GENERALIZED BESOV SPACES AND THEIR APPLICATIONS

TAKESHI KAWAZOE AND HATEM MEJJAOLI

Dedicated to Khalifa Trimèche for his 65 birthday

Abstract. We define and study the Bessel potential and inhomogeneous Besov spaces associated with the Dunkl operators on $\mathbb{R}^d$. As applications on these spaces we construct the Sobolev type embedding theorem and the paraproduct operators associated with the Dunkl operators, as similar to those defined by Bony. We also establish Strichartz type estimates for the Dunkl-Schrödinger equation and finally we study the problem of well posedness of the generalized heat equation.

1. Introduction

The Dunkl operators, which are differential-difference operators introduced by Dunkl in [3], are very important in pure mathematics and in physics. Especially, they provide a useful tool in the study of special functions related with root systems (cf. [4]). In the previous paper [7], we study some function spaces associated with Dunkl operators. We have begun a general theory on Littlewood-Paley decompositions associated with Dunkl operators and introduced generalized Sobolev spaces, generalized Hölder spaces and BMO associated with the Dunkl operators.

In this second paper of a series of our study we continue our investigation of function spaces; generalized Bessel potential spaces, inhomogeneous Besov spaces and Triebel-Lizorkin spaces associated with Dunkl operators. We obtain their basic properties and apply them to estimate the solutions of the Dunkl-Schrödinger and the Dunkl heat equations. In their recent paper [1], Abdelkefi, Anker, Sassi and Sifi also obtain some basic properties of the Besov spaces and integrability for the Dunkl transform.

The contents of the paper is as follows. In §2 we recall some basic results about the harmonic analysis associated with the Dunkl operators. In §3 we introduce the Littlewood-Paley decomposition associated with the Dunkl operators. We shall obtain Bernstein's inequalities. §4 is devoted to study the Dunkl-Bessel potential spaces, the inhomogeneous Dunkl-Besov spaces and the Dunkl-Triebel-Lizorkin spaces. According to a standard process in the Euclidean case (cf. [15]), we shall consider equivalent norms, lifting properties, interpolations and dualities of these spaces. In §5 we summarize some results on embeddings and paraproduct operators, which depend on the index γ associated to the multiplicity function of the root system. In the last §6 we consider some applications of the Dunkl-Besov spaces to differential-difference equations. We shall obtain Strichartz type estimates of the solutions of the Dunkl-Schrödinger equation and finally a space-time estimate of the solutions of the Dunkl heat equation.

Throughout this paper by c, C we always represent positive constants not necessarily the same in each occurrence.

1991 *Mathematics Subject Classification.* Primary 35L05. Secondary 46E35, 35J25, 22E30.

Key words and phrases. Dunkl operators, Littlewood-Paley decomposition, inhomogeneous Dunkl-Besov space, inhomogeneous Dunkl-Triebel-Lizorkin space, paraproduct operator, differential-difference equations.

2. Preliminaries

In order to confirm the basic and standard notations we briefly overview the theory of Dunkl operators and related harmonic analysis. Main references are [3, 4, 5, 6, 7, 11, 12, 14, 17, 18].

2.1. **Root system, reflection group and multiplicity function.** Let $\mathbb{R}^d$ be the Euclidean space equipped with a scalar product $\langle , \rangle$ and let $||x|| = \sqrt{\langle x, x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, σ_α denotes the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ perpendicular to α, i.e., for $x \in \mathbb{R}^d$, $\sigma_\alpha(x) = x - 2\|\alpha\|^{-2}\langle\alpha, x\rangle\alpha$. A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. We normalize each $\alpha \in R$ as $\langle\alpha, \alpha\rangle = 2$. We fix a $\beta \in \mathbb{R}^d\backslash\cup_{\alpha\in R}H_\alpha$ and define a positive root system R_+ of R as $R_+ = \{\alpha \in R \mid \langle\alpha, \beta\rangle > 0\}$. The reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group. A function $k : R \to \mathbb{C}$ on R is called a multiplicity function if it is invariant under the action of W. We introduce the index γ as

$$\gamma = \gamma(k) = \sum_{\alpha\in R_+} k(\alpha).$$

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in R$. We denote by ω_k the weight function on $\mathbb{R}^d$ given by

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle\alpha, x\rangle|^{2k(\alpha)},$$

which is invariant and homogeneous of degree 2γ. In the case that the reflection group W is the group $\mathbb{Z}_2^d$ of sign changes, the weight function ω_k is a product function of the form $\prod_{j=1}^d |x_j|^{2k_j}$, $k_j \geq 0$. We denote by c_k the Mehta-type constant defined by

$$c_k = \int_{\mathbb{R}^d} e^{\frac{-||x||^2}{2}} \omega_k(x)dx.$$

In the following we denote by

$C(\mathbb{R}^d)$	the space of continuous functions on $\mathbb{R}^d$.
$C_0(\mathbb{R}^d)$	the space of continuous functions on $\mathbb{R}^d$ vanishing at infinity.
$C^p(\mathbb{R}^d)$	the space of functions of class C^p on $\mathbb{R}^d$.
$C_b^p(\mathbb{R}^d)$	the space of bounded functions of class C^p.
$\mathcal{E}(\mathbb{R}^d)$	the space of C^∞-functions on $\mathbb{R}^d$.
$\mathcal{S}(\mathbb{R}^d)$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.
$D(\mathbb{R}^d)$	the space of C^∞-functions on $\mathbb{R}^d$ which are of compact support.
$\mathcal{S}'(\mathbb{R}^d)$	the space of temperate distributions on $\mathbb{R}^d$.

2.2. **The Dunkl operators.** Let $k : R \to \mathbb{C}$ be a multiplicity function on R and R_+ a fixed positive root system of R. Then the Dunkl operators T_j, $1 \leq j \leq d$, are defined on $C^1(\mathbb{R}^d)$ by

$$T_jf(x) = \frac{\partial}{\partial x_j}f(x) + \sum_{\alpha\in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle\alpha, x\rangle},$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$. Similarly as ordinary derivatives, each T_j satisfies for all f, g in $C^1(\mathbb{R}^d)$ and at least one of them is W-invariant,

$$T_j(fg) = (T_jf)g + f(T_jg)$$

and for all f in $C_b^1(\mathbb{R}^d)$ and g in $\mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} T_j f(x) g(x)\omega_k(x)dx = -\int_{\mathbb{R}^d} f(x) T_j g(x)\omega_k(x)dx.$$

Furthermore, according to [3, 4], the Dunkl operators T_j, $1 \leq j \leq d$ commute and there exists the so-called Dunkl's intertwining operator V_k such that $T_j V_k = V_k(\partial/\partial x_j)$ for $1 \leq j \leq d$ and $V_k(1) = 1$. We define the Dunkl-Laplace operator $\triangle_k$ on $\mathbb{R}^d$ by

$$\triangle_k f(x) = \sum_{j=1}^{d} T_j^2 f(x) = \triangle f(x) + 2\sum_{\alpha\in R^+} k(\alpha)\Big(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}\Big),$$

where $\triangle$ and ∇ are the usual Euclidean Laplacian and nabla operators on $\mathbb{R}^d$ respectively. Since the Dunkl operators commute, their joint eigenvalue problem is significant, and for each $y \in \mathbb{R}^d$, the system

$$T_j u(x, y) = y_j u(x, y),\ j = 1, ..., d, \quad \text{and} \quad u(0, y) = 1$$

admits a unique analytic solution $K(x, y)$, $x \in \mathbb{R}^d$, called the Dunkl kernel. which has a holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. For $x, y \in \mathbb{C}^d$, the kernel satisfies

(a) $K(x, y) = K(y, x)$,
(b) $K(\lambda x, y) = K(x, \lambda y)$ for $\lambda \in \mathbb{C}$,
(c) $K(wx, wy) = K(x, y)$ for $w \in W$.

2.3. **The Dunkl transform.** For functions f on $\mathbb{R}^d$ we define L^p-norms of f with respect to $\omega_k(x)dx$ as

$$\|f\|_{L^p_k(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |f(x)|^p \omega_k(x)dx\Big)^{\frac{1}{p}},$$

if $1 \leq p < \infty$ and $\|f\|_{L^\infty_k(\mathbb{R}^d)} = \text{ess}\ \sup_{x\in\mathbb{R}^d} |f(x)|$. We denote by $L^p_k(\mathbb{R}^d)$ the space of all measurable functions f on $\mathbb{R}^d$ with finite L^p_k-norm.

The Dunkl transform $\mathcal{F}_D$ on $L^1_k(\mathbb{R}^d)$ is given by

$$\mathcal{F}_D(f)(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(x) K(x, -iy)\omega_k(x)dx.$$

Some basic properties are the following (cf. [5] and [6]): For all $f \in L^1_k(\mathbb{R}^d)$,

(a) $\|\mathcal{F}_D(f)\|_{L^\infty_k(\mathbb{R}^d)} \leq c_k^{-1}\|f\|_{L^1_k(\mathbb{R}^d)}$,
(b) $\mathcal{F}_D(f(\cdot/\lambda)))(y) = \lambda^{2\gamma+d} F_D(f)(\lambda y)$ for $\lambda > 0$,
(c) if $\mathcal{F}_D(f)$ belongs to $L^1_k(\mathbb{R}^d)$, then

$$f(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} \mathcal{F}_D(f)(x) K(ix, y)\omega_k(x)dx,$$

and moreover, for all $f \in \mathcal{S}(\mathbb{R}^d)$,

(d) $\mathcal{F}_D(T_j f)(y) = iy_j \mathcal{F}_D(f)(y)$,
(e) if we define $\overline{\mathcal{F}_D}(f)(y) = \mathcal{F}_D(f)(-y)$, then

$$\mathcal{F}_D\overline{\mathcal{F}_D} = \overline{\mathcal{F}_D}\mathcal{F}_D = Id.$$

Proposition 2.1. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^d)$ onto itself and for all f in $\mathcal{S}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |f(x)|^2 \omega_k(x)dx = \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi)d\xi.$$

In particular, the Dunkl transform $f \to \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.

We define the tempered distribution $\mathcal{T}_f$ associated with $f \in L^p_k(\mathbb{R}^d)$ by

$$\langle \mathcal{T}_f, \phi\rangle = \int_{\mathbb{R}^d} f(x)\phi(x)\omega_k(x)dx \tag{2.1}$$

for $\phi \in \mathcal{S}(\mathbb{R}^d)$ and denote by $\langle f, \phi\rangle_k$ the integral in the righthand side.

Definition 2.1. The Dunkl transform $\mathcal{F}_D(\tau)$ of a distribution $\tau \in \mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\langle \mathcal{F}_D(\tau), \phi\rangle = \langle \tau, \mathcal{F}_D(\phi)\rangle$$

for $\phi \in \mathcal{S}(\mathbb{R}^d)$.

In particular, for $f \in L^p_k(\mathbb{R}^d)$, it follows that for $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle \mathcal{F}_D(f), \phi\rangle = \langle \mathcal{F}_D(\mathcal{T}_f), \phi\rangle = \langle \mathcal{T}_f, \mathcal{F}_D(\phi)\rangle = \langle f, \mathcal{F}_D(\phi)\rangle_k.$$

Theorem 2.2. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}'(\mathbb{R}^d)$ onto itself.*

2.4. **The Dunkl convolution.** By using the Dunkl kernel in 2.2, we introduce a generalized translation and a convolution structure in our Dunkl setting. For a function $f \in \mathcal{S}(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$ the Dunkl translation $\tau_y f$ is defined by

$$\tau_y f(x) = \frac{1}{c_k}\int_{\mathbb{R}^d} \mathcal{F}_D(f)(z)K(ix,z)K(iy,z)\omega_k(z)dz.$$

Clearly $\tau_y f(x) = \tau_x f(y)$ and by using the Dunkl's intertwinig operator V_k, $\tau_y f$ is related to the usual translation as $\tau_y f(x) = (V_k)_x(V_k)_y((V_k)^{-1}(f)(x+y))$ (cf. [11, 18]), where the subscript x of $(V_k)_x$ means that V_k is applied to the x variable. Hence, τ_y can also be defined for $f \in \mathcal{E}(\mathbb{R}^d)$. We define the Dunkl convolution product $f *_D g$ of functions $f, g \in \mathcal{S}(\mathbb{R}^d)$ as follows.

$$f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y)g(y)\omega_k(y)dy.$$

This convolution is commutative and associative (cf. [18]). Since $\mathcal{F}_D(\tau_y f)(x) = K(ix,y)$ $\mathcal{F}_D(f)(x)$ by the above definition of $\tau_y f$, it follows that

(a) For all $f, g \in D(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$), $f *_D g$ belongs to $D(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$) and

$$\mathcal{F}_D(f *_D g)(y) = \mathcal{F}_D(f)(y)\mathcal{F}_D(g)(y). \tag{2.2}$$

Moreover, as pointed in [14], §4 and §7, the operator $f \to f *_D g$ is bounded on $L^p_k(\mathbb{R}^d)$, $1 \le p \le \infty$, provided that g is a radial function in $L^1_k(\mathbb{R}^d)$ or an arbitrary function in $L^1_k(\mathbb{R}^d)$ for $W = \mathbb{Z}_2^d$. Hence the standard argument yields the following Young's inequality.

(b) Let $1 \le p, q, r \le \infty$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If $f \in L^p_k(\mathbb{R}^d)$ and $g \in L^q_k(\mathbb{R}^d)$ is radial or arbitrary for $W = \mathbb{Z}_2^d$, then $f *_D g \in L^r_k(\mathbb{R}^d)$ and

$$\|f *_D g\|_{L^r_k(\mathbb{R}^d)} \le 2^{\frac{d}{2}} \|f\|_{L^p_k(\mathbb{R}^d)} \|g\|_{L^q_k(\mathbb{R}^d)}. \tag{2.3}$$

Definition 2.2. The Dunkl convolution product of a distribution S in $\mathcal{S}'(\mathbb{R}^d)$ and a function ϕ in $\mathcal{S}(\mathbb{R}^d)$ is the function $S *_D \phi$ defined by

$$S *_D \phi(x) = \langle S_y, \tau_{-y}\phi(x)\rangle.$$

Proposition 2.3. *Let f be in $L^p_k(\mathbb{R}^d)$, $1 \leq p \leq \infty$, and ϕ in $\mathcal{S}(\mathbb{R}^d)$. Then the distribution $\mathcal{T}_f *_D \phi$ is given by the function $f *_D \phi$. If we assume that ϕ is arbitrary for $d = 1$ and radial for $d \geq 2$, then $\mathcal{T}_f *_D \phi$ belongs to $L^p_k(\mathbb{R}^d)$. Moreover, for all $\psi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\langle \mathcal{T}_f *_D \phi, \psi \rangle = \langle \check{f}, \phi *_D \check{\psi} \rangle_k, \tag{2.4}$$

where $\check{\psi}(x) = \psi(-x)$, and

$$\mathcal{F}_D(\mathcal{T}_f *_D \phi) = \mathcal{F}_D(\mathcal{T}_f)\mathcal{F}_D(\phi). \tag{2.5}$$

Proof. It follows that

$$\begin{aligned}\mathcal{T}_f *_D \phi(x) &= \langle (\mathcal{T}_f)_y, \tau_x \phi(-y) \rangle \\ &= \langle f, \tau_x \phi(-y) \rangle_k = f *_D \phi(x).\end{aligned}$$

Let us suppose that ϕ is arbitrary for $d = 1$ and radial for $d \geq 2$. Then by (2.3), $\mathcal{T}_f *_D \phi$ belongs to $L^p_k(\mathbb{R}^d)$. By Fubini-Tonelli's theorem the function $(x, y) \mapsto f(-y)\tau_x\phi_1(y)\phi_2(x)$ is integrable on $\mathbb{R}^d \times \mathbb{R}^d$ with respect to $\omega_k(y)dy\,\omega_k(x)dx$. Then for any $\psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}\langle \mathcal{T}_f *_D \phi, \psi \rangle &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(-y)\tau_x\phi(y)\psi(x)\omega_k(y)dy\omega_k(x)dx \\ &= \int_{\mathbb{R}^d} f(-y)\Big(\int_{\mathbb{R}^d} \tau_y\phi(x)\psi(x)\omega_k(x)dx\Big)\omega_k(y)dy \\ &= \int_{\mathbb{R}^d} f(-y)\phi *_D \check{\psi}(y)\omega_k(y)dy = \langle \check{f}, \phi *_D \check{\psi} \rangle_k.\end{aligned}$$

Moreover, from (1), (2.2) and (2.4) it follows that

$$\begin{aligned}\langle \mathcal{F}_D(\mathcal{T}_f *_D \phi), \psi \rangle &= \langle \mathcal{T}_f *_D \phi, \mathcal{F}_D(\psi) \rangle \\ &= \langle \check{f}, \phi *_D \check{\mathcal{F}_D(\psi)} \rangle_k \\ &= \langle f, \mathcal{F}_D(\mathcal{F}_D(\phi)\psi) \rangle_k = \langle \mathcal{F}_D(\mathcal{T}_f)\mathcal{F}_D(\phi), \psi \rangle.\end{aligned}$$

□

For each $u \in \mathcal{S}'(\mathbb{R}^d)$, we define the distributions $T_j u$, $1 \leq j \leq d$, by

$$\langle T_j u, \psi \rangle = -\langle u, T_j \psi \rangle$$

for all $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then $\langle \triangle_k u, \psi \rangle = \langle u, \triangle_k \psi \rangle$ and these distributions satisfy the following properties (see 2.3 (d)):

$$\begin{aligned}\mathcal{F}_D(T_j u) &= i y_j \mathcal{F}_D(u), \\ \mathcal{F}_D(\triangle_k u) &= -\|y\|^2 \mathcal{F}_D(u).\end{aligned} \tag{2.6}$$

In the following we denote $\mathcal{T}_f$ given by (2.1) by f for simplicity.

3. Dunkl-Littlewood-Paley decomposition

One of the main tools in this paper is the Dunkl-Littlewood-Paley decompositions of distributions on $\mathbb{R}^d$ into dyadic blocks of frequencies. Let ψ be a non-negative function in $D(\mathbb{R}^d)$, which is radial for $d \geq 2$, satisfying $\psi(\xi) \equiv 1$ for $\|\xi\| \leq \frac{1}{2}$ and $\psi(\xi) \equiv 0$ for $\|\xi\| \geq 1$. We define a function φ on $\mathbb{R}^d$ by

$$\varphi(\xi) = \psi(\frac{\xi}{2}) - \psi(\xi).$$

Then we see that $\psi(\xi) + \sum_{j=0}^{\infty} \varphi(2^{-j}\xi) = 1$.

Definition 3.1. For $j = 0, 1, 2, \cdots$, the operators S_j and Δ_j on $\mathcal{S}'(\mathbb{R}^d)$ are defined by

$$\mathcal{F}_D(S_j f) = \psi(2^{-j}\xi)\mathcal{F}_D(f),$$
$$\mathcal{F}_D(\Delta_j f) = \varphi(2^{-j}\xi)\mathcal{F}_D(f),$$

and put $\Delta_{-1} = S_0$.

We see that $f = \sum_{j=-1}^{\infty} \Delta_j f$ in the sense of $\mathcal{S}'(\mathbb{R}^d)$. We call $\Delta_j f$ the j-th dyadic block of the Dunkl-Littlewood-Paley decomposition of f. Similarly, the operators $\widetilde{S}_j$ and $\widetilde{\Delta}_j$ on $\mathcal{S}'(\mathbb{R}^d)$ are defined by replacing ψ and φ by $\widetilde{\psi}(\xi) = \psi(\frac{\xi}{2})$ and $\widetilde{\varphi}(\xi) = \psi(\frac{\xi}{4}) - \psi(4\xi)$ respectively. Throughout this paper we define the functions χ, $\widetilde{\chi}$, ϕ and $\widetilde{\phi}$ on $\mathbb{R}^d$ respectively by

$$\chi = \mathcal{F}_D^{-1}(\psi),\ \widetilde{\chi} = \mathcal{F}_D^{-1}(\widetilde{\psi}),\ \phi = \mathcal{F}_D^{-1}(\varphi),\ \widetilde{\phi} = \mathcal{F}_D^{-1}(\widetilde{\varphi}).$$

Proposition 3.1. (**Bernstein inequalities**) *For all $\mu \in \mathbb{N}^d$, $\sigma \in \mathbb{R}$, $j \in \mathbb{N}$, $1 \le p, q \le \infty$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, and $f \in \mathcal{S}'(\mathbb{R}^d)$, we have*

(1) $\|\Delta_j f\|_{L_k^q(\mathbb{R}^d)} \le 2^{j(d+2\gamma)(\frac{1}{p}-\frac{1}{q})}\|\widetilde{\phi}\|_{L_k^r(\mathbb{R}^d)}\|\Delta_j f\|_{L_k^p(\mathbb{R}^d)}$,

(2) $\|S_j f\|_{L_k^q(\mathbb{R}^d)} \le 2^{j(d+2\gamma)(\frac{1}{p}-\frac{1}{q})}\|\widetilde{\chi}\|_{L_k^r(\mathbb{R}^d)}\|S_j f\|_{L_k^p(\mathbb{R}^d)}$,

(3) $\|(\sqrt{-\triangle_k})^\sigma \Delta_j f\|_{L_k^p(\mathbb{R}^d)} \le 2^{j\sigma}\|\mathcal{F}_D^{-1}(\|\xi\|^\sigma\widetilde{\varphi})\|_{L_k^1(\mathbb{R}^d)}\|\Delta_j f\|_{L_k^p(\mathbb{R}^d)}$.

Moreover, if $W = \mathbb{Z}_2^d$, then each $T^\mu = T_1^{\mu_1} \circ \cdots \circ T_d^{\mu_d}$ satisfies

(4) $\|T^\mu \Delta_j f\|_{L_k^p(\mathbb{R}^d)} \le c2^{j|\mu|}\|T^\mu\widetilde{\phi}\|_{L_k^1(\mathbb{R}^d)}\|\Delta_j f\|_{L_k^p(\mathbb{R}^d)}$,

(5) $\|T^\mu S_j f\|_{L_k^p(\mathbb{R}^d)} \le c2^{j|\mu|}\|T^\mu\widetilde{\chi}\|_{L_k^1(\mathbb{R}^d)}\|S_j f\|_{L_k^p(\mathbb{R}^d)}$.

Proof. Proposition 2.3 implies that

$$S_j f = 2^{j(d+2\gamma)}\widetilde{\chi}(2^j\cdot) *_D S_j f, \quad \Delta_j f = 2^{j(d+2\gamma)}\widetilde{\phi}(2^j\cdot) *_D \Delta_j f. \tag{3.1}$$

Therefore, (1), (2), (3) follow from (2.3) and (4), (5) follow from (2.3) and (2.6). □

Lemma 3.2. *Assume that N is an integer such that $N > \gamma + \frac{d}{2}$ and that $\varrho \in L_k^2(\mathbb{R}^d)$ satisfies $T^\mu\varrho \in L_k^2(\mathbb{R}^d)$ for $|\mu| = N$. Then $\mathcal{F}_D^{-1}(\rho) \in L_k^1(\mathbb{R}^d)$ and*

$$\|\mathcal{F}_D^{-1}(\rho)\|_{L_k^1(\mathbb{R}^d)} \le C\|\rho\|_{L_k^2(\mathbb{R}^d)}^{1-\theta} \sup_{|\mu|=N} \|T^\mu\rho\|_{L_k^2(\mathbb{R}^d)}^{\theta},$$

where $\theta = \frac{d+2\gamma}{2N}$.

Proof. The proof is similar to the classical case (cf. [16]). □

Definition 3.2. For $s \in \mathbb{R}$, the operator $\mathcal{J}_k^s$ from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\mathcal{J}_k^s(f) = \mathcal{F}_D^{-1}((1 + \|\cdot\|^2)^{\frac{s}{2}}\mathcal{F}_D f).$$

We call $\mathcal{J}_k^{-s}$ the Dunkl-Bessel potential operator.

Proposition 3.3. *Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. If $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfies $\Delta_j f \in L_k^p(\mathbb{R}^d)$ for $j = -1, 0, 1, 2, \cdots$, then*

$$\|\mathcal{J}_k^s(\Delta_j f)\|_{L_k^p(\mathbb{R}^d)} \le C2^{sj}\|\Delta_j f\|_{L_k^p(\mathbb{R}^d)}, \tag{3.2}$$

where C is independent of p and j.

Proof. We note that for all $j = 0, 1, 2, \cdots$,

$$\Delta_j f = \sum_{l=-1}^{1} \Delta_{j+l}\Delta_j f = \sum_{l=-1}^{1} \phi_{j+l} *_D \Delta_j f,$$

where $\phi = \mathcal{F}_D^{-1}(\varphi)$ and $\phi_{j+l}(\xi) = 2^{(j+l)(d+2\gamma)}\phi(2^{j+l}\xi)$. This gives that

$$\mathcal{J}_k^s(\Delta_j f) = \sum_{l=-1}^{1} \mathcal{J}_k^s(\phi_{j+l}) *_D \Delta_j f.$$

Since the $L_k^2(\mathbb{R}^d)$-norms of $\mathcal{F}_D(\mathcal{J}_k^s(\phi_{j+l}))(\xi) = (1 + ||\xi||^2)^{\frac{s}{2}}\phi_{j+l}(\xi)$ and $2^{(j+l)s}\ (2^{-2(j+l)} + \|\xi\|^2)^{\frac{s}{2}}\varphi(\xi)$ are same, it follows from Lemma 3.2 that

$$\|\mathcal{J}_k^s(\phi_{j+l})\|_{L_k^1(\mathbb{R}^d)} \le C2^{js}, \quad l = 0, \pm 1. \tag{3.3}$$

Hence (3.2) follows from (2.3). The case of $j = -1$ is proved by the similar way. □

4. $B_{p,q}^{s,k}$, $F_{p,q}^{s,k}$, $H_{p,k}^s$ SPACES AND BASIC PROPERTIES

In this section we define analogues of the Besov, Tribel-Lizorkin and Bessel potential spaces associated with the Dunkl operators on $\mathbb{R}^d$ and obtain their basic properties. In particular, we use the Dunkl-Littlewood-Paley decomposition of f in $\mathcal{S}'(\mathbb{R}^d)$, obtained in the previous section, and apply the standard process used in the Euclidean case. Hence, we expect that, according to routine, we obtain analogous results in our Dunkl setting. However, we have some obstacles to carry out the Euclidean process, which are stated in Remarks 4.1 and 4.2 below.

4.1. **Definitions.** From now, we make the convention that for all non-negative sequence $\{a_q\}_{q\in\mathbb{Z}}$, the notation $(\sum_q a_q^r)^{\frac{1}{r}}$ stands for $\sup_q a_q$ in the case $r = \infty$. Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. For a sequence $\{u_j\}_{j=0,1,2,\cdots}$ of functions on $\mathbb{R}^d$, we define

$$\|\{u_j\}\|_{l_q^s(L_k^p)} = \|u_0\|_{L_k^p(\mathbb{R}^d)} + (\sum_{j>0}(2^{js}\|u_j\|_{L_k^p(\mathbb{R}^d)})^q)^{\frac{1}{q}},$$

$$\|\{u_j\}\|_{L_k^p(l_q^s)} = \|u_0\|_{L_k^p(\mathbb{R}^d)} + \|\sum_{j>0}(2^{js}|u_j(x)|)^q)^{\frac{1}{q}}\|_{L_k^p(\mathbb{R}^d)}.$$

Let Δ_j, $j = -1, 0, 1, 2, \cdots$, be the operators given in Definition 3.1. For convenience we replace the indices j by $j + 1$. That is, $\Delta_0 = S_0$, $\mathcal{F}_D(\Delta_j f) = \varphi(2^{-j+1}\xi)\mathcal{F}_D(f)$ and $f = \sum_{j=0}^{\infty} \Delta_j f$ in the sense of $\mathcal{S}'(\mathbb{R}^d)$.

Definition 4.1. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the inhomogeneous Dunkl-Besov space $B_{p,q}^{s,k}(\mathbb{R}^d)$ is defined by

$$B_{p,q}^{s,k}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) \mid \|f\|_{B_{p,q}^{s,k}(\mathbb{R}^d)} = \|\{\Delta_j f\}\|_{l_q^s(L_k^p)} < \infty\}.$$

Definition 4.2. Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the inhomogeneous Dunkl-Triebel-Lizorkin space $F_{p,q}^{s,k}(\mathbb{R}^d)$ is defined by

$$F_{p,q}^{s,k}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) \mid \|f\|_{F_{p,q}^{s,k}(\mathbb{R}^d)} = \|\{\Delta_j f\}\|_{L_k^p(l_q^s)} < \infty\}.$$

Definition 4.3. For $s \in \mathbb{R}$ and $1 \le p \le \infty$, the Dunkl-Bessel potential space $H_{p,k}^s(\mathbb{R}^d)$ is defined as the space $\mathcal{J}_k^{-s}(L_k^p(\mathbb{R}^d))$, equipped with the norm $\|f\|_{H_{p,k}^s(\mathbb{R}^d)} = \|\mathcal{J}_k^s(f)\|_{L_k^p(\mathbb{R}^d)}$.

Remark 4.1. We can define these spaces for $0 < p < 1$ in the same way. In order to study the case of $0 < p < 1$, (vector-valued) Hardy spaces are useful, that is, the theory of maximal operators is necessary. In our Dunkl setting, it is not accomplished generally, because of the difficulty arisen from the facts that an explicit formula for a generalized translation operator τ_y is unknown and τ_y is not a positive operator.

4.2. **Equivalent norms.** Let $f \in \mathcal{S}'(\mathbb{R}^d)$. We say that f has a general Dunkl-Littlewood-Paley decomposition if f is decomposed as $f = \sum_{j=0}^{\infty} u_j$, where each u_j is a functions on $\mathbb{R}^d$ satisfying

$$\text{supp}\mathcal{F}_D(u_0) \subset \{\xi \mid |\xi| \leq 1\},$$
$$\text{supp}\mathcal{F}_D(u_j) \subset \{\xi \mid 2^{j-2} \leq |\xi| \leq 2^j\},\ j = 1, 2, \cdots.$$

Obviously, the Dunkl-Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty} \Delta_j f$ is an example of the generalized decomposition.

Theorem 4.1. *(1) Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Then*

$$\|f\|_{B^{s,k}_{p,q}(\mathbb{R}^d)} \sim \inf \|\{u_j\}\|_{l^s_q(L^p_k)},$$

where the infimum is taken over all general Dunkl-Littlewood-Paley decompositions $f = \sum_{j\geq 0} u_j \in \mathcal{S}'(\mathbb{R}^d)$ with $\|\{u_j\}\|_{l^s_q(L^p_k)} < \infty$.

(2) Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then

$$\|f\|_{F^{s,k}_{p,2}(\mathbb{R}^d)} \sim \inf \|\{u_j\}\|_{L^p_k(l^s_2)},$$

where the infimum is taken over all general Dunkl-Littlewood-Paley decompositions $f = \sum_{j\geq 0} u_j \in \mathcal{S}'(\mathbb{R}^d)$ with $\|\{u_j\}\|_{L^p_k(l^s_2)} < \infty$.

Proof. Since $\mathcal{F}_D$ satisfies (2.2), we can apply the same argument used in the proof of Theorem 4.2.2 in [15]. We note that

$$\Delta_k f = \sum_{|l-k|\leq 2} \Delta_k u_l = \sum_{r=0,1,2} \Delta_k u_{k+r}.$$

Hence (1) follows from the inequality $\|\phi_k *_D u_{k+r}\|_{L^p_k(\mathbb{R}^d)} \leq c\|u_{k+r}\|_{L^p_k(\mathbb{R}^d)}$ for $1 \leq p \leq \infty$, where c is independent of k (see (2.3)). (2) follows from the inequality $\|\{\phi_k *_D u_{k+r}\}\|_{L^p_k(l^s_2)} \leq c\|\{u_{k+r}\}\|_{L^p_k(l^s_2)}$ for $1 < p < \infty$, where c is independent of k, which is obtained in Theorem 3.13 in [7]. □

Remark 4.2. In the Euclidean case, (2) holds for $F^{s,k}_{p,q}(\mathbb{R}^d)$ with $1 < q < \infty$, because the inequality $\|\{\phi_k *_D u_{k+r}\}\|_{L^p_k(l^s_q)} \leq c\|\{u_{k+r}\}\|_{L^p_k(l^s_q)}$ follows from the Hörmander multiplier theorem. However, in our Dunkl setting, we have no Hörmander type multiplier theorem. When $q = 2$, we can apply the Plancherel formula for the Dunkl transform $\mathcal{F}_D$ and thereby we can obtain (2).

Corollary 4.2. *Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of functions such that $\|\{u_j\}\|_{l^s_q(L^p_k)} < \infty$.*

(1) If supp$\mathcal{F}_D(u_j) \subset 2^j R$ for some annulus R centered at the origin, then $f = \sum_{j=0}^{\infty} u_j$ belongs to $B^{s,k}_{p,q}(\mathbb{R}^d)$ and there exists a positive constant $C(s)$ such that $\|f\|_{B^{s,k}_{p,q}(\mathbb{R}^d)} \leq C(s)\|\{u_j\}\|_{l^s_q(L^p_k)}$.

(2) If $s > 0$ and supp$\mathcal{F}_D(u_j) \subset 2^j B$ for some ball B centered at the origin, then $f = \sum_{j=0}^{\infty} u_j$ belongs to $B^{s,k}_{p,q}(\mathbb{R}^d)$ and there exists a positive constant $C(s)$ such that $\|f\|_{B^{s,k}_{p,q}(\mathbb{R}^d)} \leq C(s)\|\{u_j\}\|_{l^s_q(L^p_k)}$.

Proof. We can find an integer N such that $\Delta_k f = \sum_{|l-k|\leq N} \Delta_k u_l$ in the case of (1) and $\Delta_k f = \sum_{l\geq k-N} \Delta_k u_l$ in the case of (2). Hence (1) follows as in Theorem 4.1 (1) (see [7], Proposition 3.6) and (2) follows as in [7], Proposition 3.7. □

Corollary 4.3. *Let p, q be as above. The definitions of the spaces $B^{s,k}_{p,q}(\mathbb{R}^d)$ and $F^{s,k}_{p,2}(\mathbb{R}^d)$ do not depend on the choice of the couple (φ, ψ) defining the Dunkl-Littlewood-Paley decomposition.*

In the following, we denote by $\widetilde{F}^{s,k}_{p,q}(\mathbb{R}^d)$ the space of all $f \in \mathcal{S}'(\mathbb{R}^d)$ which has a general Dunkl-Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty} u_j$ with $\|\{u_j\}\|_{L^p_k(l^s_q)} < \infty$. Clearly, $F^{s,k}_{p,q}(\mathbb{R}^d) \subset \widetilde{F}^{s,k}_{p,q}(\mathbb{R}^d)$ and $F^{s,k}_{p,2}(\mathbb{R}^d) = \widetilde{F}^{s,k}_{p,2}(\mathbb{R}^d)$ by Theorem 4.1 (2).

Theorem 4.4. *Let $1 < p < \infty$ and $s \in \mathbb{R}$, we have*

$$F^{s,k}_{p,2}(\mathbb{R}^d) = H^s_{p,k}(\mathbb{R}^d).$$

Proof. Because of $F^{s,k}_{p,2}(\mathbb{R}^d) = \widetilde{F}^{s,k}_{p,2}(\mathbb{R}^d)$, it is enough to show that $\widetilde{F}^{s,k}_{p,2}(\mathbb{R}^d) = H^s_{p,k}(\mathbb{R}^d)$. When $s = 0$, this is nothing but a theorem of Littlewood-Paley type. The general case of $s \neq 0$ follows form the lifting property (see Theorem 4.7 below). □

Corollary 4.5. *Let $s \in \mathbb{N}$ and $1 < p < \infty$ then*

$$F^{s,k}_{p,2}(\mathbb{R}^d) = W^{s,p}_k(\mathbb{R}^d),$$

where $W^{s,p}_k(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) \mid T^\mu u \in L^p_k(\mathbb{R}^d)$ for all $\mu \in \mathbb{N}^d$ with $|\mu| = s\}$.

4.3. **Lifting property.** We recall that for $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$\mathcal{F}_D(T_j\Delta_n f)(\xi) = 2^n\tilde{\phi}(2^{-n}\xi)\mathcal{F}_D(f)(\xi), \quad \tilde{\phi}(\xi) = i\xi_j\phi(\xi).$$

Then we can obtain

Theorem 4.6. *Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. The operator T_j is a linear continuous operator from $B^{s,k}_{p,q}(\mathbb{R}^d)$ into $B^{s-1,k}_{p,q}(\mathbb{R}^d)$, from $\widetilde{F}^{s,k}_{p,q}(\mathbb{R}^d)$ into $\widetilde{F}^{s-1,k}_{p,q}(\mathbb{R}^d)$, and from $H^{s,k}_{p,q}(\mathbb{R}^d)$ into $H^{s-1,k}_{p,q}(\mathbb{R}^d)$.*

Similarly, we recall that $\mathcal{J}^t_k$, $t \in \mathbb{R}$, is a linear continuous injective operator from $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$ and is extended to a linear continuous operator from $\mathcal{S}'(\mathbb{R}^d)$ onto $\mathcal{S}'(\mathbb{R}^d)$ with $(\mathcal{J}^t_k)^{-1} = \mathcal{J}^{-t}_k$.

Theorem 4.7. *Let $s, t \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. The operator $\mathcal{J}^t_k$ is a linear continuous injective operator from $B^{s,k}_{p,q}(\mathbb{R}^d)$ onto $B^{s-t,k}_{p,q}(\mathbb{R}^d)$, from $\widetilde{F}^{s,k}_{p,q}(\mathbb{R}^d)$ onto $\widetilde{F}^{s-t,k}_{p,q}(\mathbb{R}^d)$, and from $H^{s,k}_{p,q}(\mathbb{R}^d)$ onto $H^{s-t,k}_{p,q}(\mathbb{R}^d)$.*

Proof. Since $\mathcal{F}_D$ satisfies (2.2), we can apply the same arguments used in the proof of Theorem 5.1.1 in [15]. □

4.4. **Embeddings.** As in the Euclidean case (see [15], 5.2), the monotone character of l_q-spaces and Minkowski's inequality yield the following.

Theorem 4.8. *(1) If $s_1 < s_2$ and $1 \leq p, q \leq \infty$, then*

$$B^{s_2,k}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s_1,k}_{p,q}(\mathbb{R}^d),$$
$$F^{s_2,k}_{p,q}(\mathbb{R}^d) \hookrightarrow F^{s_1,k}_{p,q}(\mathbb{R}^d).$$

(2) If $s \in \mathbb{R}, 1 \leq p \leq \infty$ and $1 \leq q_1 < q_2 \leq \infty$, then

$$B^{s,k}_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^{s,k}_{p,q_2}(\mathbb{R}^d),$$
$$F^{s,k}_{p,q_1}(\mathbb{R}^d) \hookrightarrow F^{s,k}_{p,q_2}(\mathbb{R}^d).$$

(3) For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, let $r = \min\{p, q\}, t = \max\{p, q\}$. Then

$$B^{s,k}_{p,r}(\mathbb{R}^d) \hookrightarrow F^{s,k}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s,k}_{p,t}(\mathbb{R}^d). \tag{4.1}$$

As in [15], §6, we can obtain

Theorem 4.9. *Let $s \in \mathbb{R}$ and $1 \le p, q < \infty$. Then $D(\mathbb{R}^d)$ is dense in $B^{s,k}_{p,q}(\mathbb{R}^d)$ and $F^{s,k}_{p,q}(\mathbb{R}^d)$.*

4.5. **Duality.** In the Euclidean case we see that $(B^s_{p,q}(\mathbb{R}^d))' = B^{-s}_{p',q'}(\mathbb{R}^d)$ and $(F^s_{p,q}(\mathbb{R}^d))' = F^{-s}_{p',q'}(\mathbb{R}^d)$, where p', q' are conjugate numbers of p, q respectively (see [15], §7). For the $B^{s,k}_{p,q}$-spaces, we can apply the same argument used in [15], §7. However, we can not do for the $F^{s,k}_{p,q}$-spaces, because Hörmander's type multiplier theorem is used in the Euclidean case (see Remark 4.2). For the $H^s_{p,k}$-space, the duality follows from the one of $L^p_k(\mathbb{R}^d)$.

Theorem 4.10. *(1) If $s \in \mathbb{R}$ and $1 \le p < \infty$, then*

$$(H^s_{p,k}(\mathbb{R}^d))' = H^{-s}_{p',k}(\mathbb{R}^d).$$

(2) If $s \in \mathbb{R}$ and $1 \le p, q < \infty$, then

$$(B^{s,k}_{p,q}(\mathbb{R}^d))' = B^{-s,k}_{p',q'}(\mathbb{R}^d).$$

4.6. **Interpolation.** We can apply the real method used in [15], §8. In this process, the duality is used frequently. In our Dunkl setting, as shown in Theorem 4.10 the duality holds only for $B^{s,k}_{p,q}$-spaces and $H^s_{p,k}$-spaces. Hence, we have the following.

Theorem 4.11. *(1) Let $s_0, s_1 \in \mathbb{R}$, $s_0 \ne s_1$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $1 \le p, q, q_0, q_1 \le \infty$. Then*

$$(B^{s_0,k}_{p,q_0}(\mathbb{R}^d), B^{s_1,k}_{p,q_1}(\mathbb{R}^d))_{\theta,q} = B^{s,k}_{p,q}(\mathbb{R}^d).$$

(2) Let $s \in \mathbb{R}$, $1 \le p_0, p_1 \le \infty$, $p_0 \ne p_1$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then

$$(F^{s,k}_{p_0,2}(\mathbb{R}^d), F^{s,k}_{p_1,2}(\mathbb{R}^d))_{\theta,p} = F^{s,k}_{p,2}(\mathbb{R}^d).$$

(3) Let $s_0, s_1 \in \mathbb{R}$, $s_0 \ne s_1$, $1 \le p_0, p_1 \le \infty$, $p_0 \ne p_1$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then

$$(F^{s_0,k}_{p_0,2}(\mathbb{R}^d), F^{s_1,k}_{p_1,2}(\mathbb{R}^d))_{\theta,p} = B^{s,k}_{p,p}(\mathbb{R}^d).$$

(4) Let $s_0, s_1 \in \mathbb{R}$, $s_0 \ne s_1$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $1 \le p, q, q_0, q_1 \le \infty$. Then

$$(F^{s_0,k}_{p,q_0}(\mathbb{R}^d), F^{s_1,k}_{p,q_1}(\mathbb{R}^d))_{\theta,q} = B^{s,k}_{p,q}(\mathbb{R}^d). \tag{4.2}$$

Proof. (1), (2), (3) follows from the arguments used in Theorem 8.1.3 and Theorem 8.3.3 in [15]. (4) follows from (1) and (4.1). □

As a consequence of real and complex interpolations, we can deduce multiplicative inequalities, which will be needed in the theory of differential operators.

Theorem 4.12. *(1) If u belongs to $B^{s,k}_{p,q}(\mathbb{R}^d) \cap B^{t,k}_{p,q}(\mathbb{R}^d)$, then u belongs to $B^{\theta s+(1-\theta)t,k}_{p,q}(\mathbb{R}^d)$ for all $\theta \in [0,1]$ and*

$$\|u\|_{B^{\theta s+(1-\theta)t,k}_{p,q}(\mathbb{R}^d)} \le \|u\|^{\theta}_{B^{s,k}_{p,q}(\mathbb{R}^d)} \|u\|^{1-\theta}_{B^{t,k}_{p,q}(\mathbb{R}^d)}.$$

(2) If u belongs to $B^{s,k}_{p,\infty}(\mathbb{R}^d) \cap B^{t,k}_{p,\infty}(\mathbb{R}^d)$ and $s < t$, then u belongs to $B^{\theta s+(1-\theta)t,k}_{p,1}(\mathbb{R}^d)$ for all $\theta \in (0,1)$ and there exists a positive constant $C(t,s)$ such that

$$\|u\|_{B^{\theta s+(1-\theta)t,k}_{p,1}(\mathbb{R}^d)} \le C(t,s) \|u\|^{\theta}_{B^{s,k}_{p,\infty}(\mathbb{R}^d)} \|u\|^{1-\theta}_{B^{t,k}_{p,\infty}(\mathbb{R}^d)}.$$

(3) If u belongs to $B^{s,k}_{p,\infty}(\mathbb{R}^d)\cap B^{s+\varepsilon,k}_{p,\infty}(\mathbb{R}^d)$ and $\varepsilon>0$, then u belongs to $B^{s,k}_{p,1}(\mathbb{R}^d)$ and there exists a positive constant C such that

$$\|u\|_{B^{s,k}_{p,1}(\mathbb{R}^d)} \le \frac{C}{\varepsilon}\|u\|_{B^{s,k}_{p,\infty}(\mathbb{R}^d)} \log_2\Big(e+\frac{\|u\|_{B^{s+\varepsilon,k}_{p,\infty}(\mathbb{R}^d)}}{\|u\|_{B^{s,k}_{p,\infty}(\mathbb{R}^d)}}\Big).$$

Proof. (1) is obvious from Hölder's inequality. As for (2), we write $\|u\|_{B^{\theta s+(1-\theta)t,k}_{p,1}(\mathbb{R}^d)}$ as

$$\sum_{j\le N} 2^{j(\theta s+(1-\theta)t)}\|\Delta_j u\|_{L^p_k(\mathbb{R}^d)} + \sum_{j>N} 2^{j(\theta s+(1-\theta)t)}\|\Delta_j u\|_{L^p_k(\mathbb{R}^d)},$$

where N is chosen here after. By the definition of the Dunkl-Besov norms, we see that

$$2^{j(\theta s+(1-\theta)t)}\|\Delta_j u\|_{L^p_k(\mathbb{R}^d)} \le 2^{j(1-\theta)(t-s)}\|u\|_{B^{s,k}_{p,\infty}(\mathbb{R}^d)},$$
$$2^{j(\theta s+(1-\theta)t)}\|\Delta_j u\|_{L^p_k(\mathbb{R}^d)} \le 2^{-j\theta(t-s)}\|u\|_{B^{t,k}_{p,\infty}(\mathbb{R}^d)}$$

and thus, $\|u\|_{B^{\theta s+(1-\theta)t,k}_{p,1}(\mathbb{R}^d)}$ is dominated by

$$\|u\|_{B^{s,k}_{p,\infty}(\mathbb{R}^d)}\sum_{j\le N}2^{j(1-\theta)(t-s)} + \|u\|_{B^{t,k}_{p,\infty}(\mathbb{R}^d)}\sum_{j>N}2^{-j\theta(t-s)}$$
$$\le C\|u\|_{B^{s,k}_{p,\infty}(\mathbb{R}^d)}\frac{2^{(N+1)(1-\theta)(t-s)}}{2^{(1-\theta)(t-s)}-1} + \|u\|_{B^{t,k}_{p,\infty}(\mathbb{R}^d)}\frac{2^{-N\theta(t-s)}}{1-2^{-\theta(t-s)}}.$$

Hence, in order to complete the proof of (2), it suffices to choose N such that

$$\frac{\|u\|_{B^{t,k}_{p,\infty}(\mathbb{R}^d)}}{\|u\|_{B^{s,k}_{p,\infty}(\mathbb{R}^d)}} \le 2^{N(t-s)} < 2\frac{\|u\|_{B^{t,k}_{p,\infty}(\mathbb{R}^d)}}{\|u\|_{B^{s,k}_{p,\infty}(\mathbb{R}^d)}}.$$

As for (3), it is easy to see that $\|u\|_{B^{s,k}_{p,1}(\mathbb{R}^d)}$ is dominated as

$$\sum_{j\le N-1}2^{js}\|\Delta_j u\|_{L^p_k(\mathbb{R}^d)} + \sum_{j\ge N}2^{js}\|\Delta_j u\|_{L^p_k(\mathbb{R}^d)}$$
$$\le (N+1)\|u\|_{B^{s,k}_{p,\infty}(\mathbb{R}^d)} + \frac{2^{-(N-1)\varepsilon}}{2^{\varepsilon}-1}\|u\|_{B^{s+\varepsilon,k}_{p,\infty}(\mathbb{R}^d)}.$$

Hence, letting

$$N = 1+\Big[\frac{1}{\varepsilon}\log_2\frac{\|u\|_{B^{s+\varepsilon,k}_{p,\infty}(\mathbb{R}^d)}}{\|u\|_{B^{s,k}_{p,\infty}(\mathbb{R}^d)}}\Big],$$

we can obtain the desired estimate. □

5. Some properties related with the index

We continue to study the $B^{k,s}_{p,q}$ spaces. The results obtained in the previous section are exactly same as in the Euclidean case. In this section we obtain some properties related with the index γ.

5.1. **Embeddings.**

Theorem 5.1. *If $s_0, s_1\in\mathbb{R}$, $s_1\le s_0$, $1\le p\le p_1\le\infty$, $1\le q\le q_1\le\infty$, $s_0-\frac{d+2\gamma}{p} = s_1-\frac{d+2\gamma}{p_1}$, then*

$$B^{s,k}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s_1,k}_{p_1,q_1}(\mathbb{R}^d).$$

Proof. In order to prove the inclusion, we use the identities

$$\Delta_j f = \widetilde{\phi}_j *_D \Delta_j f,\ j = 1, 2, \cdots, \quad \Delta_0 f = \widetilde{\chi} *_D \Delta_0 f.$$

Then the Bernstein inequality (Proposition 3.1 (1)) gives that for $j = 0, 1, 2, \cdots$,

$$\|\Delta_j f\|_{L_k^{p_1}(\mathbb{R}^d)} \leq C 2^{j(d+2\gamma)(\frac{1}{p}-\frac{1}{p_1})} \|\Delta_j f\|_{L_k^p(\mathbb{R}^d)}.$$

Thus, by definition of the inhomogeneous Dunkl-Besov spaces, we see that

$$\begin{aligned}\|f\|_{B^{s_1,k}_{p_1,q_1}(\mathbb{R}^d)} &\leq C(\|\Delta_0 f\|_{L_k^p(\mathbb{R}^d} + (\sum_{j\in\mathbb{N}} (2^{js_1} 2^{j(d+2\gamma)(\frac{1}{p}-\frac{1}{p_1})} \|\Delta_j f\|_{L_k^p(\mathbb{R}^d)})^{q_1})^{\frac{1}{q_1}} \\ &\leq C(\|\Delta_0 f\|_{L_k^p(\mathbb{R}^d)} + (\sum_{j\in\mathbb{N}} (2^{js} \|\Delta_j f\|_{L_k^p(\mathbb{R}^d)})^{q_1})^{\frac{1}{q_1}} \\ &\leq C\|f\|_{B^{s,k}_{p,q}(\mathbb{R}^d)},\end{aligned}$$

because $q \leq q_1$. □

Proposition 5.2. *If $1 < p < \infty$, then $B^{\frac{d+2\gamma}{p},k}_{p,1}(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$ and $B^{0,k}_{\infty,1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)$.*

Proof. To prove that $B^{\frac{d+2\gamma}{p},k}_{\infty,1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)$ for $1 < p \leq \infty$, we use again Bernstein inequalities (see Proposition 3.1) to deduce that

$$\|\Delta_j u\|_{L_k^\infty(\mathbb{R}^d)} \leq C 2^{j\frac{d+2\gamma}{p}} \|\Delta_j u\|_{L_k^p(\mathbb{R}^d)}.$$

This ensures that the series $\sum_j \Delta_j u$ of continuous bounded functions converges uniformly to a continuous bounded function on $\mathbb{R}^d$. Hence u is a bounded continuous function on $\mathbb{R}^d$. If p is finite, one can use in addition that $D(\mathbb{R}^d)$ is dense in $B^{\frac{d+2\gamma}{p},k}_{p,1}(\mathbb{R}^d)$ (see Theorem 4.9). Then we can conclude that u decays at infinity. □

5.2. **Sobolev type embedding.** In the previous paper [7], Theorem 4.3, the second author proved the Sobolev embedding theorem; if $s > \frac{2\gamma+d}{2}$, then

$$B^{s,k}_{2,2}(\mathbb{R}^d) = H^s_{2,k}(\mathbb{R}^d) \hookrightarrow B^{s-\gamma-\frac{d}{2},k}_{\infty,\infty}(\mathbb{R}^d).$$

In this subsection we consider the case $s < \frac{2\gamma+d}{2}$. We recall that $B^{0,k}_{1,1}(\mathbb{R}^d) \subset L^1_k(\mathbb{R}^d)$ by the definition and $B^{s,k}_{r,r}(\mathbb{R}^d) \hookrightarrow F^{s,k}_{r,2}(\mathbb{R}^d) = H^s_{r,k}(\mathbb{R}^d) \hookrightarrow L^r_k(\mathbb{R}^d)$ for $1 < r \leq 2$ by (4.1). We here obtain a stronger integrability in the case of $0 < s < \frac{2\gamma+d}{r}$.

Theorem 5.3. *If $s \in \mathbb{R}$, $1 \leq r \leq \infty$, $0 < s < \frac{2\gamma+d}{r}$, then we have a continuous embedding*

$$B^{s,k}_{r,r}(\mathbb{R}^d) \hookrightarrow L^p_k(\mathbb{R}^d),$$

where $p = \dfrac{r(2\gamma+d)}{2\gamma+d-rs}$.

Proof. We recall that for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|f\|^p_{L^p_k(\mathbb{R}^d)} = p\int_0^\infty \lambda^{p-1} m_k(\{x \mid |f(x)| \geq \lambda\}) d\lambda,$$

where $m_k(E)$ is the volume of $E \subset \mathbb{R}^d$ with respect $\omega_k(x)dx$. For $A > 0$, we put $f = f_{1,A} + f_{2,A}$ with $f_{1,A} = \sum_{2^j<A} \triangle_j f$, and $f_{2,A} = \sum_{2^j\geq A} \triangle_j f$. Then by using Proposition

3.1 we deduce that

$$\|f_{1,A}\|_{L^\infty_k(\mathbb{R}^d)} \leq \sum_{2^j<A} 2^{js}\|\triangle_j f\|_{L^r_k(\mathbb{R}^d)} 2^{j(\frac{d+2\gamma}{r}-s)}$$

$$\leq CA^{\frac{d+2\gamma}{r}-s}\|f\|_{B^{s,k}_{r,r}(\mathbb{R}^d)}. \tag{5.1}$$

We take now $A = A_\lambda$ such that $CA_\lambda^{\frac{d+2\gamma}{r}-s}\|f\|_{B^{s,k}_{r,r}(\mathbb{R}^d)} = \frac{\lambda}{4}$. Then for all $\lambda > 0$, we see that

$$\begin{aligned} m_k(\{x \mid |f(x)| \geq \lambda\}) &\leq m_k(\{x \mid |f_{1,A}(x)| \geq \tfrac{\lambda}{2}\}) + m_k(\{x \mid |f_{2,A}(x)| \geq \tfrac{\lambda}{2}\}) \\ &\leq m_k(\{x \mid |f_{2,A_\lambda}(x)| \geq \tfrac{\lambda}{2}\}) \\ &\leq 2^r\lambda^{-r}\|f_{2,A_\lambda}\|^r_{L^r_k(\mathbb{R}^d)} \end{aligned}$$

and moreover, for $\varepsilon > 0$,

$$\begin{aligned} \|f_{2,A_\lambda}\|^r_{L^r_k(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \Big|\sum_{2^j>A} \triangle_j f(x)\Big|^r \omega_k(x)dx \\ &= \int_{\mathbb{R}^d} \sum_{2^j>A} \Big|2^{j\varepsilon}\triangle_j f(x)\Big|^r \omega_k(x)dx \cdot (\sum_{2^j>A} 2^{-j\varepsilon r'})^{\frac{r}{r'}} \\ &\leq cA_\lambda^{-\varepsilon r} \sum_{2^j\geq A_\lambda} 2^{j\varepsilon r}\|\triangle_j f\|^r_{L^r_k(\mathbb{R}^d)}. \end{aligned}$$

Hence by Fubini's theorem we can deduce that

$$\begin{aligned} \|f\|^p_{L^p_k(\mathbb{R}^d)} &\leq c\int_0^\infty \lambda^{p-1-r}A_\lambda^{-\varepsilon r}\sum_{2^j\geq A_\lambda} 2^{j\varepsilon r}\|\triangle_j f\|^r_{L^r_k(\mathbb{R}^d)}d\lambda \\ &\leq c\sum_{j\geq -1}\int_0^{4c2^{j(\frac{2\gamma+d}{r}-s)}\|f\|_{B^{s,k}_{r,r}(\mathbb{R}^d)}} \lambda^{p-r-1-\frac{\varepsilon r^2}{2\gamma+d-rs}}d\lambda \\ &\qquad \times (4c\|f\|_{B^{s,k}_{r,r}(\mathbb{R}^d)})^{\frac{\varepsilon r^2}{2\gamma+d-rs}} 2^{j\varepsilon r}\|\triangle_j f\|^r_{L^r_k(\mathbb{R}^d)} \\ &\leq c\|f\|^{p-r}_{B^{s,k}_{r,r}(\mathbb{R}^d)}\sum_{j\geq -1} 2^{j(p-r)(\frac{2\gamma+d-rs}{r})}\|\triangle_j f\|^r_{L^r_k(\mathbb{R}^d)} \\ &\leq c\|f\|^{p-r}_{B^{s,k}_{r,r}(\mathbb{R}^d)}\sum_{j\geq -1} 2^{rjs}\|\triangle_j f\|^r_{L^r_k(\mathbb{R}^d)} = c\|f\|^p_{B^{s,k}_{r,r}(\mathbb{R}^d)} \end{aligned}$$

This implies the desired result. □

Theorem 5.4. *If $1 \leq r \leq \infty$ and $s \in \mathbb{R}$ such that $0 < s < \frac{2\gamma+d}{r}$, then we have*

$$\|f\|_{L^p_k(\mathbb{R}^d)} \leq C\|f\|^{1-\frac{r}{p}}_{B^{-(\frac{2\gamma+d}{r}-s),k}_{\infty,\infty}(\mathbb{R}^d)}\|f\|^{\frac{r}{p}}_{B^{s,k}_{r,r}(\mathbb{R}^d)},$$

where $p = \dfrac{r(2\gamma+d)}{2\gamma+d-rs}$.

Proof. The precedent proof is available. In fact it suffices to modify the calculation in (5.1) by

$$\|f_{1,A}\|_{L^\infty_k(\mathbb{R}^d)} \leq CA^{\frac{d+2\gamma}{r}-s}\|f\|_{B^{-(\frac{2\gamma+d}{r}-s),k}_{\infty,\infty}(\mathbb{R}^d)}$$

and by taking $A = A_\lambda$ with $CA_\lambda^{\frac{d+2\gamma}{r}-s}\|f\|_{B^{-(\frac{2\gamma+d}{r}-s),k}_{\infty,\infty}(\mathbb{R}^d)} = \frac{\lambda}{4}$. □

5.3. **Paraproduct algorithm.** In this subsection we study how the product of uv, $u, v \in \mathcal{S}'(\mathbb{R})$ acts on Dunkl-Besov spaces. This could be well useful in nonlinear partial differential-difference equations. Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $u = \sum_p \triangle_p u$ be the Dunkl-Littlewood-Paley decomposition of u. This implies that the partial sum

$$S_q u = \sum_{p \leq q-1} \triangle_p u$$

converges to $u \in \mathcal{S}'(\mathbb{R}^d)$. Let us consider two tempered distributions

$$u = \sum_p \Delta_p u \quad \text{and} \quad v = \sum_q \Delta_q v.$$

Formally, the product uv can be written as

$$uv = \sum_{p,q} \Delta_p u \Delta_q v.$$

We introduce the paraproduct and the remainder operators associated with the Dunkl operators.

Definition 5.1. (1) The paraproduct operator $\Pi(u, v)$: $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is defined by $\Pi(u, v) = \Pi_u(v)$ and

$$\Pi_u v = \sum_{q \geq 1} S_{q-2} u \cdot \Delta_q v.$$

(2) The remainder operator $R(u, v) : \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is defined by

$$R(u, v) = \sum_{|p-q| \leq 1} \Delta_p u \Delta_q v, \text{ for all.}$$

Then Bony's paraproduct decomposition of uv is given as

$$uv = \Pi_u v + \Pi_v u + R(u, v).$$

The following theorems describe paraproduct estimates in the Dunkl-Besov spaces, that is, the estimates of the action of the paraproduct and remainder operators on the Dunkl-Besov spaces. Their proofs are given by using the equivalent norms of the Dunkl-Besov spaces and Bernstein's estimates in (3.1) (see [7]).

Theorem 5.5. *Let $1 \leq p, r \leq \infty$ and $s \in \mathbb{R}$.*

(i) If $s > 0$, then Π is a bilinear continuous from $L^\infty_k(\mathbb{R}^d) \times B^{s,k}_{p,r}(\mathbb{R}^d)$ to $B^{s,k}_{p,r}(\mathbb{R}^d)$ and there exists a positive constant C such that

$$\|\Pi\|_{\mathcal{L}(L^\infty_k(\mathbb{R}^d) \times B^{s,k}_{p,r}(\mathbb{R}^d), B^{s,k}_{p,r}(\mathbb{R}^d))} \leq C^{s+1}.$$

(ii) If $s > 0, t < 0, s + t > 0$ and $1 \leq r, r_1, r_2 \leq \infty$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, then Π is a bilinear continuous from $B^{t,k}_{\infty,r_1}(\mathbb{R}^d) \times B^{s,k}_{p,r_2}(\mathbb{R}^d)$ to $B^{s+t,k}_{p,r}(\mathbb{R}^d)$ and there exists a positive constant C such that

$$\|\Pi\|_{\mathcal{L}(B^{t,k}_{\infty,r_1}(\mathbb{R}^d) \times B^{s,k}_{p,r_2}(\mathbb{R}^d), B^{s+t,k}_{p,r}(\mathbb{R}^d))} \leq \frac{C^{s+t}}{-t}.$$

Theorem 5.6. (**Morse type estimate**) *Let $(s_1, s_2) \in \mathbb{R}^2$ and $1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$. Assume that*

$$\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{r} \leq \frac{1}{r_1} + \frac{1}{r_2} \leq 1 \quad \text{and} \quad s_1 + s_2 > (d + 2\gamma)(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}).$$

Then R is a bilinear continuous from $B^{s_1,k}_{p_1,r_1}(\mathbb{R}^d)\times B^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)$ to $B^{s_{1,2},k}_{p,r}(\mathbb{R}^d)$ and there exists a positive constant C such that

$$\|R\|_{\mathcal{L}(B^{s_1,k}_{p_1,r_1}(\mathbb{R}^d)\times, B^{s_2,k}_{p_2,r_2}(\mathbb{R}^d), B^{s_{1,2},k}_{p,r}(\mathbb{R}^d))} \leq \frac{C^{s_1+s_2+1}}{s_1+s_2},$$

where $s_{1,2} = s_1+s_2-(d+2\gamma)(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})$.

Combining these estimates of the paraproducts and the remainders, we can deduce the following.

Corollary 5.7. *(1) Let $s>0$ and $1\leq p,r\leq\infty$. Then $B^{s,k}_{p,r}(\mathbb{R}^d)\cap L^\infty_k(\mathbb{R}^d)$ is an algebra and there exists a positive constant C such that*

$$\|uv\|_{B^{s,k}_{p,r}(\mathbb{R}^d)} \leq C\Big(\|u\|_{L^\infty_k(\mathbb{R}^d)}\|v\|_{B^{s,k}_{p,r}(\mathbb{R}^d)} + \|u\|_{B^{s,k}_{p,r}(\mathbb{R}^d)}\|v\|_{L^\infty_k(\mathbb{R}^d)}\Big).$$

(2) Let $(s_1,s_2)\in\mathbb{R}^2$, $1\leq p_2,r_2\leq\infty$, $s_1+s_2>\frac{d+2\gamma}{p_1}$ and $s_1<\frac{d+2\gamma}{p_1}$. Then

$$\|uv\|_{B^{s,k}_{p_2,r_2}(\mathbb{R}^d)} \leq C\Big(\|u\|_{B^{s_1,k}_{p_1,\infty}(\mathbb{R}^d)}\|v\|_{B^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)} + \|u\|_{B^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)}\|v\|_{B^{s_1,k}_{p_1,\infty}(\mathbb{R}^d)}\Big),$$

where $s = s_1+s_2-\frac{d+2\gamma}{p_1}$.

(3) Let $(s_1,s_2)\in\mathbb{R}^2$, $1\leq p_1,p_2,p,r_1,r_2\leq\infty$, $p\geq\max(p_1,p_2)$, $s_j<\frac{d+2\gamma}{p_j}$ and $s_1+s_2>(d+2\gamma)(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})$. Then

$$\|uv\|_{B^{s_{1,2},k}_{p,r}(\mathbb{R}^d)} \leq C\|u\|_{B^{s_1,k}_{p_1,r_1}(\mathbb{R}^d)}\|v\|_{B^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)},$$

where $s_{1,2}=s_1+s_2-(d+2\gamma)(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})$ and $r=\max(r_1,r_2)$.

6. Application to differential-difference equations

In this section we treat differential-difference equations, given by replacing the Laplacian Δ in a differential equation with the Dunkl-Laplacian Δ_k, and consider some basic properties of the solutions in Dunkl-Besov spaces. Though the process is a standard way, we sketch their proofs to understand the essential parts.

6.1. **The slowly hypoellipticity.** We consider the linear equation

$$-\triangle_k u + \sum_{1\leq i,j\leq d} c_{i,j}T_iuT_ju + cu = 0 \tag{6.1}$$

with $c_{i,j}\in\mathbb{R}$ and $c>0$.

Theorem 6.1. *If u is a solution of (6.1) such that u in $B^{1,k}_{1,2}(\mathbb{R}^d)\cap W^{1,\infty}_k(\mathbb{R}^d)$, then $u\in B^{n,k}_{1,2}(\mathbb{R}^d)\cap L^\infty_k(\mathbb{R}^d)$ for all $n\in\mathbb{N}$ and in particular, $u\in\mathcal{E}(\mathbb{R}^d)$.*

Proof. If u in $B^{1,k}_{1,2}(\mathbb{R}^d)$, then each $T_iu\in B^{0,k}_{1,2}(\mathbb{R}^d)$. Therefore, it follows from Corollary 5.7, (1) that $c_{i,j}T_iuT_ju\in B^{0,k}_{1,2}(\mathbb{R}^d)\cap L^\infty_k(\mathbb{R}^d)$. Hence, we can deduce that

$$-\triangle_k u + cu \in B^{0,k}_{1,2}(\mathbb{R}^d).$$

Since the operator $-\triangle_k + cI$ is isomorphism from $B^{s,k}_{p,q}(\mathbb{R}^d)$ in $B^{s-2,k}_{p,q}(\mathbb{R}^d)$ for all $s\in\mathbb{R}$ and $(p,r\in[1,\infty]^2)$, it follows that $u\in B^{2,k}_{1,2}(\mathbb{R}^d)$. By iteration we deduce that $u\in B^{n,k}_{1,2}(\mathbb{R}^d)$ for all $n\in\mathbb{N}$. Then it follows from the Theorem 5.1 that $u\in B^{n-\frac{d+2\gamma}{2},k}_{2,2}(\mathbb{R}^d)$. On the other hand, the Sobolev imbedding theorem (see [7], Theorem 4.3) yields that $H^s_{2,k}(\mathbb{R}^d) = B^{s,k}_{2,2}(\mathbb{R}^d)\hookrightarrow C^{s-\gamma-\frac{d}{2}}(\mathbb{R}^d)$ if $s>\gamma+\dfrac{d}{2}$. Thereby, the desired result follows. $\square$

6.2. **Dunkl-Schrödinger equation.** Let I be an interval of $\mathbb{R}$ (bounded or unbounded). We shall consider a space-time estimate of the solutions $u(t,x)$, $(t,x) \in I \times \mathbb{R}^d$, of the Dunkl-Schrodinger equation

$$\begin{cases} \partial_t u - i\triangle_k u = f, \\ u|_{t=0} = g \end{cases} \tag{6.2}$$

with initial data g and f. For any Banach space X, let $L^q(I,X)$ denote a mixed space-time Banach space consisting of measurable functions $u : I \to X$ such that

$$\|u\|_{L^q(I,X)} = \Big(\int_I \|u(t,\cdot)\|_X^q dt \Big)^{\frac{1}{q}} < \infty$$

if $1 \leq q < \infty$ and $\|u\|_{L^\infty(I,X)} = ess\sup_{t\in I} \|u(t,\cdot)\|_X < \infty$ if $q = \infty$. In what follows we shall consider a Strichartz type estimate of the solution u of (6.2) and obtain the $L^q(I,X)$-norm of u when $X = H^s_{r,k}(\mathbb{R}^d)$ and $B^{s,k}_{r,2}(\mathbb{R}^d)$. The special case of $X = L^r_k(\mathbb{R}^d) = H^0_{r,k}(\mathbb{R}^d)$ was treated in [8].

We suppose that $g \in X$ and $f \in L^q(I,X')$ where X, X' are $H^s_{r,k}(\mathbb{R}^d)$ and $B^{s,k}_{r,2}(\mathbb{R}^d)$. As in the Euclidean case, we use the integral formulation of u

$$\begin{aligned} u(t,x) &= \mathcal{I}_k(t)g(x) + \int_0^t \mathcal{I}_k(t-s)f(s,x)ds, \\ &= \mathcal{I}_k(g)(t,x) + \Phi_k(f)(t,x), \end{aligned} \tag{6.3}$$

where $\mathcal{I}_k(t) = e^{it\triangle_k}$, $t \in \mathbb{R}$, is the Schrödinger semi-group. Moreover, the exponents q, r are required to satisfy the so-called admissible condition:

Definition 6.1. A pair (q,r) is called $\gamma + \frac{d}{2}$-admissible if $q, r \geq 2$, $(q, r, \gamma + \frac{d}{2}) \neq (2, \infty, 1)$ and

$$\frac{1}{q} + \frac{d+2\gamma}{2r} \leq \frac{d+2\gamma}{4}.$$

In particular, when $d + 2\gamma > 2$ and $(q,r) = (2, \frac{2d+4\gamma}{d+2\gamma-2})$, the equality holds.

Theorem 6.2. (Strichartz type estimate)
(1) Let $s \in \mathbb{R}$ and (q,r) be a $\gamma + \frac{d}{2}$-admissible pair. Then there exists a constant C such that for all $g \in \mathcal{S}'(\mathbb{R}^d)$,

$$\|\mathcal{I}_k(g)\|_{L^q(I,H^s_{r,k}(\mathbb{R}^d))} \leq C\|g\|_{H^s_{r',k}(\mathbb{R}^d)},$$
$$\|\mathcal{I}_k(g)\|_{L^q(I,B^{s,k}_{r,2}(\mathbb{R}^d))} \leq C\|g\|_{B^{s,k}_{r',2}(\mathbb{R}^d)},$$

(2) Let $s \in \mathbb{R}$ and $(q,r), (q_1, s_1)$ be $\gamma + \frac{d}{2}$-admissible pairs. Then there exists a constant C such that for all $f \in \mathcal{S}'(I \times \mathbb{R}^d)$,

$$\|\Phi_k(f)\|_{L^q(I,H^s_{r,k}(\mathbb{R}^d))} \leq C\|f\|_{L^{q_1'}(I,H^s_{r_1',k}(\mathbb{R}^d))},$$
$$\|\Phi_k(f)\|_{L^q(I,B^{s,k}_{r,2}(\mathbb{R}^d))} \leq C\|f\|_{L^{q_1'}(I,B^{s,k}_{r_1',2}(\mathbb{R}^d))}.$$

Proof. Let $t \neq 0$, $s \in \mathbb{R}$ and $2 \leq p \leq \infty$. As in the Euclidean case (cf. Corollary 4.1 in [8]), we can deduce that

$$\|\mathcal{I}_k(t)g\|_{L^\infty_k(\mathbb{R}^d)} \leq \frac{1}{c_k|t|^{(\gamma+\frac{d}{2})}}\|g\|_{L^1_k(\mathbb{R}^d)}.$$

Since $\|\mathcal{I}_k(t)g\|_{L^2_k(\mathbb{R}^d)} = \|g\|_{L^2_k(\mathbb{R}^d)}$, we see by interpolation that

$$\|\mathcal{I}_k(t)g\|_{L^p_k(\mathbb{R}^d)} \leq \frac{1}{(c_k^2|t|^{2\gamma+d})^{(\frac{1}{2}-\frac{1}{p})}}\|g\|_{L^{p'}_k(\mathbb{R}^d)}. \tag{6.4}$$

On the other hand, For any $v \in \mathcal{S}'(\mathbb{R}^d)$ it is easy to see that

$$\mathcal{F}_D^{-1}(v\mathcal{F}_D(\mathcal{I}_k(g)(t,.))) = \mathcal{I}_k(t)\mathcal{F}_D^{-1}(v\mathcal{F}_D(g)). \tag{6.5}$$

In particular, it follows from (6.4) that for $2 \leq p \leq \infty$,

$$\|\mathcal{F}_D^{-1}(v\mathcal{F}_D(\mathcal{I}_k(g)(t,.)))\|_{L^p_k(\mathbb{R}^d)} \leq \frac{1}{(c_k^2|t|^{2\gamma+d})^{(\frac{1}{2}-\frac{1}{p})}}\|\mathcal{F}_D^{-1}(v\mathcal{F}_D(g))\|_{L^{p'}_k(\mathbb{R}^d)}.$$

Therefore, the definitions of the Dunkl-Bessel potential and Dunkl-Besov norms yield that

$$\|\mathcal{I}_k(t)g\|_{H^s_{p,k}(\mathbb{R}^d)} \leq \frac{1}{(c_k^2|t|^{2\gamma+d})^{(\frac{1}{2}-\frac{1}{p})}}\|g\|_{H^s_{p',k}(\mathbb{R}^d)},$$
$$\|\mathcal{I}_k(t)g\|_{B^{s,k}_{p,q}(\mathbb{R}^d)} \leq \frac{1}{(c_k^2|t|^{2\gamma+d})^{(\frac{1}{2}-\frac{1}{p})}}\|g\|_{B^{s,k}_{p',q}(\mathbb{R}^d)}.$$

Then by using the standard argument, we can deduce the desired estimates. □

6.3. **Generalized heat equation.** As in the previous section we shall obtain a space-time estimate of the solution $u(t,x)$, $(t,x) \in I \times \mathbb{R}^d$, of the generalized heat equation

$$\begin{cases} \partial_t u - \triangle_k u = f, \\ u|_{t=0} = g. \end{cases} \tag{6.6}$$

As before, to estimate the solution u of (6.6), we use the integral formulation

$$u(t,x) = H_k(t)g(x) + \int_0^t H_k(t-\tau)f(\tau,x)d\tau, \tag{6.7}$$

where $H_k(t)$ is the generalized heat semi-group. Then by using the explicit form of the kernel of $H_k(t)$ obtained by Rösler [11] and the corresponding formula (6.5), we can deduce the following (see [9]).

Lemma 6.3. *There exist positive constants κ and C such that for all $1 \leq p \leq \infty$, $\tau \geq 0$ and $j \in \mathbb{N}$,*

$$\|\Delta_j(H_k(\tau)u)\|_{L^p_k(\mathbb{R}^d)} \leq Ce^{-\kappa 2^{2j}\tau}\|\Delta_j u\|_{L^p_k(\mathbb{R}^d)}.$$

Theorem 6.4. *Let $s \in \mathbb{R}$, $T > 0$ and $1 \leq p,q,r \leq \infty$. We suppose that $g \in B^{s,k}_{p,r}(\mathbb{R}^d)$ and $f \in L^q((0,T), B^{s-2+\frac{2}{q},k}_{p,r}(\mathbb{R}^d))$. Then (6.6) has a unique solution u belongs to*

$$L^q((0,T), B^{s+\frac{2}{q},k}_{p,r}(\mathbb{R}^d)) \bigcap L^\infty((0,T), B^{s,k}_{p,r}(\mathbb{R}^d))$$

and there exists a constant C such that for all $q \leq q_1 \leq \infty$,

$$\|u\|_{L^{q_1}((0,T),B^{s+\frac{2}{q_1},k}_{p,r}(\mathbb{R}^d))} \leq C\Big((1+T^{\frac{1}{q_1}})\|g\|_{B^{s,k}_{p,r}(\mathbb{R}^d)} + (1+T^{1+\frac{1}{q_1}-\frac{1}{q}})\|f\|_{L^q((0,T),B^{s-2+\frac{2}{q},k}_{p,r}(\mathbb{R}^d))}\Big).$$

If in addition r is finite, then u belongs to $C([0,T], B^{s,k}_{p,r}(\mathbb{R}^d))$.

Proof. Since g, f are tempered, (6.6) has a unique solution u in $\mathcal{S}'((0,T)\times\mathbb{R}^d)$ satisfying

$$\mathcal{F}_D(u)(t,\xi) = e^{-t\|\xi\|^2}\mathcal{F}_D(g)(\xi) + \int_0^t e^{(\tau-t)\|\xi\|^2}\mathcal{F}_D(f)(\tau,\xi)d\tau.$$

Hence, applying Δ_j, $j \geq 1$, to (6.7), we see that

$$\Delta_j u(t,\cdot) = H_k(t)\Delta_j g + \int_0^t H_k(t-\tau)\Delta_j f(\tau,\cdot)d\tau$$

and thus, by Lemma 6.3, we can deduce that

$$\begin{aligned}\|\Delta_j u(t,\cdot)\|_{L^p_k(\mathbb{R}^d)} &\leq \|H_k(t)\Delta_j g\|_{L^p_k(\mathbb{R}^d)} + \int_0^t \|H_k(t-\tau)\Delta_j f(\tau,\cdot)\|_{L^p_k(\mathbb{R}^d)}d\tau \\ &\leq Ce^{-\kappa 2^{2j}t}\|\Delta_j g\|_{L^p_k(\mathbb{R}^d)} + \int_0^t e^{-\kappa 2^{2j}(t-\tau)}\|\Delta_j f(\tau,\cdot)\|_{L^p_k(\mathbb{R}^d)}d\tau.\end{aligned}$$

Then it follows from (2.3) that $\|\Delta_j u\|_{L^{q_1}((0,T),L^p_k(\mathbb{R}^d))}$ is dominated by

$$\Big(\frac{1-e^{-\kappa T q_1 2^{2j}}}{\kappa q_1 2^{2j}}\Big)^{\frac{1}{q_1}}\|\Delta_j g\|_{B^{s,k}_{p,r}(\mathbb{R}^d)} + \Big(\frac{1-e^{-\kappa T q_2 2^{2j}}}{\kappa q_2 2^{2j}}\Big)^{\frac{1}{q_2}}\|\Delta_j f\|_{L^q((0,T),L^p_k(\mathbb{R}^d))} \tag{6.8}$$

with $\frac{1}{q_2} = 1 + \frac{1}{q_1} - \frac{1}{q}$. Moreover, similarly as above, we can obtain that

$$\|\triangle_0 u(t,\cdot)\|_{L^p_k(\mathbb{R}^d)} \leq \|\triangle_0 g\|_{L^p_k(\mathbb{R}^d)} + \int_0^t \|\triangle_0 f(\tau,\cdot)\|_{L^p_k(\mathbb{R}^d)}d\tau,$$

and thus, if $1 \leq q \leq q_1 \leq \infty$,

$$\|\triangle_0 u\|_{L^{q_1}((0,T),L^p_k(\mathbb{R}^d))} \leq C\Big(T^{\frac{1}{q_1}}\|\triangle_0 g\|_{L^p_k(\mathbb{R}^d)} + T^{\frac{1}{q_2}}\|\triangle_0 f\|_{L^q((0,T),L^p_k(\mathbb{R}^d))}\Big). \tag{6.9}$$

Finally, taking the l^r-norm with respect to j in (6.8) and (6.9) with the usual convention if $r = \infty$, we can deduce the desired estimate. □

References

[1] **C. Abdelkefi, J-Ph. Anker, F. Sassi and M. Sifi,** *Besov-type spaces on $\mathbb{R}^d$ and integrability for the Dunkl transform,* Symmetry, Integrability and Geometry, Method and Applications, SIGMA, **5** (2009), 1-15.

[2] **J.M. Bony,** *Calcul symbolique et propogation des singularités pour les équations aux dérivées partielles non linéaires,* Annales de l'École Normale Supérieure, **14** (1981), 209-246.

[3] **C. F. Dunkl,** *Differential-difference operators associated to reflection group,* Trans. Am. Math. Soc. **311** (1989), 167-183.

[4] **C.F. Dunkl,** *Integral kernels with reflection group invariant,* Can. J. Math. **43** (1991), 1213-1227.

[5] **C.F. Dunkl,** *Hankel transforms associated to finite reflection groups,* Contemp. Math. **138** (1992), 123-138.

[6] **M.F.E.de Jeu,** *The Dunkl transform,* Invent.Math. **113** (1993), 147-162.

[7] **H. Mejjaoli,** *Littlewood-Paley decomposition associated with the Dunkl operators, and paraproduct operators,* J. Inequal. Pure App. Math. **9** (2008), 1-25

[8] **H. Mejjaoli,** *Dispersion phenomena in Dunkl-Schrödinger equation and applications,* Serdica Math. J. **35** (2009), 25-60.

[9] **H. Mejjaoli,** *Dunkl-heat equations and applications,* preprint (2010).

[10] **J. Peetre,** *Sur les espaces de Besov,* C.R.A.S. Paris Sér.I Math. **264** (1967), 281-283.

[11] **M. Rösler,** *Generalized Hermite polynomials and the heat equation for Dunkl operators,* Comm. Math. Phys. **192** (1998), 519-541.

[12] **M. Rösler,** *A positive radial product formula for the Dunkl kernel,* Trans. Amer. Math. Soc. **355** (2003), 2413-2438.

[13] **E.M. Stein,** *Singular Integrals and Differentiability Properties of Functions,* Princeton Univ. Press, 1970.

[14] **S. Thangavelu and Y. Xu,** *Convolution operator and maximal functions for Dunkl transform,* J. d'Analyse Mathematique **97** (2005), 25-56.
[15] **H. Triebel,** *Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation,* Ark. Mat. **11** (1973), 13-64.
[16] **H. Triebel,** *Interpolation theory, functions spaces differential operators,* North-Holand Amesterdam. 1978.
[17] **K. Trimèche,** *The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual,* Integ. Transf. and Special Funct. **12** (2001), 349-374.
[18] **K. Trimèche,** *Paley-Wiener theorems for Dunkl transform and Dunkl translation operators,* Integ. Transf. and Special Funct. **13** (2002), 17-38.

Department of Mathematics, Keio University at Fujisawa, Kanagawa, 252-8520 Japan.
E-mail address: *kawazoe@sfc.keio.ac.jp*

Department of Mathematics, College of sciences, King Faisal University, P.O. 380 Ahsaa 31982, Saudi Arabia.
E-mail address: *hmejjaoli@kfu.edu.sa*