Refinements of the Hardy and Morgan uncertainty principles

T. Kawazoe, J. Liu, A. Miyachi

Abstract

Various generalizations of Hardy's theorem and Morgan's theorem, which assert that a function on $\mathbb{R}$ and its Fourier transform cannot both be very small, are known. We give two theorems which improve various generalizations known so far.

1 Introduction

For an integrable function f on $\mathbb{R}$, we define the Fourier transform $\hat{f}$ by

$$\hat{f}(y) = \int_{-\infty}^{+\infty} f(x)e^{-ixy}dx, \quad y \in \mathbb{R}.$$

Classical Hardy's theorem [4] reads as follows: if $a, b > 0$, $ab = 1/4$, and if f is a measurable function on $\mathbb{R}$ such that

$$f(x)e^{ax^2} \in L^\infty(\mathbb{R}) \quad \text{and} \quad \hat{f}(y)e^{by^2} \in L^\infty(\mathbb{R}), \tag{1}$$

then f is a constant multiple of e^{-ax^2}. An immediate corollary of this theorem is the following: if $a, b > 0$, $ab > 1/4$, and if f is a measurable function on $\mathbb{R}$ satisfying (1), then $f = 0$ almost everywhere. The examples $f(x) = e^{ax^2}P(x)$ with $P(x)$ polynomials show that there are infinitely many f's that satisfy (1) for $ab < 1/4$. Morgan [6] proved the following variant of Hardy's theorem: if $1 < \beta < 2 < \alpha < \infty$, $1/\alpha + 1/\beta = 1$, $a, b > 0$, and

$$(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi(\beta-1)/2))^{1/\beta}, \tag{2}$$

and if f is a measurable function on $\mathbb{R}$ satisfying

$$f(x)e^{a|x|^\alpha} \in L^\infty(\mathbb{R}) \quad \text{and} \quad \hat{f}(y)e^{b|y|^\beta} \in L^\infty(\mathbb{R}), \tag{3}$$

then $f = 0$ almost everywhere. He also obtained that the condition (2) is optimal; if $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = (\sin(\pi(\beta-1)/2))^{1/\beta}$, then for any $m \in \mathbb{R}$ and $m' = (2m-\alpha+2)/(2\alpha-2)$, there exists a measurable function f on $\mathbb{R}$ such that $(1+|x|)^{-m}f(x)e^{a|x|^\alpha} \in L^\infty(\mathbb{R})$ and $(1+|y|)^{-m'}\hat{f}(y)e^{b|y|^\beta} \in L^\infty(\mathbb{R})$. Therefore, there are infinitely many f's that satisfy (3).

Various generalizations of Hardy's theorem and Morgan's theorem are known. Cowling and Price [2] proved that, if in Hardy's theorem the assumption (1) is replaced by

$$f(x)e^{ax^2} \in L^p(\mathbb{R}) \quad \text{and} \quad \hat{f}(y)e^{by^2} \in L^q(\mathbb{R})$$

with $1 \leqq p, q \leqq \infty$ and with at least one of p and q finite, then $f = 0$. The third author proved that (see [5], Theorem 1), if $a, b > 0$, $ab = 1/4$, and if f is a measurable function on $\mathbb{R}$ such that

$$f(x)e^{ax^2} \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \quad \text{and} \quad \int_{-\infty}^{+\infty} \log^+ \frac{|\hat{f}(y)e^{by^2}|}{C} dy < \infty$$

for some $C > 0$, then f is a constant multiple of e^{-ax^2}. Here $L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ is the set of functions of the form $f = f_1 + f_2$, $f_1 \in L^1(\mathbb{R})$, $f_2 \in L^\infty(\mathbb{R})$, and $\log^+ x = \log x$ if $x > 1$ and $\log^+ x = 0$ if $x \le 1$. Ben Farah and Mokni [1] proved that, if we replace L^∞ in the assumptions of Morgan's theorem by L^p and L^q, $1 \le p, q \le \infty$, then $f = 0$ and the condition (2) is optimal.

The purpose of the present paper is to give further generalizations of the above theorems. Our results are the following two theorems.

THEOREM 1 *Let $1 < \alpha, \beta < \infty$, $1/\alpha + 1/\beta = 1$, $a, b > 0$, and*

$$(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > c(\alpha, \beta) \tag{4}$$

with

$$c(\alpha,\beta) = \begin{cases} (\sin(\pi(\beta-1)/2))^{1/\beta} & \text{if} \quad \beta < 2, \\ (\sin(\pi(\alpha-1)/2))^{1/\alpha} & \text{if} \quad \beta > 2. \end{cases} \tag{5}$$

Suppose f is a measurable function on $\mathbb{R}$ such that

$$e^{a|x|^\alpha} f(x) \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \tag{6}$$

and

$$\int_{-\infty}^{+\infty} \log^+ \frac{|\hat{f}(y)|e^{b|y|^\beta}}{C} \frac{dy}{1+|y|} < \infty \tag{7}$$

for some $C > 0$. Then $f = 0$ almost everywhere.

THEOREM 2 *If $a, b > 0$, $ab = 1/4$, and if f is a measurable function on $\mathbb{R}$ that satisfies (6) and (7) with $\alpha = \beta = 2$, then $f(x)$ is a constant multiple of e^{-ax^2}.*

REMARK 3 (a) If the conditions (4) and (6) are satisfied and if we take $a' < a$ sufficiently near to a, then (4) is still satisfied with a' in place of a and the condition (6) implies

$$f(x)e^{a'|x|^\alpha} = f(x)e^{a|x|^\alpha}e^{(a'-a)|x|^\alpha} \in L^1(\mathbb{R}).$$

Hence the essential claim of Theorem 1 remains unchanged if the assumption (6) is replaced by the seemingly stronger assumption $f(x)e^{a|x|^\alpha} \in L^1(\mathbb{R})$.
(b) It is easy to see that (3) or its L^p-L^q-version implies (6) and (7). Therefore, L^p-L^q Morgan's theorem follows from Theorem 1.
(c) Theorem 2 is an improvement of the third author's Theorem 1 in [5], where the condition (7) was assumed with dy instead of $dy/(1 + |y|)$.
(d) Similarly as Morgan's result, the condition (4) is optimal.

In §3 we shall prove Theorems 1 and 2. Part of the argument will be only a slight modification of that of [5]. Since the paper [5] was published in a proceedings of a local seminar in Japan and is not easy to refer, we shall repeat some argument of [5] for convenience of the reader.

2 Key lemmas

For $-\infty < \alpha < \beta < \infty$, we write

$$D(\alpha, \beta) = \{z \mid \alpha < \arg z < \beta\},$$

which is the domain in the Riemann surface of $\log z$. We shall give three lemmas. The first lemma is an improvement of Lemma 1 of [5], where the integral (8) below is taken with respect to ds instead of ds/s.

LEMMA 4 *Let $-\infty < \alpha < \beta < \infty$ and f be a bounded holomorphic function on $D(\alpha, \beta)$. Then for each θ with $\alpha < \theta < \beta$,*

$$\begin{aligned} &\sup_{0<r<\infty} \log|f(re^{i\theta})| \qquad (8) \\ &\leq c_+(\alpha,\beta,\theta)\int_0^\infty \log^+|f(se^{i\alpha})|\frac{ds}{s} + c_-(\alpha,\beta,\theta)\int_0^\infty \log^+|f(se^{i\beta})|\frac{ds}{s}, \end{aligned}$$

where

$$c_\pm(\alpha,\beta,\theta) = \frac{1 \pm \cos\frac{\pi(\theta-\alpha)}{\beta-\alpha}}{2(\beta-\alpha)\sin\frac{\pi(\theta-\alpha)}{\beta-\alpha}}$$

and $f(se^{i\alpha})$ and $f(se^{i\beta})$ denote the nontangential boundary values of $f(z)$.

Proof. Let $\delta = (\beta - \alpha)/\pi$. For $z = re^{i\theta} \in D(\alpha, \beta)$, we make a change of variables as $z = e^{i\alpha} w^{\delta}$. Then $w \in D(0, \pi)$ and $g(w) = f(z) = f(e^{i\alpha} w^{\delta})$ is a bounded holomorphic function on the upper half plane. Let $P_w(t) = \Im w/(\pi |w - t|^2)$ be the Poisson kernel for the upper half plane. Then Jensen's inequality (cf. [3], Chap. II, §4, p.65) gives

$$\begin{aligned}
&\log |f(z)| = \log |g(w)| \leq \int_{-\infty}^{\infty} P_w(t) \log |g(t)| dt \\
&\leq \int_{-\infty}^{\infty} P_w(t) \log^+ |g(t)| dt \\
&= \int_{-\infty}^{\infty} P_w(t) \log^+ |f(e^{i\alpha} t^{\delta})| dt \\
&= \int_{0}^{\infty} P_w(t) \log^+ |f(e^{i\alpha} t^{\delta})| dt + \int_{0}^{\infty} P_w(-t) \log^+ |f(e^{i\beta} t^{\delta})| dt \\
&= \frac{1}{\delta} \int_{0}^{\infty} P_w(t^{1/\delta}) t^{1/\delta} \log^+ |f(e^{i\alpha} t)| \frac{dt}{t} \\
&\quad + \frac{1}{\delta} \int_{0}^{\infty} P_w(-t^{1/\delta}) t^{1/\delta} \log^+ |f(e^{i\beta} t)| \frac{dt}{t}.
\end{aligned}$$

If we write $w = (re^{i(\theta - \alpha)})^{1/\delta} = u + iv$, then

$$\begin{aligned}
&\max_{0 < s < \infty} \{ s P_w(\pm s) \} = \left[\frac{vs}{\pi((u \mp s)^2 + v^2)} \right]_{s = \sqrt{u^2 + v^2}} \\
&= \frac{v}{2\pi(\sqrt{u^2 + v^2} \mp u)} = \frac{\sqrt{u^2 + v^2} \pm u}{2\pi v} = \delta c_{\pm}(\alpha, \beta, \theta).
\end{aligned}$$

Hence the desired inequality follows. ■

LEMMA 5 *Let $0 < \beta - \alpha < \pi/\rho$ and f be a holomorphic function on $D(\alpha, \beta)$. Suppose that there exist constants $A, B > 0$ such that*

$$|f(z)| \leq A e^{B|z|^{\rho}}$$

for all $z \in D(\alpha, \beta)$. Then (8) holds for each θ with $\alpha < \theta < \beta$.

Proof. By a rotation of the variable, we may suppose that $\alpha = -\beta$ and $0 < \beta < \pi/(2\rho)$. Take a γ such that $\gamma > \rho$ and $\gamma\beta < \pi/2$. For $\epsilon > 0$, set $f_{\epsilon}(z) = f(z) e^{-\epsilon z^{\gamma}}$. Then f_{ϵ} is holomorphic on $D = D(-\beta, \beta)$. Moreover, if $z \in D$ and $\phi = \arg z$, then

$$|f_{\epsilon}(z)| = |f(z)| e^{-\epsilon |z|^{\gamma} \cos \gamma\phi} \leq A e^{B|z|^{\rho} - \epsilon |z|^{\gamma} \cos \gamma\beta}.$$

Since $\gamma > \rho$ and $\cos\gamma\beta > 0$, it follows that f_ϵ is bounded on D. Hence (8) holds with f replaced by f_ϵ. We note that $|f_\epsilon(z)| \le |f(z)|$ on D and $f_\epsilon(z) \to f(z)$ as $\epsilon \to 0$. Hence, letting $\epsilon \to 0$, we have the desired inequality. ■

The last lemma is well known as the Phragmén-Lindelöf theorem, which can be proved by an application of Lemma 5 to $f(z)/M$.

LEMMA 6 *Let α, β, ρ and f satisfy the same assumptions as in Lemma 5. Assume in addition that there exists a constant M such that $|f(z)| \le M$ on the boundary of $D(\alpha,\beta)$. Then $|f(z)| \le M$ for all $z \in D(\alpha,\beta)$.*

3 Proof of Theorem 1

We shall use the notation

$$l(\theta) = \{re^{i\theta} \mid r > 0\}, \quad \theta \in \mathbb{R}.$$

Let a, b, α, β, and f satisfy the assumptions of Theorem 1. As noted in Remark 3 (a), by replacing a with a smaller constant if necessary, we may assume that $f(t)e^{a|t|^\alpha} \in L^1(\mathbb{R})$. Thus $f(t)$, $t \in \mathbb{R}$, is of the form $f(t) = f_1(t)e^{-a|t|^\alpha}$ with $f_1 \in L^1(\mathbb{R})$.

We define $\hat{f}(z)$ for $z \in \mathbb{C}$ by

$$\hat{f}(z) = \int_{-\infty}^{+\infty} f(t)e^{-izt}dt. \tag{9}$$

For $z = x + iy \in \mathbb{C}$,

$$|\hat{f}(z)| \le \int_{-\infty}^{\infty} |f_1(t)|e^{-a|t|^\alpha}e^{yt}dt.$$

Using Young's inequality $u^\alpha/\alpha + v^\beta/\beta \ge uv$ for $u, v > 0$ with $u = (\alpha a)^{1/\alpha}|t|$ and $v = |y|/(\alpha a)^{1/\alpha}$, we have $a|t|^\alpha + |y|^\beta/(\beta(a\alpha)^{\beta/\alpha}) \ge |y||t|$ and thus

$$\int_{-\infty}^{\infty} |f_1(t)|e^{-a|t|^\alpha}e^{|y||t|}dt \le e^{|y|^\beta/(\beta(a\alpha)^{\beta/\alpha})}\|f_1\|_1.$$

Combining the above inequalities, we see that there exists a constant c such that

$$|\hat{f}(x+iy)| \le ce^{A|y|^\beta}, \quad A = 1/(\beta(a\alpha)^{\beta/\alpha}). \tag{10}$$

It is also easy to see that $\hat{f}(z)$ is an entire holomorphic function.

We shall consider the two cases $\beta < 2$ and $\beta > 2$ separately.

Case I: $1 < \beta < 2$. In this case the condition (4) with (5) implies

$$A(-\cos \pi\beta/2) < b.$$

Since $-\cos \pi\beta/2 > 0$, we can take a sufficiently small $\epsilon > 0$ such that $0 < \epsilon < \pi/2\beta$ and

$$\begin{aligned} A <&(-\cos \pi\beta/2)^{-1} b\Big(\frac{\tan(\pi\beta/2+\beta\epsilon)}{\tan \pi\beta/2}\sin^2 \pi\beta/2 + \cos^2 \pi\beta/2\Big) \\ =&-b\tan(\pi\beta/2+\beta\epsilon)\sin \pi\beta/2 - b\cos \pi\beta/2 \\ =&v\sin \pi\beta/2 - b\cos \pi\beta/2, \end{aligned} \tag{11}$$

where we set

$$v = -b\tan(\pi\beta/2+\beta\epsilon). \tag{12}$$

We set

$$\theta_\epsilon = \pi/2 - \pi/2\beta + \epsilon.$$

Notice that $0 < \theta_\epsilon < \pi/2$.

We shall prove that $\hat{f}$ is bounded on $l(\theta_\epsilon)$. To prove this, consider the function

$$g(z) = \hat{f}(z)e^{(b+iv)z^\beta}, \quad z \in D(0,\pi/2).$$

By (10), there exists $B > 0$ such that

$$|g(z)| \leq ce^{B|z|^\beta} \tag{13}$$

for $z \in D(0,\pi/2)$. Since $g(x)$, $x \in \mathbb{R}$, is bounded on a neighborhood of $x = 0$, the condition (7) implies that there exists a constant $C' > 0$ such that

$$\int_0^\infty \log^+ \frac{|g(x)|}{C'}\frac{dx}{x} < \infty. \tag{14}$$

For $z = re^{i\pi/2}$, $r > 0$, from (10) and (11) we have

$$|g(z)| \leq ce^{r^\beta(A+b\cos \pi\beta/2 - v\sin \pi\beta/2)} \leq c. \tag{15}$$

Since $\pi/2 < \pi/\beta$, we can apply Lemma 5 to g on $D(0,\pi/2)$ to see that $g(z)$ is bounded on each half line $l(\theta)$ with $0 < \theta < \pi/2$. For $z = re^{i\theta_\epsilon}$, $r > 0$, (12) gives

$$\begin{aligned} |\hat{f}(z)| &= |g(z)||e^{-(b+iv)z^\beta}| = |g(z)|e^{-r^\beta\{b\cos\beta\theta_\epsilon - v\sin\beta\theta_\epsilon\}} \\ &= |g(z)|e^{-r^\beta\{b\sin(\pi\beta/2+\beta\epsilon)+v\cos(\pi\beta/2+\beta\epsilon)\}} = |g(z)|. \end{aligned}$$

Thus, since g is bounded on $l(\theta_\epsilon)$, $\hat{f}$ is bounded on $l(\theta_\epsilon)$.

Applying the same argument to $\bar{\hat{f}}(\bar{z})$, $\hat{f}(-z)$, $\bar{\hat{f}}(-\bar{z})$, we see that $\hat{f}$ is also bounded on $l(-\theta_\epsilon)$, $l(\theta_\epsilon+\pi)$, and $l(-\theta_\epsilon+\pi)$. By (10), $\hat{f}$ is also bounded on $l(0)$ and $l(\pi)$. Notice that the 6 half lines $l(\pm\theta_\epsilon)$, $l(\pm\theta_\epsilon+\pi)$, $l(0)$, and $l(\pi)$ divide the complex plane into 6 sectors each of which has angle less than π/β. Thus using Lemma 6, we conclude that $\hat{f}$ is bounded on the whole plane. Thus by Liouville's theorem $\hat{f}$ is a constant. Obviously the constant must be 0 and hence $\hat{f}=0$ and $f=0$. This completes the proof for the case $\beta<2$.

Case II: $2<\beta<\infty$. Define v by

$$v = A(\sin\pi/2\beta)^\beta. \tag{16}$$

Consider

$$g(z) = \hat{f}(z)e^{(b+iv)z^\beta}, \quad z \in D(0,\pi/2\beta).$$

By (10) and (7), there exist constants B and C' for which g satisfies (13) for $z \in D(0,\pi/2\beta)$ and (14). For $z = re^{i\pi/2\beta}$, $r>0$, it follows from (10) and (16) that

$$|g(z)| \le ce^{r^\beta\{A(\sin\pi/2\beta)^\beta - v\}} = c.$$

Hence, by Lemma 5, g is bounded on $l(\theta)$ for each $\theta\in(0,\pi/2\beta)$. Thus we proved

$$\sup_{r>0}\{|\hat{f}(re^{i\theta})|e^{r^\beta(b\cos\beta\theta - v\sin\beta\theta)}\} < \infty \tag{17}$$

for each $\theta\in(0,\pi/2\beta)$.

Applying the same argument with $\bar{\hat{f}}(\bar{z})$ in place of $\hat{f}(z)$, we also have

$$\sup_{r>0}\{|\hat{f}(re^{-i\theta})|e^{r^\beta(b\cos\beta\theta - v\sin\beta\theta)}\} < \infty \tag{18}$$

for each $\theta\in(0,\pi/2\beta)$.

Take a θ_0 satisfying $0<\theta_0<\pi/2\beta$ and set

$$b' = b - v\tan\beta\theta_0.$$

Consider the function $h(z) = \hat{f}(z)e^{b'z^\beta}$ on $D_0 = D(-\theta_0,\theta_0)$. For $z = re^{\pm i\theta_0}$, $r>0$, we have

$$\begin{aligned}|h(z)| &= |\hat{f}(re^{\pm i\theta_0})|e^{b'r^\beta\cos\beta\theta_0}\\ &= |\hat{f}(re^{\pm i\theta_0})|e^{r^\beta(b\cos\beta\theta_0 - v\sin\beta\theta_0)}.\end{aligned}$$

Thus, by (17) and (18), the function $h(z)$ is bounded on $l(\pm\theta_0)$. By (10), $h(z)$ satisfies the global estimate $|h(z)|\le ce^{B'|z|^\beta}$ on D_0. Since $2\theta_0<\pi/\beta$, we can use Lemma 6 to see that $h(z)$ is bounded on D_0. Thus, in particular, $\hat{f}(y)e^{b'y^\beta}$ is bounded for $y > 0$.

Applying the same argument to $\hat{f}(-z)$, we see that $\hat{f}(-y)e^{b'y^\beta}$ is also bounded for $y > 0$. Thus we conclude that $\hat{f}(y)e^{b'|y|^\beta}$ is bounded for $y \in \mathbb{R}$.

Now the conditions (6) and (7) are satisfied with f, α, β, a, b replaced by $\hat{f}, \beta, \alpha, b', a$. Notice that $b' \to b$ as $\theta_0 \to 0$. Hence if we take θ_0 sufficiently small the condition (4) is satisfied with α, β, a, b replaced by β, α, b', a. Therefore, applying the result of Case I, we conclude that $f = 0$. This completes the proof of Theorem 1. ■

4 Proof of Theorem 2

By dilation of variables, we may assume that $a = b = 1/2$. We define $\hat{f}(z)$ by (9). From (6) with $a = 1/2$ and $\alpha = 2$, it follows that, for $z = x + iy \in \mathbb{C}$,

$$\begin{aligned} |\hat{f}(z)| &\leq \int_{-\infty}^{\infty} |f(t)|e^{ty}dt \\ &= e^{y^2/2}\int_{-\infty}^{\infty} |f(t)|e^{t^2/2}e^{-(t-y)^2/2}dt \leq ce^{y^2/2}, \end{aligned} \tag{19}$$

where c is a constant independent of z. It is also easy to see that $\hat{f}$ is an entire holomorphic function. We consider $g(z) = \hat{f}(z)e^{z^2/2}$, which is also an entire function. We shall prove that $g(z)$ is bounded.

For $\epsilon \in (0, \pi/2)$, we set

$$v_\epsilon = (\tan\epsilon)/4 = (\sin\epsilon)^2/2\sin 2\epsilon, \quad \theta_\epsilon = \pi/2 - \epsilon$$

and

$$g_\epsilon(z) = \hat{f}(z)e^{(1/2+iv_\epsilon)z^2}.$$

By (19), there exists a constant B_ϵ such that

$$|g_\epsilon(z)| \leq ce^{B_\epsilon|z|^2}, \quad z \in \mathbb{C}.$$

For $x \in \mathbb{R}$, $|g_\epsilon(x)| = |\hat{f}(x)e^{x^2/2}|$ satisfies (14) for some sufficiently large C' which is independent of ϵ. For $z = re^{i\theta_\epsilon}$, $r > 0$, (19) implies

$$\begin{aligned} |g_\epsilon(z)| &\leq ce^{(r^2/2)((\sin\theta_\epsilon)^2+\cos 2\theta_\epsilon-2v_\epsilon\sin 2\theta_\epsilon)} \\ &= ce^{(r^2/2)((\cos\theta_\epsilon)^2-2v_\epsilon\sin 2\theta_\epsilon)} \\ &= ce^{(r^2/2)((\sin\epsilon)^2-2v_\epsilon\sin 2\epsilon)} = c. \end{aligned}$$

If $0 < \theta < \theta_\epsilon$, then using Lemma 5 we have

$$\sup_{r>0} |g_\epsilon(re^{i\theta})| \leqq c(\theta, \epsilon),$$

where the constant $c(\theta, \epsilon)$ remains bounded if $\theta \in (0, \pi/2)$ is fixed and $\epsilon \to 0$. Since, as $\epsilon \to 0$, $v_\epsilon \to 0$ and $g_\epsilon(z) \to g(z)$, we conclude that $g(z)$ is bounded on each half line $l(\theta)$ with $0 < \theta < \pi/2$.

Applying the same argument to $\bar{g}(-\bar{z})$, $g(-z)$, $\bar{g}(\bar{z})$, we see that g is also bounded on the half lines $l(\theta)$ for $\pi/2 < \theta < \pi$, $\pi < \theta < 3\pi/2$, and $3\pi/2 < \theta < 2\pi$. Thus we can find, say, 5 half lines that divide the complex plane into 5 sectors each of which has angle less than $\pi/2$ and $g(z)$ is bounded on each half line. Thus, using Lemma 6, we can conclude that g is bounded on the whole complex plane. Since g is entire, it must be constant and thus $f(x)$ is a constant multiple of $e^{-x^2/2}$.

References

[1] S. Ben Farah and K. Mokni, Uncertainty principle and L^p-L^q-version of Morgan's theorem on some groups, Russ. J. Math. Phys. 10 (2003), 245-260.

[2] M. Cowling and J. F. Price, Generalizations of Heisenberg's inequality, Springer Lecture Notes in Math. 992, Harmonic Analysis, Proceedings of a Conference, Cortona, Italy, 1982, G. Mauceri, F. Ricci, and G. Weiss eds., Springer Verlag, 1983, pp. 443–449.

[3] J. B. Garnett, Bounded Analytic Functions, Pure and Applied Math, 96, Academic Press, Inc., 1981.

[4] G. H. Hardy, A theorem concerning Fourier transforms, J. London. Math. Soc. 8 (1933), 227-231.

[5] A. Miyachi, A generalization of theorem of Hardy, Harmonic Analysis Seminar held at Izunagaoka, Shizuoka-Ken, Japan (1997), 44-51.

[6] G. W. Morgan, A note on Fourier transform, J. London Math. Soc. 9 (1934), 187-192.

Takeshi Kawazoe
Department of Mathematics
Keio University at Fujisawa
Endo, Fujisawa, Kanagawa 252-8520
Japan

Jianming Liu
School of Mathematical Sciences
Peking University
Beijing, 100871
P. R. China.

Akihiko Miyachi
Department of Mathematics
Tokyo Woman's Christian University
Zempukuji, Suginami-ku, Tokyo 167-8585
Japan