}^{x}\int_{-\infty}^{u} A(v)dvdu$ is supported on $[-r, r]$, $\|B\|_\infty \le 2r$, and thereby $\|B\|_{L^1(\mathbb{R})} \le 4r^2$. Since integration by parts implies that $A \circledast W_+(K_t)(x) = \int_{-\infty}^{\infty} B(y)(d/dy)^2(K_t(x-y))dy$, $(G_3)$ and $(L_2)$ follow from $(g_3)$ and $(l_2)$ in Lemma 6.3 respectively. □

We return to the estimate of $g_{\mathbb{R}}(A)(x)$ for $|x| \ge 2r$. Since

$$g_{\mathbb{R}}(A)(x) \le \Big(\int_0^\infty G_r(t,x)^2 \frac{dt}{t}\Big)^{1/2} \tag{30}$$

(see (27)), applying Lemma 6.4, we have the following.

Case I: $r \ge 1$. Since $|x| \ge 2$, we can apply $(G_1)$ and $(G_3)$ in Lemma 6.4. Then $g_{\mathbb{R}}(A)^2(x)$ is dominated by

$$c|x|^{-3}\int_0^{\sqrt{r}} tdt + cr^4|x|^{-4}\int_{\sqrt{r}}^\infty t^{-5}dt \le cr|x|^{-3} + cr^2|x|^{-4} \le cr|x|^{-3}.$$

Case II: $r < 1$. When $|x| \ge 2$, we can use the same argument in Case I and obtain $g_{\mathbb{R}}(A)^2(x) \le cr|x|^{-3}$. We suppose that $|x| \le 1$. Then, if $t \le 1$, we can use $(L_1)$ and $(L_2)$, and if $t \ge 1$, we can use $(G_3)$. Hence, $g_{\mathbb{R}}(A)^2(x)$ is dominated by

$$c|x|^{-4}\int_0^r tdt + cr^4|x|^{-4}\int_r^1 t^{-3}dt + cr^4|x|^{-4}\int_1^\infty t^{-5}dt \le cr^2|x|^{-4} \le cr|x|^{-3}.$$

Therefore, in both cases we can deduce that

$$\Big(\int_0^\infty G_r(t,x)^2\frac{dt}{t}\Big)^{1/2} \le cr^{1/2}|x|^{-3/2} \quad \text{if } |x| \ge 2r. \tag{31}$$

Finally, combining (29) and (31), we see that

$$\begin{aligned} g_{\mathbb{R}}(A)(x) &\le g_{\mathbb{R}}(A)(x)\chi_{B(0,2r)}(x) + cr^{1/2}|x|^{-3/2}\chi_{B(0,2r)^c}(x) \\ &\le ca(x) + c\sum_{k=2}^\infty r^{-1}2^{-3k/2}\chi_{B(0,2^kr)}(x), \end{aligned} \tag{32}$$

where $a \ge 0$, $a$ is supported on $B(0, 2r)$ and $\|a\|_{L^2(\mathbb{R})} \le r^{-1/2}$. Hence (23) also holds for $g_{\mathbb{R}}(A)$. Therefore, the same arguments used for $M_P$ yields that $g_{\mathbb{R}}$ is bounded from $H^1_{w_\gamma}(\mathbb{R})$ to $L^1_{w_\gamma}(\mathbb{R})$ for each $\gamma \in \Gamma$.

**Theorem 6.5.** *$g$ is bounded from $H^1(\Delta)$ to $L^1(\Delta)$.*

# 7 Lusin area function

We retain the notation used in the previous section. We define the Lusin area function $S(f)$ as an analogue of the classical theory (cf. [5, p.314]). Let $B(t) = [0, t]$ and $\chi_{B(t)}$ the characteristic function of $B(t)$. We put $|B(t)| = \int_0^t \Delta(x)dx$. We define the Lusin area function $S(f)$ as

$$S(f)(x) = \Big(\int_0^\infty \frac{1}{|B(t)|}\chi_{B(t)} * \Big|f * t\frac{\partial}{\partial t}p_t\Big|^2(x)\frac{dt}{t}\Big)^{1/2}.$$

As shown in [5], $S$ is bounded on $L^2(\Delta)$. We also define a modified area function $S_\Theta(f)$ as

$$S_\Theta(f)(x)=\Big(\int_0^\infty\frac{1}{|B(t)|}\int_0^\infty\Theta(x,y)T_x\chi_{B(t)}(y)\Big|f * t\frac{\partial}{\partial t}p_t(y)\Big|^2dy\frac{dt}{t}\Big)^{1/2}, \quad (33)$$

where $\Theta(x, y)$ is the even function on $\mathbb{R}^2$, which is defined on $\mathbb{R}_+ \times \mathbb{R}_+$ as

$$\Theta(x,y) = \begin{cases} \dfrac{\Delta(y)}{\Delta(x)}\Big(\dfrac{\mathrm{th}x}{\mathrm{th}y}\Big)^{2\gamma_\alpha} & \text{if } y \geq x \\ \dfrac{\Delta(y)^2}{\Delta(x)^2} & \text{if } y < x. \end{cases}$$

We note that $\Theta(x, y) \geq 1$ if $y \geq x \geq 0$ and $\Theta(x, y) < 1$ if $y \geq x \geq 0$ and moreover, for all $0 \leq \xi \leq \gamma_\alpha$, we see that for $x, y \in \mathbb{R}_+$,

$$\Theta(x,y)\frac{\Delta(x)^2}{\Delta(y)}\Big(\frac{\mathrm{th}y}{\mathrm{th}x}\Big)^{2\xi} \leq \begin{cases} \Delta(x)\Big(\dfrac{\mathrm{th}x}{\mathrm{th}y}\Big)^{2(\gamma_\alpha-\xi)} \\ \Delta(y)\Big(\dfrac{\mathrm{th}y}{\mathrm{th}x}\Big)^{2\xi} \end{cases} \leq \min\{\Delta(x), \Delta(y)\}. \quad (34)$$

We shall consider $(H^1(\Delta), L^1(\Delta))$-boundedness of $S_\Theta$. We recall that $t(\partial/\partial t)f * p_t = W_-^1(F \circledast W_+(K_t))$ and we apply the formula (15) to (33). Here we introduce the operators $S_{\gamma,\mathbb{R}}$ for $\gamma \in \Gamma$ as follows; if $\gamma \in \Gamma_0$, then

$$\begin{aligned} S_{\gamma,\mathbb{R}}(H)(x) =&\Big(\int_0^\infty \frac{1}{|B(t)|}\int_0^\infty \Theta(x,y)T_x\chi_{B(t)}(y) \\ &\times(\mathrm{th}y)^{2\gamma}\Delta(y)^{-2}|H \circledast W_+(K_t)(y)|^2\Delta(y)dy\frac{dt}{t}\Big)^{1/2} \end{aligned} \quad (35)$$

and if $\gamma \in \Gamma_1$, then

$$\begin{aligned} S_{\gamma,\mathbb{R}}(H)(x) =&\Big(\int_0^\infty \frac{1}{|B(t)|}\int_0^\infty \Theta(x,y)T_x\chi_{B(t)}(y)(\mathrm{th}y)^{2\gamma}\Delta(y)^{-2} \\ &\times\Big|\int_y^\infty H \circledast W_+(K_t)(s)A_\gamma(y,s)ds\Big|^2\Delta(y)dy\frac{dt}{t}\Big)^{1/2}. \end{aligned} \quad (36)$$

Then we see that the $L^1(\Delta)$-norm of $S_\Theta(f)$ is dominated as

$$\begin{aligned}&\sum_{\gamma\in\Gamma}\|S_{\gamma,\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))\Delta\|_{L^1(\mathbb{R})}\\ =&\sum_{\gamma\in\Gamma}\|S_{\gamma,\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))(\mathrm{th}x)^{-\gamma}\Delta\|_{L^1_{w_\gamma}(\mathbb{R})}=\sum_{\gamma\in\Gamma}\|T_{\gamma,\mathbb{R}}(W^{\mathbb{R}}_{-\gamma}(F))\|_{L^1_{w_\gamma}(\mathbb{R})},\end{aligned}$$

where

$$T_{\gamma,\mathbb{R}}(H)(x)=S_{\gamma,\mathbb{R}}(H)(x)(\mathrm{th}x)^{-\gamma}\Delta(x).$$

As in the case of $g_{\mathbb{R}}$, $S_\Theta$ is of type $(H^1(\Delta),L^1(\Delta))$ provided that $T_{\gamma,\mathbb{R}}$ is of type $(H^1_{w_\gamma}(\mathbb{R}),L^1_{w_\gamma}(\mathbb{R}))$ for each $\gamma\in\Gamma$. Therefore, to obtain the $(H^1(\Delta),L^1(\Delta))$-boundedness of $S_\Theta$, it is enough to prove that each $T_{\gamma,\mathbb{R}}$ is $L^2$ bounded on $\mathbb{R}$ and it satisfies (30) for each centered $(1,\infty,1)$-atom $A$ on $\mathbb{R}$. Actually, these facts yield (23) for $T_{\gamma,\mathbb{R}}$ as in the case of $g_{\mathbb{R}}$ and thereby, $T_{\gamma,\mathbb{R}}$ is of type $(H^1_{w_\gamma}(\mathbb{R}),L^1_{w_\gamma}(\mathbb{R}))$ as before.

Case of $\gamma\in\Gamma_0$: First we shall prove that $T_{\gamma,\mathbb{R}}$ is bounded on $L^2(\mathbb{R})$. Let $H\in L^2(\mathbb{R})$. We apply (34) in the integrand of $T_{\gamma,\mathbb{R}}(H)^2$ and take the integration over $\mathbb{R}_+$ with respect to $\Delta(x)dx$. Then, since

$$\frac{1}{|B(t)|}\int_0^\infty T_x\chi_{B(t)}(y)\Delta(x)dx=\frac{1}{|B(t)|}\int_0^\infty\chi_{B(t)}(x)\Delta(x)dx=1,\qquad(37)$$

it follows from Proposition 6.2 that

$$\begin{aligned}\|T_{\gamma,\mathbb{R}}(H)\|^2_{L^2(\mathbb{R})}&\le c\int_0^\infty\int_0^\infty|H\circledast W_+(K_t)(y)|^2dy\frac{dt}{t}\\&=c\int_0^\infty g_{\mathbb{R}}(H)^2(y)dy\le c\|H\|_{L^2(\mathbb{R})}.\end{aligned}$$

Next we shall prove that $T_{\gamma,\mathbb{R}}$ satisfies (30). Let $A$ be a $(1,\infty,1)$-atom on $\mathbb{R}$ supported on $[-r,r]$ and let $H=A$ in (35). We suppose that $|x|\ge 2r$. $A\circledast W_+(K_t)(y)$ is given by $\int_{-\infty}^\infty A(z)W_+(K_t)(y-z)dz$ and it follows that $|x-y|\le t$ and $|z|\le r$. Since $x=(x-y)+(y-z)+z$, $|x|\le t+|y-z|+r\le t+|y-z|+|x|/2$ and thus, $|x|\le 2(t+|y-z|)$. Moreover, $|y-z|\le|y|+|z|\le t+|x|+r\le t+3|x|/2$. Hence it follows that $t+|x|\sim t+|y-z|$. Then, applying the arguments used in the proofs of Lemmas 6.3 and 6.4 to $A\circledast W_+(K_t)(y)$, we can deduce that $|A\circledast W_+(K_t)(y)|\le cG_r(t,x)$. Since (34) and

$$\frac{1}{|B(t)|}\int_0^\infty T_x\chi_{B(t)}(y)\Delta(y)dy=\frac{1}{|B(t)|}\int_0^\infty\chi_{B(t)}(y)\Delta(y)dy=1,\qquad(38)$$

it follows that

$$T_{\gamma,\mathbb{R}}(A)(x) \leq c\Big(\int_0^\infty G_r^2(t,x)\frac{dt}{t}\Big)^{1/2}.$$

Case of $\gamma \in \Gamma_1$: First we shall prove that $T_{\gamma,\mathbb{R}}$ is bounded on $L^2(\mathbb{R})$. As in the previous case, first we apply (34) and take the integration over $\mathbb{R}_+$ with respect to $\Delta(x)dx$. Since $A_\gamma(x,s)$ is of the form $A_\gamma(x,s) = Q_\gamma(x,s)Z_\gamma(s-x)$ and satisfies the properties (i) and (ii) of Theorem 3.6, it follows that

$$\begin{aligned}&\int_0^\infty T_{\gamma,\mathbb{R}}(H)^2(x)dx\\ \leq &\int_0^\infty \int_0^\infty \Big|\int_0^\infty H \circledast W_+(K_t)(s+y)\frac{(\mathrm{th}y)^\xi}{\mathrm{th}(s+y)}(\mathrm{th}s)s^{-(1+\xi)}ds\Big|^2 dy\frac{dt}{t}.\end{aligned}$$

When $0 < y < 1$ and $0 < s < 1$, it becomes

$$\begin{aligned}&\int_0^\infty \int_0^1 \Big|\int_0^1 H \circledast W_+(K_t)(s+y)\frac{y^\xi}{(s+y)}s^{-\xi}ds\Big|^2 dy\frac{dt}{t}\\ =&\int_0^\infty \int_0^1 \Big|\int_0^{1/y} H \circledast W_+(K_t)((s+1)y)(s+1)^{-1}s^{-\xi}ds\Big|^2 dy\frac{dt}{t}\\ \leq& c\|g_{\mathbb{R}}(H)\|^2_{L^2(\mathbb{R})}\Big(\int_0^\infty (s+1)^{-2}s^{-\xi}ds\Big)^2 \leq c\|H\|_{L^2(\mathbb{R})}.\end{aligned}$$

Otherwise, the integral is dominated as

$$\begin{aligned}\leq& c\int_{-\infty}^\infty \Big(\int_0^\infty g_{\mathbb{R}}(H)(s+y)Z_\gamma(s)ds\Big)^2 dy\\ \leq& c\|g_{\mathbb{R}}(H)\|^2_{L^2(\mathbb{R})}\Big(\int_0^\infty Z_\gamma(s)ds\Big)^2 \leq \|H\|_{L^2(\mathbb{R})}.\end{aligned}$$

Next we shall prove that $T_{\gamma,\mathbb{R}}$ satisfies (30) for a $(1,\infty,1)$-atom $A$ on $\mathbb{R}$ supported on $[-r,r]$. Let $|x| \geq 2r$ and let $H = A$ in (36). When $s \geq |x|$, it follows that $s \geq 2r$ and $|A \circledast W_+(K_t)(s)| \leq G_r(t,s) \leq G_r(t,|x|)$ by Lemma 6.4. When $s \leq |x|$, we note that $A \circledast W_+(K_t)(s)$ is given by $\displaystyle\int_{-\infty}^\infty A(z)W_+(K_t)(s-z)dz$ and $t + |s-z| \sim t + |x|$. Actually, we may suppose that $|z| \leq r$, $|x-y| \leq t$, and $0 \leq y \leq s \leq x$. Since $x = (x-y)+(y-s)+(s-z)+z$, we see that $x \leq 2t+|s-z|+r \leq 2t+|s-z|+x/2$ and thus, $t+x \leq 4(t+|s-z|)$. Moreover, $t+|s-z| \leq t+s+|z| \leq t+3x/2 \leq 3(t+x)/2$. Therefore, it follows from the arguments used in the proofs of Lemmas 6.3 and 6.4 yield that $A \circledast W_+(K_t)(s) \le cG_r(t,|x|)$ again. Hence it follows from the property (iv) of $A_\gamma(x,s)$ (see Theorem 3.6) that

$$\int_y^\infty A \circledast W_+(K_t)(s)A_\gamma(y,s)ds \le cG_r(t,x)\int_y^\infty A_\gamma(x,s)ds \le cG_r(t,x).$$

Then (34) and (38) imply that

$$T_{\gamma,\mathbb{R}}(A)(x) \le c\Big(\int_0^\infty G_r^2(t,x)\frac{dt}{t}\Big)^{1/2}.$$

We can obtain that $T_{\gamma,\mathbb{R}}$, $\gamma \in \Gamma$, satisfy the desired properties.

**Theorem 7.1.** *$S_\Theta$ is bounded from $H^1(\Delta)$ to $L^1(\Delta)$.*

**Remark 7.2.** Since $a, b > \dfrac{ab}{a+b}$ and $a + b \le (1+a)(1+b)$ for $a, b \ge 0$, it easily follows that $\Theta(x,y) \ge \dfrac{(\mathrm{th}x)^{4\gamma_\alpha}}{\Delta(x)^2}\cdot(\mathrm{th}y)^{2\gamma_\alpha}\Delta(y)$. Therefore, the operator defined by

$$\frac{(\mathrm{th}x)^{2\gamma_\alpha}}{\Delta(x)}\Big(\int_0^\infty \frac{1}{|B(t)|}\chi_{B(at)} * \Big|w_{\gamma_\alpha}\sqrt{\Delta}\cdot f * t\frac{\partial}{\partial t}p_t\Big|^2(x)\frac{dt}{t}\Big)^{1/2}$$

is also bounded from $H^1(\Delta)$ to $L^1(\Delta)$.

Now we shall consider a modified operator $S_{a,\gamma_\alpha}$ for $a > 0$:

$$S_{a,\gamma_0}(f)(x) = \Big(\int_0^\infty \frac{1}{|B(t)|}\chi_{B(at)} * \Big|w_{\gamma_\alpha}\cdot f * t\frac{\partial}{\partial t}p_t\Big|^2(x)\frac{dt}{t}\Big)^{1/2}.$$

By this modification, $\Theta(x,y)$ and $T_x\chi_{B(t)}$ in (33) is changed to $(\mathrm{th}y)^{2\gamma_\alpha}$ and $T_x\chi_{B(at)}$ respectively, and thereby, (34) becomes

$$\begin{aligned} W(x,y) =& (\mathrm{th}y)^{2\gamma_\alpha}\frac{\Delta(x)^2}{\Delta(y)}\Big(\frac{\mathrm{th}y}{\mathrm{th}x}\Big)^{2\gamma} \\ =& \frac{\Delta(x)}{\Delta(y)}\Big(\frac{\mathrm{th}y}{\mathrm{th}x}\Big)^{2\gamma_\alpha}\Delta(x)(\mathrm{th}y)^{2\gamma}(\mathrm{th}x)^{2(\gamma_\alpha-\gamma)} \le ce^{2\rho(x-y)}\Delta(x). \end{aligned}$$

In the previous arguments for $S_\Theta$, which yields Theorem 7.1, the key process is that (34) yields (37), (38) respectively. Therefore, if we can deduce that

$$J(y) = \frac{1}{|B(t)|}\int_{T_xy\le at} W(x,y)dx \sim 1, \tag{34a}$$

$$I(x) = \frac{1}{|B(t)|}\int_{T_xy\le at} W(x,y)dy \sim 1, \tag{34b}$$

then we can apply the previous arguments without changes.

In what follows we suppose that $0 < a \leq 1/3$.

Case I. $x \leq y$: Since $W(x, y) \leq \Delta(x) \leq \Delta(y)$, it follows that $I(x)$ and $J(y)$ are dominated by $\dfrac{|B(at)|}{|B(t)|} \leq 1$.

Case II. $x > y$: Since $a \leq 1/3$, it follows that

$$J(y) \leq e^{2\rho at}\frac{|B(at)|}{|B(t)|} \sim e^{2\rho t(2a-1)} \leq 1.$$

As for $I(x)$, we consider separately the following cases.

Case II(1). $x > y, y \geq 1$: Since $W(x, y) \leq ce^{2\rho(x-y)} \cdot \dfrac{\Delta(x)}{\Delta(y)} \cdot \Delta(y) \leq ce^{4\rho(x-y)}\Delta(y)$, it follows that $I(x) \leq ce^{4\rho at}\dfrac{|B(at)|}{|B(t)|} \sim e^{2\rho t(3a-1)} \leq 1$.

Case II(2). $x > y, y < 1, t \geq 1$: Since $x - at < y < x + at$, it follows that $x < y + at < 1 + at$ and thus, $\Delta(x) \leq \Delta(1 + at) \sim e^{2\rho at}$. Hence

$$I(x) \leq c\frac{\Delta(x)}{|B(t)|}\int_{x-at}^{x+at} e^{2\rho(x-y)}dy = c\frac{e^{4\rho at}}{|B(t)|} \sim e^{2\rho t(2a-1)} \leq 1.$$

Case II(3). $x > y, y < 1, t \leq 1, t \geq x/2$: Since $W(x, y) \leq ce^{2\rho(x-y)}\Delta(x) \leq ce^{2\rho(x-y)}\Delta(2t)$, it follows that

$$I(x) \leq ce^{2\rho at}\frac{\Delta(2t)}{|B(t)|}\int_{x-at}^{x+at} dy = ce^{2\rho at}\frac{\Delta(2t)t}{|B(t)|} \sim 1.$$

Case II(4). $x > y, y < 1, t \leq 1, t < x/2$: Since $x/2 \leq x - t \leq x - at \leq y$ and $x \leq y + at \leq y + t < 2$, we see that $\dfrac{\Delta(x)}{\Delta(y)} \leq \dfrac{\Delta(x)}{\Delta(x/2)} \sim 1$ and $W(x, y) \leq ce^{2\rho(x-y)} \cdot \dfrac{\Delta(x)}{\Delta(y)} \cdot \Delta(y) \leq ce^{2\rho at}\Delta(y)$. Hence $I(x) \leq ce^{2\rho at}\dfrac{|B(at)|}{|B(t)|} \sim 1$.

Thereby, $I(x)$ and $J(y)$ satisfy the desired estimates (34a) and (34b). Hence, by using the same arguments in the proof of Theorem 7.1, we can obtain the following.

**Theorem 7.3.** *For $0 < a < 1/3$, $S_{a,\gamma_\alpha}$ is bounded from $H^1(\Delta)$ to $L^1(\Delta)$.*

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