ON THE INVERSION FORMULA OF JACOBI TRANSFORM

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Abstract. In this paper, we define the complex Jacobi transform for functions which may not decrease rapidly at ∞ and get the corresponding inversion formula in distribution sense.

1. Introduction

For $\alpha,\ \beta, \lambda \in \mathbb{C}$, $\Re\alpha > -1$, the Jacobi function $\phi_\lambda^{\alpha,\beta}(t)$ of order (α,β) is the even C^∞ function on $\mathbb{R}$ which satisfies $\phi_\lambda^{\alpha,\beta}(0) = 1$ and the differential equation

$$(L_{\alpha,\beta} + \lambda^2 + \rho^2)\phi_\lambda^{\alpha,\beta}(t) = 0, \tag{1.1}$$

where $\rho = \alpha+\beta+1$ and

$$L_{\alpha,\beta} = \frac{d^2}{dt^2} + ((2\alpha+1)\coth t + (2\beta+1)\tanh t)\frac{d}{dt}.$$

By using the Jacobi function, for even C^∞ functions f with compact support on $\mathbb{R}$ the Fourier-Jacobi transform is defined by

$$\hat{f}(\lambda) = \int_0^\infty f(t)\phi_\lambda^{\alpha,\beta}(t)\Delta_{\alpha,\beta}(t)\,dt, \tag{1.2}$$

where

$$\Delta_{\alpha,\beta}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}. \tag{1.3}$$

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The corresponding Paley-Wiener theorem and the inversion formula were obtained by Flensted-Jensen[4] when $\alpha \geq \beta \geq -1/2$ and by Koornwinder[6] for all $\alpha, \beta \in \mathbb{C}, \Re\alpha > -1$. Noticing the order of $\phi_\lambda^{\alpha,\beta}(t)$ and $\Delta_{\alpha,\beta}(t)$ (see Lemma 2.1 and (1.3)), we easily see that the inversion formula is applicable to functions of the form $(\cosh t)^{-\mu} g(t)$ where $\mu \in \mathbb{R}$, $\mu \geq \Re\rho$ and g is an even rapidly decreasing functions on $\mathbb{R}$. However, if $\mu < \Re\rho$, the Fourier-Jacobi transform is not well defined since the integral (1.2) is not convergent and thus, the formula does not make sense.

The aim of this paper is to give an inversion formula in the case of $\mu < \Re\rho$. Since the Fourier-Jacobi transform is not well defined, we shall use the so-called complex Jacobi transform defined by

$$\tilde{f}(\lambda) = \int_0^\infty f(t)\Phi_\lambda^{\alpha,\beta}(t)\Delta_{\alpha,\beta}(t)dt,$$

where $\Phi_\lambda^{\alpha,\beta}(t)$ is another solution of (1.1) (see (2.1)). When f is an even C^∞ function with compact support, the inversion formula for the complex Jacobi transform was obtained by Koornwinder [6, 7]. For $f(t) = (\cosh t)^{-\mu} g(t)$ with $\mu < \Re\rho$ the complex Jacobi transform $\tilde{f}(\lambda)$ is well defined if $\Im\lambda > \Re\rho - \mu$, and the corresponding inversion formula holds in a distribution sense (see Theorem 3.1). Moreover, $\tilde{f}(\lambda)$ is extended to a meromorphic function in $\Im\lambda \geq 0$ by analytic continuation. Thereby, we can shift the integral in the inversion formula down to the real line. In this process some residue terms arise from the poles of $\tilde{f}(\lambda)$ and $C(-\lambda)^{-1}$ (see Theorem 3.2).

This idea of the inversion formula for $(\cosh t)^{-\mu} g(t)$ with $\mu < \Re\rho$ comes from [1], in which Dijk and Hille treat the rank one group case where α and β take certain real discrete values and $g \equiv 1$. They obtain the inversion formula for $(\cosh t)^{-\mu}$ and show that the above residue terms correspond to certain complementary series representations of the group.

2. Notations and Preliminaries

For α, $\beta, \lambda \in \mathbb{C}$, $\Re\alpha > -1$, the Jacobi functions $\phi_\lambda^{\alpha,\beta}(t)$ can be expressed by using

Gaussian hypergeometric functions as

$$\phi_\lambda^{\alpha,\beta}(t) = F(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda); \alpha + 1; -\sinh^2 t),$$

and for $\lambda \notin -i\mathbb{N}$ another solution $\Phi_\lambda^{\alpha,\beta}$ of (1.1) is given by

$$\Phi_\lambda^{\alpha,\beta}(t) = (2\sinh t)^{i\lambda-\rho} F(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(-\alpha + \beta + 1 - i\lambda); 1 - i\lambda; -\sinh^{-2} t) \tag{2.1}$$

(cf. [7]). Moreover, for $\lambda \notin i\mathbb{Z}$ we have the identity

$$\phi_\lambda^{\alpha,\beta}(t) = c_{\alpha,\beta}(\lambda)\Phi_\lambda^{\alpha,\beta}(t) + c_{\alpha,\beta}(-\lambda)\Phi_{-\lambda}^{\alpha,\beta}(t), \tag{2.2}$$

where

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\rho))\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))}. \tag{2.3}$$

In the following, if no confusion is possible, we will suppress the parameters α and β in our notations.

We have the following from [4,6,7].

Lemma 2.1. *Assume that $\alpha, \beta \in \mathbb{C}$ and $\Re\alpha > -1$.*

(1) For each fixed $t > 0$, as a function of λ, $\phi_\lambda(t)$ is an entire function. There exists $K > 0$ such that for all $t \geq 0$ and all $\lambda \in \mathbb{C}$,

$$|\phi_\lambda(t)| \leq (1 + |\lambda|)^\epsilon (1+t) e^{(|\Im\lambda| - \Re\rho)t},$$

where $\epsilon = 0$ if $\Re\alpha > -\frac{1}{2}$ and $\epsilon = 1$ for $-1 < \Re\alpha \leq -\frac{1}{2}$.

(2) For each fixed $t > 0$, as a function of λ, $\Phi_\lambda(t)$ is a holomorphic function in $\mathbb{C} - \{-i\mathbb{N}\}$. For each fixed $\lambda \in \mathbb{C} - \{-i\mathbb{N}\}$,

$$\Phi_\lambda(t) = e^{(i\lambda-\rho)t}(1 + o(1)) \text{ as } t \to \infty.$$

(3) $c(-\lambda)^{-1}$ is a meromorphic function in $\mathbb{C}$. For each $r > 0$, if λ is at distance larger than r from the poles of $c(-\lambda)^{-1}$, then there exists $K > 0$ such that

$$|c(-\lambda)^{-1}| \leq K(1 + |\lambda|)^{\Re\alpha + 1/2}.$$

The asymptotic behavior of $\Phi_\lambda(t)$ and $d\Phi_\lambda(t)/dt$ as t approaches 0 is given as follows.

Lemma 2.2. *Assume that* $\alpha, \beta \in \mathbb{C}$ *and* $\Re\alpha > -1$*. For each* $K > 0$ *there exists a function* $a(\lambda)$ *such that for* $0 < t < K$

$$|\Phi_\lambda(t)| \le a(\lambda)\begin{cases} t^{-2\Re\alpha}, & \Re\alpha > 0 \\ |\log t|, & \Re\alpha = 0 \\ 1, & -1 < \Re\alpha < 0 \end{cases} \tag{2.4}$$

and

$$\left|\frac{d}{dt}\Phi_\lambda(t)\right| \le a(\lambda) t^{-2\Re\alpha - 1}. \tag{2.5}$$

Here $a(\lambda)$ *is different in each appearance.*

Proof. We use the fact that if $\Re(c-a-b) > 0$ and $c \ne 0, -1, -2, \cdots$, then $\lim_{x\to 1_-} F(a,b,c;x)$ exists (see [2,§2.8, (46)]).

When $\Re\alpha < 0$, it follows from (2.1) that

$$\Phi_\lambda(t) = (2\cosh t)^{i\lambda-\rho} F(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\alpha - \beta + 1 - i\lambda), 1 - i\lambda; \cosh^{-2} t).$$

Since $\Re(1 - i\lambda - \frac{1}{2}(\rho - i\lambda) - \frac{1}{2}(\alpha - \beta + 1 - i\lambda)) = -\Re\alpha > 0$, if $\lambda \ne -i, -2i, \ldots$, then $\lim_{t\to 0}\Phi_\lambda(t)$ exists, i.e., for a certain function $a(\lambda)$, $|\Phi_\lambda(t)| \le a(\lambda)$ as $0 < t < K$.

When $\Re\alpha > 0$, (2.1) and the formula

$$F(a,b;c;z) = (1-z)^{-b} F(c-a, b; c; \frac{z}{z-1})$$

(see [2, §2.1, (22)]) yield that

$$\begin{aligned}\Phi_\lambda(t) = {} & 2^{i\lambda-\rho} F(1 - \frac{1}{2}(\rho + i\lambda), \frac{1}{2}(-\alpha + \beta + 1 - i\lambda); 1 - i\lambda; \cosh^{-2} t) \\ & \cdot (\cosh^{-2} t)^{(-\alpha+\beta+1-i\lambda)/2} (\sinh t)^{-2\alpha}.\end{aligned}$$

Since $\Re(1 - i\lambda - \frac{1}{2}(-\alpha + \beta + 1) - 1 - \frac{1}{2}(\rho + i\lambda)) = \Re\alpha > 0$, if $\lambda \ne -i, -2i, \ldots$, then $\lim_{t\to 0} F(\frac{1}{2}(-\alpha + \beta + 1), 1 - \frac{1}{2}(\rho + i\lambda); 1 - i\lambda; \cosh^{-2} t)$ exists and $|\Phi_\lambda(t)| \le a(\lambda) t^{-2\Re\alpha}$ as $0 < t < K$.

When $\Re\alpha = 0$, we use the formula

$$az^{a-1} F(a+1, b; c; z) = \frac{d}{dz}(z^a F(a,b;c;z))$$

(see [2, §2.8, (21)]). Then we have

$$\frac{d}{dt}\Phi_\lambda^{\alpha,\beta}(t) = 2(i\lambda - \rho)\sinh 2t \cdot \Phi_\lambda^{\alpha+1,\beta+1}(t). \tag{2.6}$$

Since $|\Phi_\lambda^{\alpha+1,\beta+1}(t)| \le a(\lambda)t^{-2}$ as $0 < t < K$ by the previous case, we get $|\Phi_\lambda^{\alpha,\beta}(t)| \le a(\lambda)|\log t|$ as $0 < t < K$. Also (2.5) follows from (2.6). □

Let $D_e(\mathbb{R})$ be the space of even C^∞ functions with compact support on $\mathbb{R}$. For $f \in D_e(\mathbb{R})$ the Fourier-Jacobi transform $\hat{f}(\lambda)$ ($\lambda \in \mathbb{C}$) is defined by (1.2). Then the following lemmas were obtained in [6,7].

Lemma 2.3. *Assume $\alpha, \beta \in \mathbb{C}$ and $\Re\alpha > -1$,*

(1) For each $f \in D_e(\mathbb{R})$ the Fourier-Jacobi transform $\hat{f}(\lambda)$ is an even entire function and there are positive constants A and C_n ($n = 0, 1, 2, \ldots$) such that

$$|\hat{f}(\lambda)| \le C_n(1 + |\lambda|)^{-n} e^{A|\Im\lambda|}.$$

(2) For each $f \in D_e(\mathbb{R})$, $\nu \ge 0$, and $\nu > -\Re(\alpha \pm \beta + 1)$,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda + i\nu)\Phi_{\lambda+i\nu}(t)c(-\lambda - i\nu)^{-1}d\lambda, \quad t > 0.$$

It follows from (2.3) that all the singular points of the meromorphic function $c(-\lambda)^{-1}$ in $\Im\lambda \ge 0$ lie in the set

$$D_{\alpha,\beta} = \{i(\varepsilon\beta - \alpha - 1 - 2m) \ne 0 : m = 0, 1, 2, \ldots, \Re(\varepsilon\beta - \alpha - 1 - 2m) \ge 0\},$$

where $\varepsilon = 1$ if $\Re\beta > 0$ and $\varepsilon = -1$ if $\Re\beta < 0$. In particular, if $\alpha, \beta \in \mathbb{R}$ and $\alpha > -1$, then

$$D_{\alpha,\beta} = \{i(|\beta| - \alpha - 1 - 2m) : m = 0, 1, 2, \ldots, |\beta| - \alpha - 1 - 2m) > 0\} \subset i\mathbb{R}_+.$$

The residue of $c(-\lambda)^{-1}$ at $\gamma = i(\varepsilon\beta - \alpha - 1 - 2m)$ is

$$\begin{aligned} R_{c^{-1}}(\gamma) &= \operatorname*{Res}_{\nu=\gamma} c(-\nu)^{-1} \\ &= \frac{2i}{(-1)^m m! 2^{(1-\varepsilon)\beta+2\alpha+2+2m}} \frac{\Gamma(\varepsilon\beta - m)}{\Gamma(\alpha+1)\Gamma(\varepsilon\beta - \alpha - 1 - 2m)}. \end{aligned}$$

Lemma 2.4. *Assume $\alpha, \beta \in \mathbb{C}$ and $\Re\alpha > -1$. If $D_{\alpha,\beta} \cap \{\Im\lambda = 0\}$ is empty, then for $f \in \mathcal{D}_e(\mathbb{R})$, the inversion Fourier-Jacobi transform is given by*

$$f(t) = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(t)(c(\lambda)c(-\lambda))^{-1}d\lambda - i\sum_{\gamma\in D_{\alpha,\beta}} \frac{\hat{f}(\gamma)}{c(\gamma)}\phi_\gamma(t)R_{c^{-1}}(\gamma).$$

3. Complex Jacobi transform

The complex Jacobi transform $f \to \tilde{f}$ is defined by

$$\tilde{f}(\lambda) = \int_0^\infty f(t)\Phi_\lambda(t)\Delta(t)\,dt$$

for all f on $\mathbb{R}_+$ and $\lambda \in \mathbb{C}$ for which the right hand side is well defined. If $\tilde{f}(\pm\lambda)$ and $\hat{f}(\lambda)$ are well defined, we have from (2.2) that

$$\hat{f}(\lambda) = c(\lambda)\tilde{f}(\lambda) + c(-\lambda)\tilde{f}(-\lambda). \tag{3.1}$$

Remark. When $\alpha = \beta = -\frac{1}{2}$, $\phi_\lambda(t) = \cos\lambda t$ and $\Phi_\lambda(t) = e^{i\lambda t}$. The Fourier-Jacobi transform $\hat{f}$ is reduced to Fourier-cosine transform and the complex Jacobi transform to the complex Fourier transform.

For $\mu \in \mathbb{R}$ and $k = 0, 1, 2, \cdots$ let $C_\mu^k(\mathbb{R})$ denote the set of functions of the form

$$(\cosh t)^{-\mu} g(\cosh^{-2} t),$$

where g is a k-th differentiable function on $[0, 1]$ and there exists $K > 0$ such that

$$\sup_{0\le t\le 1}\left|\frac{d^n g}{dt^n}(t)\right| \le K \quad (0 \le n \le k).$$

Let $f \in C_\mu^0(\mathbb{R})$, i.e., there exists $K > 0$ such that

$$|f(t)| \le K(\cosh t)^{-\mu}.$$

We see from Lemma 2.1 (2), Lemma 2.2 and (1.3) that if $\Im\lambda > \max\{\Re\rho - \mu, -1\}$, then $\tilde{f}(\lambda)$ is well defined. Moreover, it is a holomorphic function in $\Im\lambda > \max\{\Re\rho - \mu, -1\}$ and bounded in any closed horizontal strips in $\Im\lambda > \max\{\Re\rho - \mu, -1\}$.

We have the following.

Theorem 3.1. *Assume $\alpha, \beta \in \mathbb{C}$ and $\Re\alpha > -1$. Let $f \in C^0_\mu(\mathbb{R})$ and let $\nu \geq 0$, $\nu > -\Re(\alpha \pm \beta + 1)$ and $\nu > \Re\rho - \mu$. Then for all $\phi \in \mathcal{D}_e(\mathbb{R})$,*

$$\langle f, \phi \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\lambda + i\nu) \tilde{f}(\lambda + i\nu) c(-\lambda - i\nu)^{-1} d\lambda. \tag{3.2}$$

Proof. Since ν satisfies the condition in Lemma 2.3(2), we see that for all $\phi \in \mathcal{D}_e(\mathbb{R})$

$$\phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\lambda + i\nu) \Phi_{\lambda + i\nu}(t) c(-\lambda - i\nu)^{-1} d\lambda.$$

Moreover, since $\nu \geq 0$ and $\nu > \Re\rho - \mu$, the complex Jacobi transform $\tilde{f}(\lambda)$ is bounded if $\Im\lambda > \nu$. Therefore, by Fubini theorem, we have

$$\begin{aligned} \langle f, \phi \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\lambda + i\nu) \int_0^{\infty} f(t) \Phi_{\lambda + i\nu}(t) \Delta(t) dt (c(-\lambda - i\nu))^{-1} d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\lambda + i\nu) \tilde{f}(\lambda + i\nu) c(-\lambda - i\nu)^{-1} d\lambda. \end{aligned}$$

□

We suppose that $\mu \in \mathbb{R}$ and $\Re\rho - \mu > 0$. As said before, if $f \in C^0_\mu(\mathbb{R})$, the complex Jacobi transform $\tilde{f}(\lambda)$ is a holomorphic function in $\Im\lambda > \Re\rho - \mu$. Now we assume certain smoothness of f and show that $\tilde{f}(\lambda)$ has a meromorphic extension to $\Im\lambda \geq 0$.

We put

$$k_\mu = \min\{k : k = 1, 2, \cdots, \Re\rho - \mu - 2k < 0\},$$

and let $f \in C^{k_\mu}_\mu(\mathbb{R}) \subset C^0_\mu(\mathbb{R})$. Then for $\Im\lambda > \Re\rho - \mu$

$$\begin{aligned} \tilde{f}(\lambda) &= \int_0^{\infty} (\cosh t)^{-\mu} g(\cosh^{-2} t) \Phi_\lambda(t) \Delta(t) dt \\ &= 2^{i\lambda + \rho - 1} \int_0^{\infty} g(\cosh^{-2} t) F(\frac{1}{2}(-\alpha + \beta + 1 - i\lambda), 1 - \frac{1}{2}(\rho + i\lambda); 1 - i\lambda; \cosh^{-2} t) \\ &\quad \cdot (\cosh^{-2} t)^{\frac{\mu}{2} - 1 - \frac{1}{2}(i\lambda + \rho)} d(\cosh^{-2} t) \\ &= 2^{i\lambda + \rho - 1} \int_0^{1} g(y) F(\frac{1}{2}(-\alpha + \beta + 1 - i\lambda), 1 - \frac{1}{2}(\rho + i\lambda); 1 - i\lambda; y) y^{\frac{\mu}{2} - 1 - \frac{1}{2}(i\lambda + \rho)} dy \\ &= 2^{i\lambda + \rho - 1} \int_0^{1} \sum_{l=0}^{\infty} \frac{(\frac{1}{2}(-\alpha + \beta + 1 - i\lambda))_l (1 - \frac{1}{2}(\rho + i\lambda))_l}{(1 - i\lambda)_l l!} g(y) y^{l + \frac{\mu}{2} - 1 - \frac{1}{2}(i\lambda + \rho)} dy \\ &= 2^{i\lambda + \rho - 1} \sum_{l=0}^{\infty} \frac{(\frac{1}{2}(-\alpha + \beta + 1 - i\lambda))_l (1 - \frac{1}{2}(\rho + i\lambda))_l}{(1 - i\lambda)_l l!} A_\lambda(l) \end{aligned} \tag{3.3}$$

where

$$A_\lambda(l) = \int_0^1 g(y) y^{l+\frac{\mu}{2}-1-\frac{1}{2}(i\lambda+\rho)} dy.$$

Since $|A_\lambda(l)| \le K \int_0^1 y^{l-1+\frac{1}{2}(\Im\lambda - \Re\rho+\mu)} dy \sim l^{-1}$ as $l \to \infty$ and by Stirling' formula, for l large,

$$\frac{(\frac{1}{2}(-\alpha+\beta+1-i\lambda))_l (1-\frac{1}{2}(\rho+i\lambda))_l}{(1-i\lambda)_l l!} \sim l^{-1-\alpha} \text{ as } l \to \infty,$$

the series in (3.3) is convergent for $\Im\lambda > \Re\rho - \mu$.

Now we consider a meromorphic extension of $\hat{f}(\lambda)$. Since $k_\mu \ge 1$, integral by parts yields that

$$A_\lambda(l) = \frac{1}{l+\frac{\mu}{2}-\frac{1}{2}(i\lambda+\rho)} \left(g(1) - \int_0^1 g'(y) y^{l+\frac{\mu}{2}-\frac{1}{2}(i\lambda+\rho)} dy \right). \tag{3.4}$$

Here the right hand of (3.4) is well defined for $\Im\lambda > \Re\rho - \mu - 2l - 2$. Thereby, repeating integral by parts, we can extend $A_\lambda(l)$ to $\Im\lambda > \Re\rho - \mu - 2l - 2k_\mu$, with possible simple poles at $i(\rho - 2\mu - 2l - 2n)$, $n = 0, 1, 2, \ldots$. The residue of $A_\lambda(l)$ at $\gamma = i(\rho - 2\mu - 2l - 2n)$ is

$$R_g(\gamma) = \operatorname*{Res}_{\lambda=\gamma} A_\lambda(l) = \frac{2i}{n!} g^{(n)}(0).$$

Let $\Im\lambda > \max\{-1, \Re\rho - \mu - 2k_\mu\}$ and λ be not a singular point for any $A_\lambda(l)$. Since $A_\lambda(l) = O(l^{-1})$ as $l \to \infty$, the series (3.3) is absolutely convergent. Thus the right hand side of (3.3) defines a meromorphic extension of $\tilde{f}(\lambda)$ to $\Im\lambda > \max\{-1, \Re\rho - \mu - 2k_\mu\}$ (< 0). All the singular points of the meromorphic extension $\tilde{f}(\lambda)$ in $\Im\lambda \ge 0$ lie in the set

$$F^\mu_{\alpha,\beta} = \{i(\rho - \mu - 2m) : m = 0, 1, 2, \ldots, \Re(\rho - \mu - 2m) \ge 0\}$$

and the residue of $\tilde{f}(\lambda)$ at $\gamma = i(\rho - \mu - 2m)(m > k_\mu)$ is

$$R_f(\gamma) = \operatorname*{Res}_{\lambda=\gamma} \tilde{f}(\lambda) = \sum_{0 \le l \le m} 2^{\mu+2m-1} \frac{(\beta+1-\mu/2-m)_l (1-\mu/2-m)_l}{(1+\rho-\mu-2m)_l l!} R_g(\gamma).$$

Moreover, $\tilde{f}(\lambda)$ is bounded if λ is at certain distance from the poles and in any horizontal strips in $\Im\lambda > \Re\rho - \mu - 2k_\mu$.

Now we consider the shift of the integral in (3.2) down to $\Im\lambda = 0$. We retain the notation in (3.2). We recall that $\hat{\phi}(\lambda)$ is an entire function of exponential type (see Lemma 2.3 (1)), and $c(-\lambda)^{-1}$ is meromorphic in $\Im\lambda \geq 0$ and bounded at certain distance from the poles in the set $D_{\alpha,\beta}$ (see Lemma 2.1 (3)). Hence, for $f \in C_\mu^{k_\mu}(\mathbb{R})$, $\hat{\phi}(\lambda)\tilde{f}(\lambda)c(-\lambda)^{-1}$ is a meromorphic function in $\Im\lambda \geq 0$ with poles at $D_{\alpha,\beta} \cup F^\mu_{\alpha,\beta}$. Moreover, it is rapidly decreasing in any horizontal strips in $\Im\lambda \geq 0$. Therefore, if $D_{\alpha,\beta} \cup F^\mu_{\alpha,\beta}$ has no intersection with $\Im\lambda = 0$, then we can shift the integral in (3.2) down to $\Im\lambda = 0$ and obtain the following.

Theorem 3.2. *Assume α, $\beta \in \mathbb{C}$ and $\Re\alpha > -1$. Let $\mu \in \mathbb{R}$, $\mu \leq \Re\rho$ and let $f \in C_\mu^{k_\mu}(\mathbb{R})$. If $F_{\mu,g}$ and $D_{\alpha,\beta}$ have no intersection with $\Im\lambda = 0$, then for all $\phi \in \mathcal{D}_e(\mathbb{R})$,*

$$\langle f, \phi\rangle = -i \sum_{\gamma \in D_{\alpha,\beta} \cup F^\mu_{\alpha,\beta}} \hat{\phi}(\gamma) \operatorname*{Res}_{\lambda=\gamma} \frac{\tilde{f}(\lambda)}{c(-\lambda)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{\phi}(\lambda)\tilde{f}(\lambda)}{c(-\lambda)} d\lambda.$$

Moreover, if $D_{\alpha,\beta} \cap F^\mu_{\alpha,\beta}$ is empty, we have

$$\langle f, \phi\rangle = -i \sum_{\gamma \in F^\mu_{\alpha,\beta}} \frac{\hat{\phi}(\gamma)}{c(-\gamma)} R_f(\gamma) - i \sum_{\gamma \in D_{\alpha,\beta}} \hat{\phi}(\gamma)\tilde{f}(\gamma)R_{c^{-1}}(\gamma) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{\phi}(\lambda)\tilde{f}(\lambda)}{c(-\lambda)} d\lambda.$$

Remark (1) In the case $\mu > \Re\rho$ it is easy to see that, for $f \in C^0_\mu(\mathbb{R})$, the Fourier-Jacobi transform $\hat{f}(\lambda)$ is well defined for $\lambda \in \mathbb{R}$ and therefore, (3.1) is valid on $\mathbb{R}$. In the case $\mu \leq \Re\rho$, even if $f \in C_\mu^{k_\mu}(\mathbb{R})$, $\hat{f}(\lambda)$ is not well defined on $\mathbb{R}$, however, the complex Jacobi transform $\tilde{f}(\lambda)$ is defined on $\mathbb{R}$ by analytic continuation. Thereby, if we define $\hat{f}(\lambda)$ on $\mathbb{R}$ by (3.1), then the integral in the inversion formula can be rewritten as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{\phi}(\lambda)\tilde{f}(\lambda)}{c(-\lambda)} d\lambda = \frac{1}{2\pi}\int_0^{\infty} \hat{\phi}(\lambda)\hat{f}(\lambda)(c(\lambda)c(-\lambda))^{-1} d\lambda.$$

(2) We consider the special case $f(t) = f_\mu(t) = (\cosh t)^{-\mu}$. As above, if $\mu > \Re\rho$, then $\hat{f}_\mu(\lambda)$ and $\tilde{f}_\mu(\lambda)$ are well defined on $\mathbb{R}$ and (3.1) is valid on $\mathbb{R}$. Furthermore, by using the same arguments as in [1,§6 and §7], $\hat{f}_\mu$ and $\tilde{f}_\mu$ have meromorphic extensions of μ

by analytic continuation. Hence, (3.1) is valid on $\mathbb{R}$ for all μ. Actually, even if $\alpha, \beta \in \mathbb{C}$ and $\Re\alpha > -1$, $\hat{f}_\mu(\lambda)$ is explicitly given by

$$\hat{f}_\mu(\lambda) = 2^{2\rho-1}\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}(\mu-\rho+i\lambda))\Gamma(\frac{1}{2}(\mu-\rho-i\lambda))}{\Gamma(\frac{\mu}{2}-\beta)\Gamma(\frac{\mu}{2})},$$

and $\tilde{f}_\mu$ satisfies a reduction formula on μ:

$$\tilde{f}_\mu(\lambda) = \frac{b(\lambda) + \mu(\mu-2\beta)\tilde{f}_{\mu+2}(\lambda)}{\lambda^2 + (\rho-\mu)^2}$$

where

$$b(\lambda) = \lim_{t\to 0}\frac{\partial\Phi_\lambda}{\partial t}(t)\Delta(t).$$

This limit exists from (1.3) and (2.5).

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