# Characterizations of some function spaces by the discrete Radon transform on $\mathbb{Z}^n$.

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**Abstract**

Let $\mathbb{Z}^n$ be the lattice in $\mathbb{R}^n$ and $\mathbb{G}$ the set of all discrete hyperplanes in $\mathbb{Z}^n$. Similarly as in the Euclidean case, for a function $f$ on $\mathbb{Z}^n$, the discrete Radon transform $Rf$ is defined by the integral of $f$ over hyperplanes, and $R$ maps functions on $\mathbb{Z}^n$ to functions on $\mathbb{G}$. In this paper we determine the Radon transform images of the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$, the space of compactly supported functions on $\mathbb{Z}^n$, and a discrete Hardy space $H^1(\mathbb{Z}^n)$.

## 1 Introduction

Let $\mathbb{Z}^n$ be the lattice in $\mathbb{R}^n$. For $a = (a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n \backslash \{0\}$ and $k \in \mathbb{Z}$, the linear diophantine equation $a \cdot x = k$ has an infinity of solutions in $\mathbb{Z}^n$ if and only if $k$ is an integral multiple of the greatest common divisor $d(a)$ of $a_1, a_2, \cdots, a_n$. Therefore, for $a \in \mathcal{P} = \{z \in \mathbb{Z}^n \mid d(z) = 1\}$, the set of solutions $H(a, k) = \{x \in \mathbb{Z}^n \mid a \cdot x = k\}$ forms a discrete hyperplane in $\mathbb{Z}^n$. Let $\mathbb{G}$ be the set of all discrete hyperplanes in $\mathbb{Z}^n$. Then $\mathbb{G}$ can be parametrized as $\mathcal{P} \times \mathbb{Z}/\{\pm 1\}$ (see §2 in [1]). As an analogue of the Euclidean case, the discrete Radon transform $R$, which maps functions on $\mathbb{Z}^n$ to functions on $\mathbb{G}$, is given by

$$Rf(H(a,k)) = \sum_{m \in H(a,k)} f(m)$$

for a suitable function $f$ on $\mathbb{Z}^n$. In their previous paper [1] the first author and A. Ihsane investigated the basic properties of $R$. Especially, the Strichartz type inversion formula for $R$ and the support theorem were obtained. Moreover, they showed that the discrete Radon transform $R$ is a continuous linear mapping of $\mathcal{S}(\mathbb{Z}^n)$ into $\mathcal{S}(\mathbb{G})$ (see §2 for the definitions of the Schwartz spaces $\mathbb{Z}^n$ and $\mathbb{G}$). Our natural question, that is a starting point of this paper, is whether this map is bijective or not. Let us suppose that $f$ belongs to $\mathcal{S}(\mathbb{Z}^n)$ and has a suitable decay. Then it follows from Corollary 3.10 in [1] that for $\theta \in \mathbb{T}$,

$$\mathcal{F}_1 Rf(H(a, \cdot))(\theta) = \mathcal{F}f(\theta a), \tag{1}$$

wehre $\mathcal{F}$ and $\mathcal{F}_1$ are the classical Fourier (inverse) transforms on $\mathbb{Z}^n$ and $\mathbb{Z}$ respectively. We note that $\mathcal{F}f$ is a $C^\infty$ function on $\mathbb{T}^n$. Therefore, in order to show the surjectivity of the above mapping, we have to construct a $C^\infty$ function on $\mathbb{T}^n$ from $\theta a$ variables with $\theta \in \mathbb{T}$ and $a \in \mathcal{P}$. However, it is impossible, because $\theta a$ varies in a dence subset of $\mathbb{T}^n$ (see Remark 8). Hence, the map $R$ from $\mathcal{S}(\mathbb{Z}^n)$ into $\mathcal{S}(\mathbb{G})$ is not bijective and, to characterize the image of $\mathcal{S}(\mathbb{Z}^n)$, a condition corresponding to (1) is required. When we fix an $a \in \mathcal{P}$ and we restrict our attention to a local area $\mathbb{Z}^n_a$ in $\mathbb{Z}^n$ and the set of hyperplanes with direction $a$: $\mathbb{G}_a = \{H(a,k) \mid k \in \mathbb{Z}\}$ in $\mathbb{G}$, we use a terminology "local"; for example, we call $\mathcal{S}(\mathbb{Z}^n_a)$ and $\mathcal{S}(\mathbb{G}_a)$ the local Shcwartz spaces on $\mathbb{Z}^n_a$ and $\mathbb{G}_a$ respectively (see §3). This paper is organized as follows. A characterization of the local Shcwartz space $\mathcal{S}(\mathbb{Z}^n_a)$ is given in §3 and a Paley-Wiener type theorem is obtained in §4. We determine the Radon transform image of the global Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ in §5. We introduce discrete Hardy spaces on $\mathbb{Z}^n$ and $\mathbb{G}$, locally and globally, in §6 and then characterize their Radon transform images.

## 2 Notations

Let $\mathbb{Z}^n$ be the vector space of all $a = (a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n$ equipped with the norm $\|a\|^2 = a_1^2 + a_2^2 + \cdots + a_n^2$ and the inner product $a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$. For $1 \le p < \infty$ let $l^p(\mathbb{Z}^n)$ denote the space of all complex valued functions $f$ on $\mathbb{Z}^n$ with finite norm:

$$\|f\|_p = \Big( \sum_{m \in \mathbb{Z}^n} |f(m)|^p \Big)^{1/p} < \infty.$$

We introduce a space $\mathbb{G}$ of hyperplanes in $\mathbb{Z}^n$. For $a = (a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n$ let $d(a) = d(a_1, a_2, \cdots, a_n)$ denote the greatest common divisor of $a_1, a_2, \cdots, a_n$ and put $\mathcal{P} = \{a \in \mathbb{Z}^n \backslash \{0\} \mid d(a) = 1\}$. For each $(a,k) \in \mathcal{P} \times \mathbb{Z}$ we define a discrete hyperplane $H(a,k)$ by

$$H(a,k) = \{m \in \mathbb{Z}^n \mid a \cdot m = k\}.$$

Let $\mathbb{G}$ be the set of all hyperplanes $H(a,k)$ with $(a,k) \in \mathcal{P} \times \mathbb{Z}$, which is parameterized as $\mathcal{P} \times \mathbb{Z}/\{\pm 1\}$. For $1 \le p < \infty$ let $l^p(\mathbb{G})$ denote the space of all complex valued functions $F$ on $\mathbb{G}$ with finite norm:

$$\|F\|_{a,p} = \Big( \sum_{k \in \mathbb{Z}} |F(H(a,k))|^p \Big)^{1/p} < \infty$$

for all $a \in \mathcal{P}$.

For $f \in l^1(\mathbb{Z}^n)$, the Radon transform $Rf$ on $\mathbb{G}$ is given by

$$Rf(H(a,k)) = \sum_{a \cdot m = k} f(m).$$

Then $R(l^1(\mathbb{Z}^n)) \subset l^1(\mathbb{G})$ (see Remark 3.8 in [1]) and the Strichartz type inversion of $R$ is given as follows: For each $m \in \mathbb{Z}^n$,

$$f(m) = \lim_{j \to \infty} Rf(H(a_j, a_j \cdot m)),$$

where $a_j = (1, j, j^2, \cdots, j^{n-1})$ (see Theorem 4.1 in [1]). We define the Fourier (inverse) transform $\mathcal{F}f(t)$ of $f$ as

$$\mathcal{F}f(t) = \sum_{m\in\mathbb{Z}^n} f(m)e^{-im\cdot t}, \quad t \in \mathbb{T}^n.$$

Similarly, $\mathcal{F}_1$ denotes the one-dimensional Fourier (inverse) transform on $\mathbb{Z}$. Then Corollary 3.6 in [1] asserts that $Rf$ satisfies that for all $a \in \mathcal{P}$ and $\theta \in \mathbb{T}$,

$$\mathcal{F}_1 Rf(H(a,\cdot))(\theta) = \mathcal{F}f(\theta a). \tag{2}$$

We introduce Schwartz spaces on $\mathbb{Z}^n$ and $\mathbb{G}$ as follows. Let $\mathcal{S}(\mathbb{Z}^n)$ denote the space of all complex valued functions $f$ on $\mathbb{Z}^n$ such that for all $N \in \mathbb{N}$,

$$p_N(f) = \sup_{m\in\mathbb{Z}^n} (1 + \|m\|^2)^N |f(m)| < \infty$$

and $\mathcal{S}(\mathbb{G})$ the space of all complex valued functions $F$ on $\mathbb{G}$ such that for all $N \in \mathbb{N}$,

$$q_N(f) = \sup_{a\in\mathcal{P},k\in\mathbb{Z}} \Big(\frac{1+k^2}{1+\|a\|^2}\Big)^N |F(H(a,k))| < \infty.$$

Then $R(\mathcal{S}(\mathbb{Z}^n)) \subset \mathcal{S}(\mathbb{G})$ and

$$\sum_{k\in\mathbb{Z}} F(H(a,k))k^p = \sum_{m\in\mathbb{Z}^n} f(m)(a\cdot m)^p.$$

(see Corollary 3.10 in [1]). In particular, as a function of $a \in \mathcal{P}$,

$$\textstyle\sum_{k\in\mathbb{Z}} F(H(a,k))k^p \text{ is a homogeneous polynomial of degree } p \tag{3}$$

for all $p = 0, 1, 2, \cdots$. The dual space $\mathcal{S}'(\mathbb{Z}^n)$ of $\mathcal{S}(\mathbb{Z}^n)$, the space of distributions on $\mathbb{Z}^n$, is defined as the space of all complex valued functions $f$ on $\mathbb{Z}^n$ for which there exists $N \in \mathbb{N}$ such that $\sup_{m\in\mathbb{Z}^n}(1 + \|m\|)^{-N}|f(m)| < \infty$.

# 3 Characterization of local Schwartz spaces

As in the classical case (cf. Theorem 2.4 in [3]), we shall consider local Schwartz spaces on $\mathbb{Z}^n$ and $\mathbb{G}$ respectively. In what follows we fix an $a \in \mathcal{P}$. For each hyperplane $H(a,k)$ let $p_{a,k} \in \mathbb{Z}^n$ be the point in $H(a,k)$ that is nearest from the origin. We set

$$\mathbb{Z}^n_a = \{p_{a,k} \mid k \in \mathbb{Z}\}.$$

We define a local Schwartz space $\mathcal{S}(\mathbb{Z}^n_a)$ on $\mathbb{Z}^n$ by

$$\mathcal{S}(\mathbb{Z}^n_a) = \{f \in \mathcal{S}(\mathbb{Z}^n) \mid \mathrm{supp}(f) \subset \mathbb{Z}^n_a\}.$$

We denote by $\mathbb{G}_a$ the set of all hyperplanes with direction $a$, that is,

$$\mathbb{G}_a = \{H(a,k) \mid k \in \mathbb{Z}\}.$$

Since $H(a,k) \cap H(a,k') = \emptyset$ if $k \neq k'$ and $\cup_{k\in\mathbb{Z}} H(a,k) = \mathbb{Z}^n$, it is clear that $\mathbb{G}_a \cap \mathbb{G}_{a'} = \emptyset$ if $a \neq a'$ and $\cup_{a\in\mathcal{P}} \mathbb{G}_a = \mathbb{G}$ (see [1] for more details). We define a local Schwartz space $\mathcal{S}(\mathbb{G}_a)$ on $\mathbb{G}$ by

$$\mathcal{S}(\mathbb{G}_a) = \{F \in \mathcal{S}(\mathbb{G}) \mid \mathrm{supp}(F) \subset \mathbb{G}_a\}.$$

For a function $F$ on $\mathbb{G}$ we denote by $P_{\mathbb{G}_a}(F)$ the function on $\mathbb{G}$ such that $P_{\mathbb{G}_a}(F)(H) = F(H)$ if $H \in \mathbb{G}_a$ and 0 otherwise.

**Theorem 1.** *For all $a \in \mathcal{P}$,*

$$P_{\mathbb{G}_a} \circ R(\mathcal{S}(\mathbb{Z}^n_a)) = \mathcal{S}(\mathbb{G}_a).$$

*Proof.* Since The argument used in the proof that $R(\mathcal{S}(\mathbb{Z}^n)) \subset \mathcal{S}(\mathbb{G})$ (see Theorem 3.7 in [1]) yields that $P_{\mathbb{G}_a} \circ R(\mathcal{S}(\mathbb{Z}^n_a)) \subset \mathcal{S}(G_a)$. We shall prove the converse. For $F \in \mathcal{S}(\mathbb{G}_a)$ we put

$$f(m) = \sum_{k\in\mathbb{Z}} F(H(a,k))\chi_{p_{a,k}}(m).$$

Then it follows Proposition 3.4 in [1] that

$$Rf(H) = \sum_{k\in\mathbb{Z}} F(a,k)\chi^{\mathbb{G}}_{p_{a,k}}(H).$$

If $H = H(a,k) \in \mathbb{G}_a$, then $Rf(H(a,k)) = F(H(a,k))$ and $Rf(H) = 0$ otherwise. Therefore, it follows that $P_{\mathbb{G}_a} \circ R(f) = F$. Hence, to complete the proof of the surjectivity, it is enough to prove that $f \in \mathcal{S}(\mathbb{Z}^n)$. Since $f$ is supported on $\mathbb{Z}^n_a$ and $\|p_{a,k}\| \leq |k|$, it follows that for all $m = p_{a,k}$,

$$\begin{aligned}(1 + \|m\|^2)^N |f(m)| &= (1 + \|p_{a,k}\|^2)^N |f(p_{a,k})| \\ &\leq (1 + k^2)^N |F(H(a,k))|.\end{aligned}$$

Since $F \in \mathcal{S}(\mathbb{G})$, it follows that $p_N(f) < \infty$. □

# 4 A Paley-Wiener type theorem

We shall consider a discrete Paley-Wiener theorem relatively at the discrete Radon transform, which characterizes the image of functions on $\mathbb{Z}^n$ with finite support. Let $K = \{x_1, x_2, \ldots, x_l\}$ be a finite set in $\mathbb{Z}^n$. We denote by $\mathcal{D}_K(\mathbb{Z}^n)$ the subspace of $\mathcal{S}(\mathbb{Z}^n)$ consisting of all complex-valued functions on $\mathbb{Z}^n$ such that $\mathrm{supp} f \subset K$. Let $\mathbb{G}_K = \{H \in \mathbb{G} \mid H \cap K \neq \emptyset\}$. We denote by $\mathcal{D}_K(\mathbb{G})$ the subspace of $\mathcal{S}(\mathbb{G})$ consisting of all complex-valued functions on $\mathbb{G}$ such that $\mathrm{supp} F \subset \mathbb{G}_K$. Each function $f \in \mathcal{D}_K(\mathbb{Z}^n)$ is of the form

$$f = \sum_{z\in K} f(z)\chi_z.$$

As shown in [1], Proposition 3.4 and Theorem 3.5, $Rf$ is of the form

$$Rf = \sum_{z \in K} f(z) \chi_z^{\mathbb{G}} \tag{4}$$

and $\mathrm{supp} Rf \subset \bigcup_{z \in K} \mathbb{G}_z$. Hence $Rf$ belongs to $\mathcal{D}_K(\mathbb{G})$. In what follows we shall characterize functions of the form (4). We define $\mathcal{D}_{*,K}(\mathbb{G})$ as the subspace of $\mathcal{D}_K(\mathbb{G})$ consisting of all functions $F$ with moment condition (3) and for each $m \in \mathbb{Z}^n$, there exists $j_m \in \mathbb{N}$ for which

$$F(H(a_j, a_j \cdot m)) = F(H(a_{j_m}, a_{j_m} \cdot m)) \text{ for all } j \geq j_m, \tag{5}$$

where $a_j = (1, j, j^2, \ldots, j^{n-1})$.

**Lemma 2.** *Let $F \in \mathcal{D}_K(\mathbb{G})$ and $a \in \mathcal{P}$. Assume that*

$$\sum_{k \in \mathbb{Z}} F(H(a,k)) k^p = 0 \text{ for all } p = 0, 1, 2, \cdots.$$

*Then $F(H(a,k)) = 0$ for all $k \in \mathbb{Z}$.*

*Proof.* We note that for a fixed $a \in \mathcal{P}$, $F(H(a,k)) = 0$ except finite $k$ and

$$\sum_{p=0}^{\infty} \sum_{k \in \mathbb{Z}} F(H(a,k)) \frac{(i\theta k)^p}{p!} = \sum_{k \in \mathbb{Z}} F(H(a,k)) e^{i\theta k} = 0$$

for $\theta \in \mathbb{T}$. Therefore $\mathcal{F}_1 F(H(a,\cdot))(\theta) = 0$ and thus, $F(H(a,k)) = 0$ for all $k \in \mathbb{Z}$. Hence $F = 0$. $\square$

**Lemma 3.** *Let $m \in \mathbb{Z}^n$.*

(i) *The case of $m \notin K$: Let $j_{m,K} = \sum_{z \in K} \|z - m\|^2$. If $j > j_{m,K}$, then*

$$H(a_j, a_j \cdot m) \cap K = \emptyset.$$

(ii) *The case of $m \in K$: Let $j_K = \sum_{z \in K} \|z\|^2$. If $j > 2j_K$, then*

$$H(a_j, a_j \cdot m) \cap K = \{m\}.$$

*Proof.* (i) Let $m \notin K$ and $j > j_{m,K}$. We assume that $H(a_j, a_j \cdot m) \cap K \neq \emptyset$ and $z \in H(a_j, a_j \cdot m) \cap K$. This implies that

$$(z_1 - m_1) + (z_2 - m_2) j + \cdots + (z_n - m_n) j^{n-1} = 0.$$

Hence $j \mid (z_1 - m_1)$ and thus, there exists $\alpha \in \mathbb{Z}$ such that $z_1 - m_1 = \alpha j$. Then it follows that

$$(\alpha j)^2 = (z_1 - m_1)^2 \leq \|z - m\|^2 \leq j_{m,K}.$$

Therefore, $\alpha$ must be 0 and $z_1 = m_1$. We repeat the same argument for $z_2 - m_2$ and so on. Then $z = m$. This contradicts to $m \notin K$ and $z \in K$. Hence $H(a_j, a_j \cdot m) \cap K = \emptyset$ for all $j > j_{m,K}$. (ii) Let $m \in K$ and $j > j_K$. We assume that $z \in H(a_j, a_j \cdot m) \cap K$. Similarly as above, $z_1 - m_1 = \alpha j$. Then it follows that

$$j_K \geq \|z\|^2 \geq z_1^2 = m_1^2 + (\alpha j)^2 + 2\alpha j m_1$$
$$\geq (\alpha j)^2 - 2|\alpha| j j_K = |\alpha| j (|\alpha| j - 2 j_K).$$

Therefore, $\alpha$ must be 0 and $z_1 = m_1$. We repeat the same argument for $z_2 - m_2$ and so on. Then $z = m$. □

**Theorem 4.** *$R$ is a bijection of $\mathcal{D}_K(\mathbb{Z}^n)$ onto $\mathcal{D}_{*,K}(\mathbb{G})$.*

*Proof.* Let $f \in \mathcal{D}_K(\mathbb{Z}^n)$. As said above, $Rf \in \mathcal{D}_K(\mathbb{G})$. Moreover, it follows from Corollary 3.10 in [1] that $Rf$ satisfies (3) and from the proof of Theorem 4.1 in [1] that $Rf$ satisfies (5). Therefore, $Rf$ belongs to $\mathcal{D}_{*,K}(\mathbb{G})$. Since the inversion formula of $R$ exists, it is sufficient to prove the surjectivity of $R$. Let $F \in \mathcal{D}_{*,K}(\mathbb{G})$. We define a function $g$ on $\mathbb{Z}^n$ by $g(m) = \lim_{j\to\infty} F(H(a_j, a_j \cdot m))$. Then if follows from Lemma 3 that $g \in \mathcal{D}_K(\mathbb{Z}^n)$. Moreover, since $F$ satisfies (5) and $K$ is finite, there exists $J_K \in \mathbb{N}$ such that $g(m) = F(H(a_j, a_j \cdot m))$ for all $m \in K$ and $j \geq J_K$. Then it follows from Corollary 3.10 in [1] and Lemma 3 that for all $j > j_0 = \max\{2j_K, J_K\}$ and $p = 0, 1, 2, \cdots$,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} Rg(H(a_j, k))k^p &= \sum_{m\in\mathbb{Z}^n} g(m)(a_j \cdot m)^p \\
&= \sum_{m\in K} F(H(a_j, a_j \cdot m))(a_j \cdot m)^p \\
&= \sum_{k\in\mathbb{Z}} \Big( \sum_{m\in H(a_j,k)\cap K} F(H(a_j, k)) \Big) k^p \\
&= \sum_{k\in\mathbb{Z}} F(H(a_j, k))k^p.
\end{aligned}$$

Hence $\sum_{k\in\mathbb{Z}}(Rg - F)(H(a_j, k))k^p = 0$ for all $j \geq j_0$. Since $Rf$ and $F$ satisfies the moment condition (3), as a function of $a \in \mathcal{P}$, $\sum_{k\in\mathbb{Z}}(Rg - F)(H(a, k))k^p$ is a homogeneous polynomial of degree $p$, which is equal to 0 at $a = a_j$ for all $j \geq j_0$. Therefore, $\sum_{k\in\mathbb{Z}}(Rg - F)(H(a, k))k^p = 0$ for all $a \in \mathcal{P}$. Then by Lemma 2, we see that $F = Rg$. This completes the proof of the theorem. □

# 5 Characterization of global spaces

We shall obtain a global characterization of the Radon transform images of subspaces of $l^1(\mathbb{Z}^n)$. For a subspace $X$ of $L^1(\mathbb{T}^n)$, we denote by $\hat{X}$ the space on $\mathbb{Z}^n$ consisting of all Fourier coefficients of $G \in X$, that is,

$$\hat{X} = \{f : \mathbb{Z}^n \to \mathbb{C} \mid \text{there exists } G \in X \text{ such that } f(m) = \hat{G}(m)\},$$

where

$$\hat{G}(m) = \int_{\mathbb{T}^n} G(t)e^{-im\cdot t}dt.$$

Let $X_{\mathbb{G}}$ denote a subspace of $l^1(\mathbb{G})$ consisting of all $F \in l^1(\mathbb{G})$ such that there exists a $G \in X$ for which

$$\mathcal{F}_1 F(H(a,\cdot))(\theta) = G(\theta a). \tag{6}$$

**Theorem 5.** *Suppose that $\hat{X} \subset l^1(\mathbb{Z}^n)$. Then $R$ is a bijection of $\hat{X}$ onto $X_{\mathbb{G}}$.*

*Proof.* Since $\hat{X} \subset l^1(\mathbb{Z}^n)$ and $R(l^1(\mathbb{Z}^n)) \subset l^1(\mathbb{G})$, $R$ is defined on $\hat{X}$ and each $Rf, f \in \hat{X}$, belongs to $l^1(\mathbb{G})$. Moreover, it follows from (2) that $Rf$ satisfies (6) with $G = \mathcal{F}f \in X$. Hence $R(\hat{X}) \subset X_{\mathbb{G}}$. Since the inversion formula of $R$ exists, it is sufficient to prove the surjectivity of $R$. Let $F \in X_{\mathbb{G}}$ and suppose that $\mathcal{F}_1 F(H(a,\cdot))(\theta) = G(\theta a)$ for $G \in X$. Since $\hat{X} \subset l^1(\mathbb{Z}^n)$, it follows that

$$\begin{aligned} G(\theta a) &= \sum_{m \in \mathbb{Z}^n} \hat{G}(m) e^{-im\cdot\theta a} = \sum_{m \in \mathbb{Z}^n} \hat{G}(m) e^{-i\theta m \cdot a} \\ &= \sum_{k \in \mathbb{Z}} \Big( \sum_{m \in H(a,k)} \hat{G}(m) \Big) e^{-i\theta k}. \end{aligned}$$

Hence $F(H(a,k)) = \sum_{m \in H(a,k)} \hat{G}(m)$. Therefore, if we define a function $g$ on $\mathbb{Z}^n$ by $g(m) = \hat{G}(m)$, $m \in \mathbb{Z}^n$, it follows that $F = Rg$. Clearly, $G \in X$ implies $g \in \hat{X}$. This completes the proof of the theorem. □

Let $\mathcal{S}_*(\mathbb{G})$ denote a subspace of $\mathcal{S}(\mathbb{G})$ consisting of all $F \in \mathcal{S}(\mathbb{G})$ such that there exists a $G \in \mathcal{C}^\infty(\mathbb{T}^n)$ for which $\mathcal{F}_1 F(H(a,\cdot))(\theta) = G(\theta a)$. Then

**Corollary 6.** *$R$ is a bijective continuous mapping of $\mathcal{S}(\mathbb{Z}^n)$ onto $\mathcal{S}_*(\mathbb{G})$.*

Let $A^1(\mathbb{T}^n)$ denote a subspace of $L^1(\mathbb{T}^n)$ consisting of all $G \in L^1(\mathbb{T}^n)$ such that $\sum_{m \in \mathbb{Z}^n} |G(m)| < \infty$ and let $L^1_*(\mathbb{G})$ be a subspace of $L^1(\mathbb{G})$ consisting of all $F \in L^1(\mathbb{G})$ such that there exists a $G \in A^1(\mathbb{T}^n)$ for which $\mathcal{F}_1 F(H(a,\cdot))(\theta) = G(\theta a)$. Then it follows that

**Corollary 7.** *$R$ is a bijective continuous mapping of $l^1(\mathbb{Z}^n)$ onto $L^1_*(\mathbb{G})$.*

**Remark 8.** (i) The left hand side of (6) is a function of $\theta$ with $0 \le \theta \le 2\pi$ and $G$ in the right hand side is a function on $\mathbb{T}^n$. Therefore, for a fixed $a \in \mathcal{P}$, $\theta$ in the right hand side varies in the set of $0 \le \theta \le \dfrac{L_a}{\|a\|}$, where $L_a$ is the length of the line segment with direction $a$ between the origin and the boundary of $\mathbb{T}^n$. Hence, if we rewrite (6) as

$$\mathcal{F}_1 F(H(a,\cdot))\Big(\frac{\theta}{\|a\|}\Big) = G\Big(\theta \frac{a}{\|a\|}\Big), \quad 0 \le \theta \le L_a, \tag{7}$$

then $\dfrac{a}{\|a\|}$, $a \in \mathcal{P}$ moves all rational points in $S^{n-1} \cap \mathbb{T}^n$. Therefore, the condition (7) implies that the left hand side defined on

$$\{r\omega \mid \omega \in S^{n-1} \cap \mathbb{T}^n \text{ and rational}, 0 \le r \le L_\omega\}$$

can be extended to a function $G$ on $\mathbb{T}^n$. (ii) In Corollary 6 and Corollary 7, the continuity of $R^{-1}$ is an open problem.

## 6 Hardy spaces

The lattice $\mathbb{Z}^n$ is a space of homogeneous type, because $\mathbb{Z}^n$ is equipped with a Euclidean distance and a counting measure. Hence, we can introduce real Hardy spaces on $\mathbb{Z}^n$ according to the process in G. Folland and E. Stein [2]. We shall prove that the Radon transform $R$ locally maps $H^1(\mathbb{Z}^n)$ onto $H^1(\mathbb{Z})$ and globally maps to an atomic Hardy space on $\mathbb{G}$.

We briefly overview the definition of $H^1(\mathbb{Z}^n)$ and its atomic decomposition. For $\phi \in \mathcal{S}(\mathbb{R}^n)$ we define a discrete dilation $\phi_t$, $t = 1, 2, 3, \cdots$ of $\phi$ by

$$\phi_t(x) = t^{-n} \sum_{m \in \mathbb{Z}^n} \phi(t^{-1}m)\chi_m(x), \; x \in \mathbb{Z}^n.$$

In the Euclidean case, $\phi_t$ is an approximate identity. Hence, to keep this propperty, we put

$$\phi_0(x) = \phi(0)\chi_0(x).$$

For any $f \in \mathcal{S}'(\mathbb{Z}^n)$, we define a radial maximal function $M_\phi f$ on $\mathbb{Z}^n$ by

$$M_\phi f(x) = \sup_{t=0,1,2,\cdots} |f * \phi_t(x)|, \; x \in \mathbb{Z}^n,$$

where $*$ is the discrete convolution on $\mathbb{Z}^n$. We say that a distribution $f \in \mathcal{S}'(\mathbb{Z}^n)$ belongs to $H^1(\mathbb{Z}^n)$ if there is a $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx \neq 0$ so that $M_\phi f \in L^1(\mathbb{Z}^n)$. We put $\|f\|_{H^1} = \|M_\phi f\|_1$. Since $|f(x)| \leq cM_\phi f(x)$, it follows that $H^1(\mathbb{Z}^n) \subset l^1(\mathbb{Z}^n)$. According to the process in [2], Chap. 3, we can obtain an atomic decomposition of $H^1(\mathbb{Z}^n)$ as follows. We say that a function $b$ on $\mathbb{Z}^n$ is a $(1, \infty, 0)$-atom if it satisfies

$$\begin{aligned} &(i) \quad \mathrm{supp} b \subset B(m_0, r), \\ &(ii) \; \|b\|_\infty \leq r^{-n} \\ &(iii) \sum_{m \in \mathbb{Z}^n} b(m) = 0, \end{aligned}$$

where $B(m_0, r)$ is a closed ball centered at $m_0 \in \mathbb{Z}^n$ and radius $r \in \mathbb{N}$, which depends on $b$. Then $f \in H^1(\mathbb{Z}^n)$ if and only if there exist a collection $\{b_i\}$ of $(1, \infty, 0)$-atoms on $\mathbb{Z}^n$ and a sequence $\{\lambda_i\}$ of complex numbers with $\sum_i |\lambda_i| < \infty$ so that $f = \sum_i \lambda_i b_i$. Moreover, $\|f\|_{H^1} \sim \inf \sum_i |\lambda_i|$, where the infimum is taken over all atomic decomposition of $f$. Similarly, we can define $H^1(\mathbb{Z})$ on $\mathbb{Z}$. For $l \in \mathbb{N}$, we denote by $H_l(\mathbb{Z})$ the subspace of $H(\mathbb{Z})$ which is constructed by using $(1, \infty, 0)$-atoms supported on intervals with length $2lr$, $r \in \mathbb{N}$.

Now we shall characterize the image $Rb$ on $\mathbb{G}$ of a $(1, \infty, 0)$-atom $b$ on $\mathbb{Z}^n$. As pointed in §3, $Rb = \sum_{z \in B(m_0,r)} b(z)\chi_z^{\mathbb{G}}$. Therefore, for each fixed $a \in \mathcal{P}$, as a function of $k \in \mathbb{Z}$, $Rb(H(a, k))$ is supported on $\{a \cdot z \mid z \in B(m_0, r)\}$. We denote by $k_0 = a \cdot m_0$ the middle point of this support. Then we can easily deduce that

$$\mathrm{supp} Rb(H(a, \cdot)) \subset [k_0 - \|a\|r, k_0 + \|a\|r],$$

because if $k = a\cdot z$ is in the support of $Rb(H(a,\cdot))$, $|k-k_0| = |a\cdot(z-m_0)| \leq \|a\|r$. We note that

$$
\begin{aligned}
|Rb(H(a,k))| &\leq \sum_{m\in H(a,k)} |b(m)| \\
&\leq r^{-n}|H(a,k)\cap B(m_0,r)| \qquad (8)\\
&\leq r^{-n}\frac{cr^{n-1}}{\|a\|} = c(\|a\|r)^{-1}
\end{aligned}
$$

and

$$\sum_{k\in\mathbb{Z}} Rb(H(a,k)) = \sum_{m\in\mathbb{Z}^n} b(m) = 0.$$

These properties imply that $c^{-1}Rb(H(a,k)$ is a $(1,\infty,0)$-atom on $\mathbb{Z}$ with radius $\|a\|r$. Therefore, if we define $R_a$ by

$$R_a f(k) = Rf(H(a,k)),$$

we can obtain the following.

**Proposition 9.** *For each $a \in \mathcal{P}$, $R_a$ continuously maps $H^1(\mathbb{Z}^n)$ into $H^1_{\|a\|}(\mathbb{Z})$.*

For the surjectivity we shall prove the following lemmas.

**Lemma 10.** *Let $a \in \mathcal{P}$ and $Q$ be a cube in $\mathbb{R}^n$ such that it is centered at the origin, each side length is $2\|a\|r$ and a face parallels to $H(a,0)$. Then, for each integer $l \in [-\|a\|r, \|a\|r]$, $|Q\cap H(a,l)|$ contains at least $(2r)^{n-1}$ elements.*

*Proof.* Since $Q$ is cubic, we may suppose that $l = 0$. When $n = 2$, the assertion is clear. Let $a = (a_1, a_2, \cdots, a_n)$, $n \geq 3$ and $b_i = (0,\cdots,0,a_i,a_{i+1},0,\cdots,0)$ for $1 \leq i \leq n-1$. Then it follows from the case of $n = 2$ that there exist at least $2r$ elements $n_{i,j} = (0,\cdots,0,m_{i,j},m_{i+1,j},0,\cdots,0) \in Q$ for which $m_{i,j}a_i + m_{i+1,j}a_{i+1} = 0$, $1 \leq j \leq 2r$. Hence each $n_{i,j}$ belongs to $Q\cap H(a,0)$. Since $n_{i,j}$, $1 \leq i \leq n-1$, are linearly independent, the desired result follows. □

**Lemma 11.** *Let $a \in \mathcal{P}$. For each $(1,\infty,0)$-atom $B$ on $\mathbb{Z}$ with radius $\|a\|r$, $r \in \mathbb{N}$, there exist a $(1,\infty,0)$-atom $b$ on $\mathbb{Z}^n$ and a constant $C$ for which $B = CR_ab$, where $C$ depends only on $n$ and $a$.*

*Proof.* We may suppose that $B$ is supported on $[-\|a\|r, \|a\|r]$. Let $Q$ be a cube in $\mathbb{R}^n$ such that $Q$ is centered at the origin, each side length $2\|a\|r$and a face parallels to $H(a,0)$. By Lemma 10, for each integer $l \in [-\|a\|r, \|a\|r]$, $Q\cap H(a,l)$ contains $(2r)^{n-1}$ integral points, say $\{m_l^q\}$, $1 \leq q \leq (2r)^{n-1}$. We define a function $b$ on $\mathbb{Z}^n$ as

$$b(m_l^q) = (2r)^{-(n-1)}B(l)$$

for $\|a\|r \le l \le \|a\|r, 1 \le q \le (2r)^{n-1}$ and 0 otherwise. Then $b$ is supported on $Q \subset B(\mathbf{0}, ([\sqrt{n}]+1)\|a\|r)$. We note that

$$\sum_{m\in\mathbb{Z}^n} b(m) = \sum_{l,q} b(m_l^q) = \sum_l B(l) = 0$$

by the moment condition of $B$. Moreover,

$$|b(m_l^q)| \le (2r)^{-(n-1)}|B(l)| \le 2\|a\|^{-1}(2r)^{-n}.$$

Therefore, it is easy to see that $2^{n-1}([\sqrt{n}]+1)^{-n}\|a\|^{-n+1}b$ is a $(1,\infty,0)$-atom on $\mathbb{Z}^n$. Last we note that

$$R_a b(l) = \sum_{m\in H(a,l)} b(m) = \sum_q b(m_l^q) = B(l).$$

Hence $B = R_a b = C_{n,a}R_a(C_{n,a}^{-1}b)$, where $C_{n,a} = 2^{-n+1}([\sqrt{n}]+1)^n\|a\|^{n-1}$. □

Let $F \in H^1_{\|a\|}(\mathbb{Z})$ and $F = \sum_i \lambda_i B_i$ an atomic decomposition of $F$, where each atom is supported on an interval with length $2\|a\|r$. Then it follows from Lemma 11 that there exist a collection of $(1,\infty,0)$-atoms $b_i$ on $\mathbb{Z}^n$ so that $B_i = C_{n,a}R_a b_i$. Therefore, $F = R_a(\sum_i \lambda_i C_{n,a} b_i)$ and $\sum_i |\lambda_i| C_{n,a} \le C_{n,a}\|F\|_{H^1}$. Hence, $F$ belongs to the image of $H^1(\mathbb{Z}^n)$.

**Theorem 12.** *For each $a \in \mathcal{P}$, $R_a$ continuously maps $H^1(\mathbb{Z}^n)$ onto $H^1_{\|a\|}(\mathbb{Z})$.*

Now we shall introduce an atomic Hardy space on $\mathbb{G}$. Let $B(m,r) \subset \mathbb{Z}^n$ be a closed ball centered at $m \in \mathbb{Z}^n$ with radius $r \in \mathbb{N}$. We use the notations in §3 and we recall Theorem 4 and (8). We say that a function $B$ on $\mathbb{G}$ is a $(1,\infty,0)$-atom on $\mathbb{G}$ if it satisfies

$$\begin{aligned} &(i)\quad B \in \mathcal{D}_{*,B(m,r)}(\mathbb{G}),\\ &(ii)\quad |B(H(a,k))| \le |H(a,k)\cap B(m,r)|r^{-n} \text{ for all } H(a,k)\in\mathbb{G},\\ &(iii)\quad \sum_{k\in\mathbb{Z}} B(H(a,k)) = 0 \text{ for all } a \in \mathcal{P}, \end{aligned} \tag{9}$$

where $B(m,r)$ depends on $B$. Clearly, if $b$ is a $(1,\infty,0)$-atom on $\mathbb{Z}^n$, then $Rb$ is a $(1,\infty,0)$-atom on $\mathbb{G}$ (see the argument before Proposition 9). Let $B$ be a $(1,\infty,0)$-atom on $\mathbb{G}$. Then by Theorem 4 and its proof, $R^{-1}B$ is supported on $B(m,r)$ and, if $x \in B(m,r)$, then for a sufficiently large $j$, $H(a_j, a_j\cdot x)\cap B(m,r) = \{x\}$ (see Lemma 3). Hence, it follows from (9) that $|R^{-1}B(x)| = |B(H(a_j, a_j\cdot x))| \le r^{-n}$ and thus, $\|R^{-1}B\|_\infty \le r^{-n}$. Moreover,

$$\begin{aligned} \sum_{m\in\mathbb{Z}^n} R^{-1}B(m) &= \sum_{k\in\mathbb{Z}}\sum_{m\in H(a,k)} R^{-1}B(m)\\ &= \sum_{k\in\mathbb{Z}} B(H(a,k)) = 0. \end{aligned}$$

Therefore, it is easy to see that $R^{-1}B$ is a $(1,\infty,0)$-atom on $\mathbb{Z}^n$.

Finally, we define the atomic Hardy space $H^1_{\infty,0}(\mathbb{G})$ on $\mathbb{G}$ as $F \in H^1_{\infty,0}(\mathbb{G})$ if and only if there exist a collection $\{B_i\}$ of $(1,\infty,0)$-atoms on $\mathbb{G}$ and a sequence $\{\lambda_i\}$ of complex numbers with $\sum_i |\lambda_i| < \infty$ so that $F = \sum_i \lambda_i B_i$. We put $\|F\|_{H^1_{\infty,0}} = \inf \sum_i |\lambda_i|$, where the infimum is taken over all atomic decomposition of $F$. Then the previous argument yields the following.

**Theorem 13.** *$R$ is an isomorphism of $H^1(\mathbb{Z}^n)$ onto $H^1_{\infty,0}(\mathbb{G})$.*

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