Fractional calculus and analytic continuation of the complex Fourier-Jacobi transform

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Abstract

By using the Riemann-Liouville type fractional integral operators we shall reduce the complex Fourier-Jacobi transforms of even functions on $\mathbf{R}$ to the Euclidean Fourier transforms. As an application of the reduction formula, Parseval's formula and an inversion formula of the complex Jacobi transform are easily obtained. Moreover, we shall introduce a class of even functions, not C^∞ and not compactly supported on $\mathbf{R}$, whose transforms have meromorphic extensions on the upper half plane.

1. Introduction.

Let $\alpha, \beta, \lambda \in \mathbf{C}$ and $t \in \mathbf{R}$. For $\alpha \neq -1, -2, -3, \cdots$, $\phi_\lambda^{\alpha,\beta}(t)$ denotes the Jacobi function of the first kind, and for $\lambda \neq -i, -2i, -3i, \cdots$, $\Phi_\lambda^{\alpha,\beta}(t)$ the one of the second kind. Let $C_0^\infty(\mathbf{R})$ denote the space of all even C^∞ functions on $\mathbf{R}$ with compact support. For $f \in C_0^\infty(\mathbf{R})$ and $\Re\alpha > -1$ the Fourier-Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$ and the complex Fourier-Jacobi one $\tilde{f}_{\alpha,\beta}(\lambda)$ are defined by

$$\hat{f}_{\alpha,\beta}(\lambda) = \frac{\sqrt{2}}{\Gamma(\alpha+1)} \int_0^\infty f(t)\phi_\lambda^{\alpha,\beta}(t)\Delta_{\alpha,\beta}(t)dt \tag{1}$$

and

$$\tilde{f}_{\alpha,\beta}(\lambda) = \int_0^\infty f(t)\Phi_\lambda^{\alpha,\beta}(t)\Delta_{\alpha,\beta}(t)dt \tag{2}$$

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respectively, where

$$\Delta_{\alpha,\beta}(t) = (2\mathrm{sh}t)^{2\alpha+1}(2\mathrm{ch}t)^{2\beta+1}. \tag{3}$$

The Fourier-Jacobi transform $f \to \hat{f}_{\alpha,\beta}$ is well-understood. For example, the Paley-Wiener theorem and the inversion formula for $C_0^\infty(\mathbf{R})$ are obtained by Flensted-Jensen [2] and Koornwinder [3]. In particular, Koornwinder reduces the transform $\hat{f}_{\alpha,\beta}$ to the Fourier Cosine transform, which corresponds to the case of $\alpha = \beta = -1/2$:

$$\begin{aligned} \hat{f}_{\alpha,\beta}(\lambda) &= 2^{3(\alpha+1/2)} \frac{2}{\sqrt{2\pi}} \left(W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(f) \right)^{\wedge}_{-1/2,-1/2}(\lambda) \\ &= 2^{3(\alpha+1/2)} \left(W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(f) \right)^{\wedge}(\lambda). \end{aligned} \tag{4}$$

Here $W^\sigma_\mu(f)$, $\mu \in \mathbf{C}$, $\sigma > 0$, is the Weyl type fractional integral of f, which is for $\Re\mu > 0$ defined by

$$W^\sigma_\mu(f)(y) = \Gamma(\mu)^{-1} \int_y^\infty f(x)(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu-1} d(\mathrm{ch}\sigma x) \tag{5}$$

and extended to an entire function in μ. Moreover, in the second line of (4) $W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(f)$ is regarded as an even function on $\mathbf{R}$ and $(\ \cdot\ )^\wedge$ is the Euclidean Fourier transform on $\mathbf{R}$ (see [3, (2.7), (3,7), (3.12)]). One of the aim of this paper is to obtain an analogous formula for the complex Fourier-Jacobi transform $\tilde{f}_{\alpha,\beta}(\lambda)$. Actually, we shall reduce $\tilde{f}_{\alpha,\beta}$ to the Euclidean Fourier transform, which corresponds to the case of $\alpha = \beta = -1/2$ (see [2, (2.7)]). In order to obtain the reduction formula we introduce the Riemann-Liouville type fractional integral $\tilde{W}^\sigma_\mu(f)$: For $f \in C_0^\infty(\mathbf{R})$ and $\Re\mu > 0$, $\tilde{W}^\sigma_\mu(f)$ is defined by

$$\tilde{W}^\sigma_\mu(f)(y) = \sigma\Gamma(\mu)^{-1} \int_0^y f(x)(\mathrm{ch}\sigma y - \mathrm{ch}\sigma x)^{\mu-1} dx \cdot \mathrm{sh}\sigma y \tag{6}$$

and extended to an entire function in μ (see Lemma 3.2). Then the relation between the complex Fourier-Jacobi transform and the Euclidean Fourier one is given by

$$\tilde{f}_{\alpha,\beta}(\lambda) = 2^{-3(\alpha+1/2)} C_{\alpha,\beta}(-\lambda) \left(\tilde{W}^1_{-(\alpha-\beta)} \circ \tilde{W}^2_{-(\beta+1/2)}(f\Delta_{\alpha,\beta}) \right)^{\sim}_{-1/2,-1/2}(\lambda),$$

where $C_{\alpha,\beta}$ is the C-function (see (11) and Proposition 4.2). In this formula, if $\Re\alpha > \Re\beta > -1/2$, two operators $\tilde{W}^1_{-(\alpha-\beta)}$ and $\tilde{W}^2_{-(\beta+1/2)}$ correspond to fractional derivatives.

As an application of this formula, Parseval's formula for $C_0^\infty(\mathbf{R})$, which characterizes the inner product $\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta})}$ in terms of $\hat{f}_{\alpha,\beta}$ and $\tilde{g}_{\alpha,\beta}$, easily follows from the one for $L^2(\mathbf{R})$ (see Theorem 5.1). Next, we shall consider analytic continuation of $\tilde{g}_{\alpha,\beta}(\lambda)$ when g is not C^∞ and not compactly supported. We note that, if $\Im\lambda$ is sufficiently large, then $\Phi_\lambda^{\alpha,\beta}$ has exponential decay and thus, $\tilde{g}_{\alpha,\beta}(\lambda)$ is well-defined for a large class of even functions. We shall introduce a class of even functions g on $\mathbf{R}$ for which $\tilde{g}_{\alpha,\beta}(\lambda)$ has a meromorphic extension on $\Im\lambda \geq 0$. Then we can deduce an inversion formula of the complex Fourier-Jacobi transform $g \to \tilde{g}_{\alpha,\beta}$ in a distribution sense (see Theorem 6.5).

Similar result is obtained in [5] by a different and direct approach without using the reduction arguments. Moreover, in [1] the Fourier-Jacobi transform $\tilde{g}_{\alpha,\beta}$ of $g(x) = (\mathrm{ch}x)^\eta$ is explicitly calculated for the group case of $SU(n,1)$ $(\alpha = n-1, \beta = 0)$. This function $(\mathrm{ch}x)^\eta$ is a simple example of unbounded functions whose Fourier-Jacobi transform has a meromorphic extension on $\Im\lambda \geq 0$. Compared with these direct approach, if $\Re\alpha > \Re\beta > -1/2$, then the same result follows in our approach, otherwise, some extra conditions on g are required to carry out our reduction method. However, under these extra conditions we see that all poles appeared in our inversion formula are simple and we can distinguish between poles arisen from the C-function and ones from the analytic continuation (see Theorem 6.5 and Remark 6.6).

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2. Notations.

Let $\alpha, \beta, \lambda \in \mathbf{C}$ and $t \in \mathbf{R}$. We shall consider the differential equation

$$(L_{\alpha,\beta} + \lambda^2 + \rho^2) f(t) = 0, \tag{7}$$

where $\rho = \alpha + \beta + 1$ and

$$L_{\alpha,\beta} = \frac{d^2}{dt^2} + ((2\alpha+1)\mathrm{cth}t + (2\beta+1)\mathrm{th}t)\frac{d}{dt}.$$

Then, for $\alpha \neq -\mathbf{N}$, the Jacobi function of the first kind with order (α, β)

$$\phi_\lambda^{\alpha,\beta}(t) = F\left(\frac{\rho + i\lambda}{2}, \frac{\rho - i\lambda}{2}; \alpha + 1; -\mathrm{sh}^2 t\right) \tag{8}$$

is a unique solution of (7) satisfying $\phi_\lambda^{\alpha,\beta}(0) = 1$ and $d\phi_\lambda^{\alpha,\beta}/dt(0) = 0$. For $\lambda \neq -i\mathbf{N}$, the Jacobi function of the second kind with order (α, β)

$$\Phi_\lambda^{\alpha,\beta}(t) = (e^t - e^{-t})^{i\lambda - \rho} F\left(\frac{\rho - 2\alpha - i\lambda}{2}, \frac{\rho - i\lambda}{2}; 1 - i\lambda; -\mathrm{sh}^{-2} t\right) \tag{9}$$

is another solution of (7). Then $\Gamma(\alpha+1)^{-1}\phi_\lambda^{\alpha,\beta}$ is entire of α,β, and for $\lambda \notin i\mathbf{Z}$, we have the identity

$$\sqrt{\pi}\Gamma(\alpha+1)^{-1}\phi_\lambda^{\alpha,\beta}(t) = \frac{1}{2}C_{\alpha,\beta}(\lambda)\Phi_\lambda^{\alpha,\beta}(t) + \frac{1}{2}C_{\alpha,\beta}(-\lambda)\Phi_{-\lambda}^{\alpha,\beta}(t), \qquad (10)$$

where $C_{\alpha,\beta}$ is the C-function given by

$$C_{\alpha,\beta}(\lambda) = \frac{2^\rho\Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(\frac{1+i\lambda}{2}\right)}{\Gamma\left(\frac{i\lambda+\rho}{2}\right)\Gamma\left(\frac{i\lambda+\rho-2\beta}{2}\right)}. \qquad (11)$$

We recall the following properties of these functions (cf. [3] and [4]).

Lemma 2.1. *Assume that $\alpha,\beta \in \mathbf{C}$ and $\Re\alpha > -1$.*

(1) *For each fixed $t > 0$, as a function of λ, $\phi_\lambda^{\alpha,\beta}(t)$ is an entire function. There exists a constant $K > 0$ such that for all $t \geq 0$ and all $\lambda \in \mathbf{C}$,*

$$|\phi_\lambda^{\alpha,\beta}(t)| \leq K(1+|\lambda|)^\epsilon(1+t)e^{(|\Im\lambda|-\Re\rho)t},$$

where $\epsilon = 0$ if $\Re\alpha > -1/2$ and $\epsilon = 1$ for $-1 < \Re\alpha \leq -1/2$.

(2) *For each fixed $t > 0$, as a function of λ, $\Phi_\lambda^{\alpha,\beta}(t)$ is a holomorphic function in $\mathbf{C}\backslash\{-i\mathbf{N}\}$. For each $c > 0$ there exists a constant $K > 0$ such that for all $t \geq c$ and all $\Im\lambda \geq 0$,*

$$|\Phi_\lambda^{\alpha,\beta}(t)| \leq Ke^{(-\Im\lambda-\Re\rho)t}$$

and for all $0 < t < c$ and all $\Im\lambda \geq 0$,

$$|\Phi_\lambda^{\alpha,\beta}(t)| \leq K\begin{cases} t^{-(2\Re\alpha+1)} & \text{if } \Re\alpha > -1/2, \\ \log|t| & \text{if } \Re\alpha = -1/2, \\ 1 & \text{if } -1 < \Re\alpha < -1/2. \end{cases}$$

(3) *For each $r > 0$, there exists a constant $K > 0$ such that, if $\lambda \in \mathbf{C}$, $\Im\lambda \geq 0$ and λ is at distance larger than r from the poles of $C_{\alpha,\beta}(-\lambda)^{-1}$, then*

$$|C_{\alpha,\beta}(-\lambda)^{-1}| \leq K(1+|\lambda|)^{\Re\alpha+1/2}.$$

Let $C_0^\infty(\mathbf{R})$ denote the set of even C^∞ functions on $\mathbf{R}$ with compact support. For $f \in C_0^\infty(\mathbf{R})$ we define the Fourier-Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$ and the complex Fourier-Jacobi transform $\tilde{f}_{\alpha,\beta}(\lambda)$ by (1) and (2) respectively. From Lemma 2.1 it follows that $\hat{f}_{\alpha,\beta}(\lambda)$ is entire and $\tilde{f}_{\alpha,\beta}(\lambda)$ is holomorphic for $\lambda \neq -i\mathbf{N}$. Especially, (10) implies that for all $\lambda \notin i\mathbf{Z}$,

$$\sqrt{2\pi}\hat{f}_{\alpha,\beta}(\lambda) = C_{\alpha,\beta}(\lambda)\tilde{f}_{\alpha,\beta}(\lambda) + C_{\alpha,\beta}(-\lambda)\tilde{f}_{\alpha,\beta}(-\lambda). \tag{12}$$

In the following we define the Gauss symbol $[z]$ for $z \in \mathbf{C}$ as $[\Re z]$.

3. Fractional integrals.

3.1. Let $C_c^\infty(\mathbf{R}_a)$, $\mathbf{R}_a = [a, \infty)$, $a \in \mathbf{R}$, denote the set of all C^∞ functions F_a on $\mathbf{R}$ with compact support, where F is right differentiable at a. For $F \in C_c^\infty(\mathbf{R}_a)$ and $-n < \Re\mu$, $n = 0, 1, 2, \cdots$, we shall define the Weyl type fractional integral operator $W_\mu^{\mathbf{R}}$ by

$$(W_\mu^{\mathbf{R}}(F))(y) = \frac{(-1)^n}{\Gamma(\mu+n)} \int_y^\infty \frac{d^n F(x)}{dx^n}(x-y)^{\mu+n-1}dx. \tag{13}$$

We extend it as an entire function in μ. Then $W_0^{\mathbf{R}}$ is the identity operator, $W_\mu^{\mathbf{R}} \circ W_\nu^{\mathbf{R}} = W_{\mu+\nu}^{\mathbf{R}}$, and

$$W_\nu^{\mathbf{R}} : C_c^\infty(\mathbf{R}_a) \to C_c^\infty(\mathbf{R}_a)$$

is bijection. We also define the Riemann-Liouville type fractional integral operator $\tilde{W}_\mu^{\mathbf{R}}$ by

$$(\tilde{W}_\mu^{\mathbf{R}}(F))(y) = \frac{1}{\Gamma(\mu+n)} \frac{d^n}{dy^n} \int_a^y F(x)(y-x)^{\mu+n-1}dx \tag{14}$$

and extend it as an entire function in μ. We note that $\tilde{W}_0^{\mathbf{R}}$ is the identity operator and $\tilde{W}_{-\mu}^{\mathbf{R}} \circ \tilde{W}_\mu^{\mathbf{R}}(F) = F$ if $\Re\mu > 0$. For $\Re\mu \leq 0$, $\tilde{W}_{-\mu}^{\mathbf{R}} \circ \tilde{W}_\mu^{\mathbf{R}}(F) = F$ provided $F(a) = F'(a) = \cdots = F^{(n-1)}(a) = 0$. On $C_c^\infty(\mathbf{R}_b)$, $b > a$, $\tilde{W}_\mu^{\mathbf{R}} \circ \tilde{W}_\nu^{\mathbf{R}} = \tilde{W}_{\mu+\nu}^{\mathbf{R}}$. For $\tau, \eta \in \mathbf{C}$ and $m = 0, 1, 2, \cdots$, we define $A_{\tau,\eta}^m(\mathbf{R}_a)$ the class of C^m functions F on $\mathbf{R}_a$ of the form $F = F_0 + F_1$;

$$F_0(x) = (x-a)^\tau G(x), \quad G \in C^m([a, a+2)) \tag{15}$$

and

$$F_1(x) = x^\eta H(x), \quad H \in C^m((a+1, \infty)), \tag{16}$$

where

$$\sup_{0 \leq k \leq m, a+1 \leq x < \infty} \left| x^k \frac{d^k H(x)}{dx^k} \right| \leq c. \tag{17}$$

Moreover, $A^m_{\tau*,\eta}(\mathbf{R}_a)$ denote the class defined by replacing $(x-a)^\tau$ in (15) with $\log(x-a)\cdot(x-a)^\tau$ and $A^m_{\tau,\eta*}(\mathbf{R}_a)$ the one defined by replacing $\sup_{a+1\le x<\infty}|H(x)|\le c$, $k=0$ in (17), with $\sup_{a+1\le x<\infty}|(\log x)H(x)|\le c$.

Lemma 3.1. *For $m=0,1,2,\cdots$ and $\mu,\tau,\eta\in\mathbf{C}$ the fractional operators $W^{\mathbf{R}}_\mu$ and $\tilde W^{\mathbf{R}}_\mu$ satisfy the following.*

(1) *If $m+[\mu]-1\ge 0$, $\Re\eta<0$ and $\Re(\eta+\mu)<0$, then*

$$W^{\mathbf{R}}_\mu : A^m_{\tau,\eta}(\mathbf{R}_a)\to A^{m+[\mu]}_{\delta,\eta+\mu}(\mathbf{R}_a),$$

where $\delta=\tau+\mu$ if $\mu=0,-1,-2,\cdots$, and otherwise

$$\delta=\begin{cases}0 & \text{if } \Re(\tau+\mu)>0,\\ 0* & \text{if } \Re(\tau+\mu)=0,\\ \tau+\mu & \text{if } \Re(\tau+\mu)<0.\end{cases}\qquad (18)$$

(2) *If $m+[\mu]\ge 0$ and $\Re\tau>-1$, then*

$$\tilde W^{\mathbf{R}}_\mu : A^m_{\tau,\eta}(\mathbf{R}_a)\to A^{m+[\mu]}_{\tau+\mu,\delta}(\mathbf{R}_a),$$

where $\delta=\eta+\mu$ if $\mu=0,-1,-2,\cdots$, and otherwise

$$\delta=\begin{cases}\eta+\mu & \text{if } \Re\eta>-1,\\ (\eta+\mu)* & \text{if } \Re\eta=-1,\\ \mu-1 & \text{if } \Re\eta<-1.\end{cases}$$

Proof. (1) When $\mu=0,-1,-2,\cdots$, $W^{\mathbf{R}}_\mu(F)(y)=c[F^{(-\mu)}]^\infty_y=cF^{(-\mu)}(y)$, because $\Re(\eta+\mu)<0$. Therefore, the assertion for $\mu=0,-1,-2,\cdots$ easily follows. Let $\mu\ne 0,-1,-2,\cdots$. Also we may assume that $\Re\mu>0$. Actually, if $\Re\mu\le 0$, let $W^{\mathbf{R}}_\mu=W^{\mathbf{R}}_{\mu-[\mu]}\circ W^{\mathbf{R}}_{[\mu]}$ and note that $0<\Re(\mu-[\mu])<1$ and $[\mu-[\mu]]+[\mu]=[\mu]$. Hence, the assertion for $\Re\mu\le 0$ follows from the cases of $\Re\mu>0$ and $\mu=0,-1,-2,\cdots$. Let $F\in A^m_{\tau,\eta}(\mathbf{R}_a)$ be of the form $F=F_0+F_1$ in (15) and (16). If $y\ge a+1$, then $W^{\mathbf{R}}_\mu(F)$ is defined as

$$\begin{aligned}W^{\mathbf{R}}_\mu(F)(y) &= c\int_y^\infty (x-a)^\tau G(x)(x-y)^{\mu-1}dx + c\int_y^\infty x^\eta H(x)(x-y)^{\mu-1}dx\\ &= I_1(y)+I_2(y).\end{aligned}$$

Clearly, $I_1(y)=0$ if $y\ge a+2$ and $I_1\in C^{(m+[\mu])}$. Moreover,

$$\begin{aligned}I_2(y) &= cy^{\eta+\mu}\int_1^\infty x^\eta H(yx)(x-1)^{\mu-1}dx\\ &= cy^{\eta+\mu}H_\mu(y).\end{aligned}$$

For $0 \leq l \leq m$,

$$\begin{aligned} H_\mu^{(l)}(y) &= \int_1^\infty x^{\eta+l} H^{(l)}(yx)(x-1)^{\mu-1} dx \\ &= y^{-(\eta+\mu+l)} \int_y^\infty x^{\eta+l} H^{(l)}(x)(x-y)^{\mu-1} dx \end{aligned}$$

and for $0 \leq l' \leq [\mu]$,

$$\begin{aligned} H_\mu^{(l+l')}(y) &\sim \sum_{k=0}^{l'} y^{-(\eta+\mu+l+k)} \int_y^\infty x^{\eta+l} H^{(l)}(x)(x-y)^{\mu-1-(l'-k)} dx \\ &\sim \sum_{k=0}^{[\mu]} y^{-l'} \int_1^\infty x^{\eta+l} H^{(l)}(yx)(x-1)^{\mu-1-([\mu]-k)} dx \\ &\sim \sum_{k=0}^{[\mu]} y^{-(l+l')} \int_1^\infty x^{\eta} (xy)^l H^{(l)}(yx)(x-1)^{\mu-1-([\mu]-k)} dx, \end{aligned}$$

where, if μ is positive integer, the term corresponding to $l' = [\mu], k = 0$ equals $y^{-l'} H^{(l)}(y) = y^{-(l+l')} \cdot y^l H^{(l)}(y)$ (see the first line). Hence, (17) implies that $y^{l+l'} H_\mu^{(l+l')}$, $0 \leq l + l' \leq m + [\mu]$, is bounded on $(a+1, \infty)$. Therefore, $H_\mu(y)$ satisfies (17) replaced m with $m + [\mu]$. If $a < y < a+1$, then $W_\mu^{\mathbf{R}}(F)$ is estimated as

$$\begin{aligned} &\int_y^{a+1} (x-a)^\tau G(x)(x-y)^{\mu-1} dx + \int_{a+1}^\infty x^\eta H(x)(x-y)^{\mu-1} dx \\ \sim\ & (y-a)^{\tau+\mu} \int_1^{1/(y-a)} x^\tau G((y-a)x+a)(x-1)^{\mu-1} dx + y^{\eta+\mu} \\ \sim\ & (y-a)^{\tau+\mu} \left\{ \begin{array}{ll} (y-a)^{-(\tau+\mu)} & \text{if } \Re(\tau+\mu) > 0 \\ \log(y-a) & \text{if } \Re(\tau+\mu) = 0 \\ 1 & \text{if } \Re(\tau+\mu) < 0 \end{array} \right\} + 1 \\ \sim\ & (x-a)^\delta G_\mu(x). \end{aligned}$$

Noting $0 < (y-a) < 1$ and the argument in the previous case, we see that $G_\mu \in C^{m+[\mu]}$. Therefore, $W_\mu^{\mathbf{R}}(F)$ is of the desired form.

(2) When $\mu = 0, -1, -2, \cdots$, $\tilde{W}_\mu^{\mathbf{R}}(F)$ coincides with $cF^{(-\mu)}$ provided $\Re\tau > -1$. Since $\tilde{W}_\mu^{\mathbf{R}} = \tilde{W}_{[\mu]}^{\mathbf{R}} \circ \tilde{W}_{\mu-[\mu]}^{\mathbf{R}}$ if $\Re\mu \leq 0$, as in the first case, we may assume that $\Re\mu > 0$. We note that, if $a < y < a+1$, then

$$\begin{aligned} \tilde{W}_\mu^{\mathbf{R}}(F)(y) &= \int_a^y (x-a)^\tau G(x)(y-x)^{\mu-1} dx \\ &= (y-a)^{\tau+\mu} \int_0^1 x^\tau G((y-a)x+a)(1-x)^{\mu-1} dx \\ &= (y-a)^{\tau+\mu} G_\mu(x), \end{aligned}$$

and if $y \geq a+1$, then $\tilde{W}^{\mathbf{R}}_\mu(F)(y)$ is estimated as

$$\begin{aligned}
& \int_a^{a+1} (x-a)^\tau G(x)(y-x)^{\mu-1}dx + \int_{a+1}^y x^\eta H(x)(y-x)^{\mu-1}dx \\
\sim \quad & (y-a)^{\tau+\mu}\int_0^{1/(y-a)} x^\tau G((y-a)x+a)(1-x)^{\mu-1}dx \\
& +y^{\eta+\mu}\int_{(a+1)/y}^1 x^\eta H(yx)(x-1)^{\mu-1}dx \\
\sim \quad & (y-a)^{\tau+\mu}\int_0^{1/(y-a)} x^\tau dx + y^{\eta+\mu}\int_{(a+1)/y}^1 x^\eta dx \\
\sim \quad & (y-a)^{\mu-1} + y^{\eta+\mu}\left\{\begin{array}{ll} 1 & \text{if } \Re\eta > -1 \\ \log(y-a) & \text{if } \Re\eta = -1 \\ (y-a)^{-\eta-1} & \text{if } \Re\eta < -1 \end{array}\right\} \\
\sim \quad & y^\delta H_\mu(x).
\end{aligned}$$

Noting $(y-1) \geq 1$ and $-\Re(\tau+\mu-1) < 0$, as in the first case, we see that $G_\mu \in C^{m+[\mu]}$ and H_μ satisfies (17) replaced m with $m+[\mu]$. ■

Remark 3.2. In Lemma 3.1 we note that, if $\Re(\tau+\mu) \geq 0$ and $\mu \neq 1, 2, 3, \cdots$, then the Weyl type fractional operator $W^{\mathbf{R}}_\mu$ does not keep the zero of F at $x=a$ even if F has sufficiently higher order of zero.

3.2. We shall transfer the operators $W^{\mathbf{R}}_\mu$ and $\tilde{W}^{\mathbf{R}}_\mu$ on $C^\infty_c(\mathbf{R}_1)$, $\mathbf{R}_1 = [1,\infty)$, to ones for $C^\infty_0(\mathbf{R})$. For $f \in C^\infty_0(\mathbf{R})$, $\sigma > 0$ and $-n < \Re\mu$, $n = 0, 1, 2, \cdots$, we shall define the Weyl type and the Riemann-Liouville type fractional integral operators W^σ_μ and $\tilde{W}^\sigma_\mu$ respectively as follows:

$$W^\sigma_\mu(f)(y) = \frac{(-1)^n}{\Gamma(\mu+n)}\int_y^\infty \frac{d^n f(x)}{d(\mathrm{ch}\sigma x)^n}(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu+n-1}d(\mathrm{ch}\sigma x) \qquad (19)$$

and

$$\tilde{W}^\sigma_\mu(f)(y) = \frac{\sigma^{-1}}{\Gamma(\mu+n)}\frac{d^n}{d(\mathrm{ch}\sigma y)^n}\int_0^y f(x)(\mathrm{ch}\sigma y - \mathrm{ch}\sigma x)^{\mu+n-1}dx \cdot \mathrm{sh}\sigma y. \quad (20)$$

Then the change of varibale:

$$f(x) = [f]^\sigma(\mathrm{ch}\sigma x)$$

yields the relation between $W^{\mathbf{R}}_\mu$ and W^σ_μ:

$$W^\sigma_\mu(f)(y) = W^{\mathbf{R}}_\mu([f]^\sigma)(\mathrm{ch}\sigma y) \qquad (21)$$

and the one between $\tilde{W}_\mu^{\mathbf{R}}$ and $\tilde{W}_\mu^\sigma$:

$$\tilde{W}_\mu^\sigma(f)(y) = \tilde{W}_\mu^{\mathbf{R}}([f \cdot (\mathrm{sh}\sigma x)^{-1}]^\sigma)(\mathrm{ch}\sigma y) \cdot \mathrm{sh}\sigma y. \tag{22}$$

For $\tau, \eta \in \mathbf{C}$ and $m = 0, 1, 2, \cdots$, let $\mathcal{A}_{\tau,\eta}^{\sigma,m}(\mathbf{R})$ denote the space of all even functions f on $\mathbf{R}$ of the form $f = f_0 + f_1$;

$$f_0(x) = (\mathrm{sh}\sigma x)^{2\tau} g(\mathrm{ch}\sigma x), \quad g \in C^m([1,3)) \tag{23}$$

and

$$f_1(x) = (\mathrm{ch}\sigma x)^\eta h(\mathrm{ch}\sigma x), \quad h \in C^m((2,\infty)), \tag{24}$$

where

$$\sup_{0 \le k \le m, 2 \le x < \infty} \left| x^k \frac{d^k h(x)}{dx^k} \right| \le c. \tag{25}$$

Moreover, $\mathcal{A}_{\tau*,\eta}^m(\mathbf{R})$ denote the class defined by replacing $(\mathrm{sh}\sigma x)^{2\tau}$ in (23) with $(\log x)(\mathrm{sh}\sigma x)^{2\tau}$ and $\mathcal{A}_{\tau,\eta*}^m(\mathbf{R})$ by replacing $\sup_{2 \le x < \infty} |h(x)| \le c$, $k = 0$ in (25), with $\sup_{2 \le x < \infty} |(\log x) h(x)| \le c$. Then, using the relations (21) and (22), we can rewrite Lemma 3.1 for $W_\mu^{\mathbf{R}}$ and $\tilde{W}_\mu^{\mathbf{R}}$ to the one for W_μ^σ and $\tilde{W}_\mu^\sigma$:

Lemma 3.3. *Let $\mu, \tau, \eta \in \mathbf{C}$ and $m = 0, 1, 2, \cdots$.*

(1) *If $m + [\mu] - 1 \ge 0$, $\Re\eta < 0$ and $\Re(\eta + \mu) < 0$, then*

$$W_\mu^\sigma : \mathcal{A}_{\tau,\eta}^{\sigma,m}(\mathbf{R}) \to \mathcal{A}_{\delta,\eta+\mu}^{\sigma,m+[\mu]}(\mathbf{R}),$$

where $\delta = \tau + \mu$ if $\mu = 0, -1, -2, \cdots$, and otherwise δ is the same as (18).

(2) *If $m + [\mu] \ge 0$ and $\Re\tau > -1/2$, then*

$$\tilde{W}_\mu^\sigma : \mathcal{A}_{\tau,\eta}^{\sigma,m}(\mathbf{R}) \to \mathcal{A}_{\tau+\mu,\delta}^{\sigma,m+[\mu]}(\mathbf{R}),$$

where $\delta = \eta + \mu$ if $\mu = 0, -1, -2, \cdots$, and otherwise

$$\delta = \begin{cases} \eta + \mu & \text{if } \Re\eta > 0 \\ (\eta + \mu)* & \text{if } \Re\eta = 0 \\ \mu & \text{if } \Re\eta < 0. \end{cases} \tag{26}$$

3.3. As an application of Lemma 3.3, we shall consider the inner product of $f \in A_{\tau_1,\eta_1}^{\sigma,n_1}$ and $g \in A_{\tau_2,\eta_2}^{\sigma,n_2}$, and obtain an adjoint relation between W_μ^σ and $\tilde{W}_\mu^\sigma$.

Proposition 3.4. *Let $\sigma > 0$, $\mu \in \mathbf{C}$ and $\tau_i, \eta_i \in \mathbf{C}, n_i \in \mathbf{N}$ for $i = 1, 2$. Suppose that $\Re\rho \geq 0$, $n_1 + [\mu] - 1 \geq 0$, $n_2 + [\mu] \geq 0$ and*

$$\begin{array}{ll} (a) & \Re(\eta_1 + \mu) < 0, \Re\eta_1 < 0, \\ (b) & \Re(\eta_1 + \eta_2 + \mu + 2\rho/\sigma) < 0 \\ (c) & \Re(\tau_1 + \tau_2 + \alpha + \mu) > -1 \\ (d) & \Re(\tau_2 + \alpha) > -1 \\ (e) & \Re(\tau_2 + \alpha + \mu) > -1. \end{array}$$

Then for $f \in A^{\sigma,n_1}_{\tau_1,\eta_1}$ and $g \in A^{\sigma,n_2}_{\tau_2,\eta_2}$,

$$\langle W^\sigma_\mu(f), g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} = \langle f, \tilde{W}^\sigma_{\bar{\mu}}(g\Delta_{\bar{\alpha},\bar{\beta}})\rangle_{L^2(\mathbf{R}_+,dx)}. \tag{27}$$

Proof. First we check the both sides of (27) are finite. Lemma 3.3 (1) with (a) implies that $W^\sigma_\mu(f) \in \mathcal{A}^{\sigma,n_1+[\mu]}_{\delta,\eta_1+\mu}(\mathbf{R})$ with δ in (18). Since $g\Delta_{\alpha,\beta} \in A^{\sigma,n_2}_{\tau_2+\alpha+1/2,\eta_2+2\rho/\sigma}(\mathbf{R})$, the left hand side of (27) is finite from $(b), (c), (d)$. As for the right hand side, Lemma 3.3 (2) with (d) implies that

$$\tilde{W}^\sigma_\mu(g\Delta_{\alpha,\beta}) \in \mathcal{A}^{\sigma,n_2+[\mu]}_{\tau_2+\alpha+1/2+\mu,\delta}(\mathbf{R})$$

with δ in (26). Then the right hand side of (27) is also finite from (a) and (b). We shall prove the equality. When $\Re\mu > 0$, (27) is clear by changing the order of integration. Let us suppose that $-n < \Re\mu \leq -n+1$, $n = 1, 2, 3, \cdots$. Then, it follows from (19) that

$$\begin{aligned} &\langle W^\sigma_\mu(f), g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} \\ = &\int_0^\infty \frac{(-1)^n}{\Gamma(\mu+n)} \int_y^\infty \frac{d^n f(x)}{d(\text{ch}\sigma x)^n} (\text{ch}\sigma x - \text{ch}\sigma y)^{\mu+n-1} d(\text{ch}\sigma x) \cdot \overline{g(y)}\Delta_{\alpha,\beta}(y)dy \\ = &\int_0^\infty \frac{d^n f(x)}{d(\text{ch}\sigma x)^n} \frac{(-1)^n}{\Gamma(\mu+n)} \int_0^x \overline{g(y)}\Delta_{\alpha,\beta}(y)(\text{ch}\sigma x - \text{ch}\sigma y)^{\mu+n-1} dy \cdot d(\text{ch}\sigma x). \end{aligned}$$

Since $g(y)\Delta_{\alpha,\beta}(y) = O(x^{2\tau_2+2\alpha+1})$ if $0 < x < 1$, the last integral with respect to dy is $O(x^{2(\tau_2+\alpha+\mu+n)})$ if $0 < x < 1$. Thereby, since (e) implies that $2\Re(\tau_2 + \alpha + \mu + n) > -2 + 2n \geq 0$, we can repeat n-times integration by parts with respect to $d(\text{ch}\sigma x)$. This process shifts the differential operator $d/d(\text{ch}\sigma x)$ acting on f to the one acting on the inner integral with respect to dy. Therefore, the desired equality follows from (20). ■

4. Reduction formula.

In order to obtain a reduction formula of $\tilde{f}_{\alpha,\beta}$, we recall some reduction formulas of $\Phi_\lambda^{\alpha,\beta}$ obtained by Koornwinder [3]. Let $\Re\mu > 0$ and $\Im\lambda > -\Re\rho$. Then for $x > 0$,

$$C_{\alpha,\beta}(-\lambda)^{-1}\Phi_\lambda^{\alpha,\beta} = 2^{3\mu+1}C_{\alpha+\mu,\beta+\mu}(-\lambda)^{-1}W_\mu^2(\Phi_\lambda^{\alpha+\mu,\beta+\mu})$$

(see [3, (2.15)]). Hence, applying Proposition 3.4 with Lemma 2.1 (2), we see that for $f \in C_0^\infty(\mathbf{R})$,

$$\begin{aligned} & C_{\alpha,\beta}(-\lambda)^{-1}\tilde{f}_{\alpha,\beta}(\lambda) \\ = \; & C_{\alpha,\beta}(-\lambda)^{-1}\langle f, \overline{\Phi_\lambda^{\alpha,\beta}}\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} \\ = \; & 2^{3\mu+1}C_{\alpha+\mu,\beta+\mu}(-\lambda)^{-1}\langle f, W_{\bar{\mu}}^2(\overline{\Phi_\lambda^{\alpha+\mu,\beta+\mu}})\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} \\ = \; & 2^{3\mu+1}C_{\alpha+\mu,\beta+\mu}(-\lambda)^{-1}\langle \tilde{W}_\mu^2(f\Delta_{\alpha,\beta}), \overline{\Phi_\lambda^{\alpha+\mu,\beta+\mu}}\rangle_{L^2(\mathbf{R}_+,dx)} \\ = \; & 2^{3\mu+1}C_{\alpha+\mu,\beta+\mu}(-\lambda)^{-1}\left(\tilde{W}_\mu^2(f\Delta_{\alpha,\beta})\Delta_{\alpha+\mu,\beta+\mu}^{-1}\right)^{\sim}_{\alpha+\mu,\beta+\mu}(\lambda). \end{aligned}$$

Clearly, this equation is meromorphically extended to $\alpha, \beta, \lambda, \mu \in \mathbf{C}$.

Proposition 4.1. *Let $\Re\alpha > -1$ and $f \in C_0^\infty(\mathbf{R})$. As a meromorphic function of $\alpha, \beta, \lambda, \mu \in \mathbf{C}$,*

$$\tilde{f}_{\alpha,\beta}(\lambda) = 2^{3\mu+1}\frac{C_{\alpha,\beta}(-\lambda)}{C_{\alpha+\mu,\beta+\mu}(-\lambda)}\left(\frac{\tilde{W}_\mu^2(f\Delta_{\alpha,\beta})}{\Delta_{\alpha+\mu,\beta+\mu}}\right)^{\sim}_{\alpha+\mu,\beta+\mu}(\lambda).$$

Now we shall reduce the complex Fourier-Jacobi transform $\tilde{f}_{\alpha,\beta}$ to the Euclidean Fourier transform.

One way to obtain the reduction is to use Proposition 4.1 repeatedly and to reduce the parameters (α, β) to $(-1/2, -1/2)$. We here apply another way, but essentially it is the same way. We note the following formula: Let $\Re\alpha > \Re\beta > -1/2$, $s > 0$ and $\Im\lambda > 0$. Then

$$e^{i\lambda s} = C_{\alpha,\beta}(-\lambda)^{-1}\int_s^\infty \Phi_\lambda^{\alpha,\beta}(t)A_{\alpha,\beta}(s,t)dt, \tag{28}$$

where $A_{\alpha,\beta}(s,t)$ is given by

$$\frac{2^{3(\alpha+1/2)+1}\mathrm{sh}2t}{\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}\int_s^t(\mathrm{ch}2t - \mathrm{ch}2w)^{\beta-1/2}(\mathrm{ch}w - \mathrm{ch}s)^{\alpha-\beta-1}\mathrm{sh}w\,dw.$$

(see [2, (2.17)]). In particular, it follows from [2, (3.5), (3.12)] that (28) can be rewritten as

$$e^{i\lambda s} = C_{\alpha,\beta}(-\lambda)^{-1}2^{3(\alpha+1/2)}W_{\alpha-\beta}^1 \circ W_{\beta+1/2}^2(\Phi_\lambda^{\alpha,\beta})(s).$$

Since $(\alpha-\beta)+2(\beta-1/2)=\rho$, Lemma 2.1 (2) and Lemma 3.3 (1) imply that the right hand side is well-defined if $\Im\lambda > 0$. Furthermore, it follows from Lemma 3.3 (1) that, if $\Im\lambda$ is sufficiently large, then

$$\Phi_\lambda^{\alpha,\beta} = C_{\alpha,\beta}(-\lambda)2^{-3(\alpha+1/2)}W^2_{-(\beta+1/2)}\circ W^1_{-(\alpha-\beta)}(e^{i\lambda(\cdot)}).$$

Since $\Re\alpha > \Re\beta > -1/2$ means that $\Re(-(\alpha-\beta))<0$ and $\Re(-(\beta+1/2))<0$, $e^{i\lambda x}$ for a sufficiently large $\Im\lambda$ and $f\in C_0^\infty(\mathbf{R})$ satisfy the assumptions on f, g in Proposition 3.4. Thereby, it is easy to see that

$$\begin{aligned}
& C_{\alpha,\beta}(-\lambda)^{-1}\tilde{f}_{\alpha,\beta}(\lambda)\\
= \ & C_{\alpha,\beta}(-\lambda)^{-1}\langle f, \overline{\Phi_\lambda^{\alpha,\beta}}\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)}\\
= \ & 2^{-3(\alpha+1/2}\langle f, W^2_{-(\bar\beta+1/2)}\circ W^1_{-(\bar\alpha-\bar\beta)}(e^{-i\lambda(\cdot)})\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)}\\
= \ & 2^{-3(\alpha+1/2}\langle \tilde W^1_{-(\alpha-\beta)}\circ\tilde W^2_{-(\beta+1/2)}(f\Delta_{\alpha,\beta}), e^{-i\lambda(\cdot)}\rangle_{L^2(\mathbf{R}_+,dx)}\\
= \ & 2^{-3(\alpha+1/2}\left(\tilde W^1_{-(\alpha-\beta)}\circ\tilde W^2_{-(\beta+1/2)}(f\Delta_{\alpha,\beta})\right)^\sim_{-1/2,-1/2}(\lambda).
\end{aligned}$$

If, for simplicity, we put

$$W_{\alpha,\beta} = W^1_{\alpha-\beta}\circ W^2_{\beta+1/2} \text{ and } \tilde W_{\alpha,\beta} = \tilde W^2_{\beta+1/2}\circ\tilde W^1_{\alpha-\beta}, \tag{29}$$

then we have the following.

Proposition 4.2. *Let $f\in C_0^\infty(\mathbf{R})$ and $\Re\alpha>\Re\beta>-1/2$. Then, as a meromorphic function of $\alpha,\beta,\lambda\in\mathbf{C}$,*

$$\tilde f_{\alpha,\beta}(\lambda) = 2^{-3(\alpha+1/2)}C_{\alpha,\beta}(-\lambda)\left(\tilde W^{-1}_{\alpha,\beta}(f\Delta_{\alpha,\beta})\right)^\sim_{-1/2,-1/2}(\lambda). \tag{30}$$

We shall extend this formula for $f\in\mathcal{A}^{\sigma,m}_{\tau,\eta}(\mathbf{R})$. We recall that $\tilde W^{-1}_{\alpha,\beta}$ is a composition of two fractional operators $\tilde W^1_{-(\alpha-\beta)}$ and $\tilde W^2_{-(\beta+1/2)}$ (see (29)) and these operators change smoothness according to Lemma 3.3 (2). We take $m=N_{\alpha,\beta}$ defined by

$$\begin{cases}
-[-(\beta+1/2)]-[-(\alpha-\beta)] & \text{if } \Re(\beta+1/2)\ge 0, \Re(\alpha-\beta)\ge 0\\
-[-(\beta+1/2)] & \text{if } \Re(\beta+1/2)\ge 0, \Re(\alpha-\beta)< 0\\
[-(\beta+1/2)]-[-(\alpha-\beta)] & \text{if } (\Re\beta+1/2)< 0, \Re(\alpha-\beta)\ge 0\\
0 & \text{if } (\Re\beta+1/2)< 0, \Re(\alpha-\beta)< 0.
\end{cases}$$

Corollary 4.3. *Let $\alpha,\beta,\tau,\eta\in\mathbf{C}$, $\Re\alpha>-1$, $\Re\tau\ge 0$, $\Re\tau > -\Re(\alpha-\beta)-1/2$ and $\Re(\eta+\rho)>\max\{-\Re\rho, -\Re(\alpha-\beta)\}$. Then for $f\in\mathcal{A}^{1,N_{\alpha,\beta}}_{\tau,\eta}(\mathbf{R})$, $\tilde f_{\alpha,\beta}(\lambda)$ is holomorphic on $\Im\lambda>\Re(\eta+\rho)$ and satisfies* (30).

Proof. It follows from Lemma 2.1 (2) that $\tilde{f}_{\alpha,\beta}(\lambda)$ is well-defined if $\Re\alpha > -1$, $\Re\tau \geq 0$ and $\Im\lambda > \Re(\eta+\rho)$. On the other hand, we note that

$$f(x)\Delta_{\alpha,\beta}(x) \sim (\mathrm{ch}x)^{\eta+2\rho}(\mathrm{th}x)^{2\tau+2\alpha+1}.$$

Since $\Re(\tau+\alpha+1/2) > \Re(\tau-1/2) \geq -1/2$ and $\Re(\eta+2\rho) > 0$, Lemma 3.3 (2) implies that

$$\tilde{W}^2_{-(\beta+1/2)}(f\Delta_{\alpha,\beta})(x) \sim (\mathrm{ch}x)^{\eta+2\rho-2(\beta+1/2)}(\mathrm{th}x)^{2\tau+2\alpha+1-2(\beta+1/2)}.$$

Since $\Re(\tau+\alpha+1/2-(\beta+1/2)) = \Re(\tau+(\alpha-\beta)) > -1/2$ and $\Re(\eta+2\rho-2(\beta+1/2)) = \Re(\eta+\rho+(\alpha-\beta)) > 0$, Lemma 3.3 (2) again implies that

$$\tilde{W}^1_{-(\alpha-\beta)} \circ \tilde{W}^2_{-(\beta+1/2)}(f\Delta_{\alpha,\beta})(x) \sim (\mathrm{ch}x)^{\eta+\rho}(\mathrm{th}x)^{2\tau}.$$

Therefore, the Euclidean Fourier transform of $\tilde{W}^{-1}_{\alpha,\beta}(f\Delta_{\alpha,\beta})$ is well-defined if $\Im\lambda > \Re(\eta+\rho)$. ■

Remark 4.4. If $-(\beta+1/2)$ and $-(\alpha-\beta)$ are $0,-1,-2,\cdots$, then the condition $\Re(\eta+\rho) > \max\{-\Re\rho, -\Re(\alpha-\beta)\}$ is not necessary.

5. Inversion formula.

Let $\alpha,\beta \in \mathbf{C}$ and $\Re\alpha > -1$. The inversion formula of the Fourier-Jacobi transform $f \to \hat{f}_{\alpha,\beta}$, $f \in C_0^\infty(\mathbf{R})$, is obtained by Flensted-Jensen [3] and Koornwinder [4]. We recall their inversion formula and give a simple proof.

Let $D_{\alpha,\beta}$ denote the set of poles of $C_{\alpha,\beta}(-\lambda)^{-1}$ located in $\Im\lambda \geq 0$:

$$D_{\alpha,\beta} = \{\gamma_m = i(\varepsilon\beta-\alpha-1-2m)\ ;\ m=0,1,2,\cdots,\Im\gamma_m \geq 0\}, \qquad (31)$$

where $\varepsilon = 1$ if $\Re\beta > 0$ and $\varepsilon = -1$ if $\Re\beta \leq 0$. Let $R_{\alpha,\beta}(\gamma_m)$ denote the residue of $C_{\alpha,\beta}(-\lambda)^{-1}$ at γ_m, explicitly given by

$$R_{\alpha,\beta}(\gamma_m) = \frac{(-1)^m 2^{-\rho+(\varepsilon\beta-\alpha-1-2m)}i}{m!\sqrt{\pi}} \frac{\Gamma(\varepsilon\beta-m)}{\Gamma(\varepsilon\beta-\alpha-1-2m)}.$$

Then it follows from [4, Theorems 2.2, 2.3, 2.4] that

Theorem 5.1 *Let $\alpha,\beta \in \mathbf{C}$ and $\nu \in \mathbf{R}$. Suppose that $\Re\alpha > -1$, $\nu \geq 0$, and $\nu > -\Re(\alpha\pm\beta+1)$.*

(1) *For each $f \in C_0^\infty(\mathbf{R})$ and $t > 0$,*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_{\alpha,\beta}(\lambda+i\nu)\Phi^{\alpha,\beta}_{\lambda+i\nu}(t)C_{\alpha,\beta}(-\lambda-i\nu)^{-1}d\lambda$$

$$= \frac{\sqrt{2}}{\Gamma(\alpha+1)} \int_0^\infty \hat{f}_{\alpha,\beta}(\lambda)\phi_\lambda^{\alpha,\beta}(t)(C_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda))^{-1}d\lambda$$
$$-\frac{2\sqrt{2}\pi i}{\Gamma(\alpha+1)} \sum_{\gamma\in D_{\alpha,\beta}} \frac{\hat{f}_{\alpha,\beta}(\gamma)}{C_{\alpha,\beta}(\gamma)}\phi_\gamma^{\alpha,\beta}(t)R_{\alpha,\beta}(\gamma).$$

(2) *For each* $f, g \in C_0^\infty(\mathbf{R})$,

$$\begin{aligned}
&\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} \\
= &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}_{\alpha,\beta}(\lambda+i\nu)\tilde{\overline{g}}_{\alpha,\beta}(\lambda+i\nu)C_{\alpha,\beta}(-\lambda-i\nu)^{-1}d\lambda \\
= &\frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}_{\alpha,\beta}(\lambda)\tilde{\overline{g}}_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda)^{-1}d\lambda \\
&-\sqrt{2\pi}i \sum_{\gamma\in D_{\alpha,\beta}} \hat{f}_{\alpha,\beta}(\gamma)\tilde{\overline{g}}_{\alpha,\beta}(\gamma)R_{\alpha,\beta}(\gamma).
\end{aligned}$$

(3) *If* $\alpha, \beta \in \mathbf{R}$ *and* $\alpha \geq \beta > -1/2$, *then* $D_{\alpha,\beta} = \emptyset$ *and for* $f, g \in C_0^\infty(\mathbf{R})$,

$$\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} = \langle \hat{f}_{\alpha,\beta}, \hat{g}_{\alpha,\beta})\rangle_{L^2(\mathbf{R}_+,|C_{\alpha,\beta}(\lambda)|^{-2}d\lambda)}.$$

Proof. We shall give a simple proof based on Proposition 3.4 and the reduction formula in Corollary 4.3. Obviously, it is enough to prove the first equation in (2). We note that $|\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)}| \leq \|f\|_2\|g\|_2$, where $\|\cdot\|_2$ is the $L^2(\mathbf{R}_+, \Delta_{\alpha,\beta}dx)$-norm, and $|\hat{f}_{\alpha,\beta}(\lambda+i\nu)| \leq \|f\|_{L^1(\mathbf{R}_+)}e^{(\nu+2\Re\rho)R}$ if $\mathrm{supp}(f) \subset [-R, R]$ and $|\hat{g}_{\alpha,\beta}(\lambda+i\nu)| \leq \|g\|_{L^1(\mathbf{R}_+)}$ (see Lemma 2.1). Therefore, by using approximation argument, we may suppose that f, g belong to $\mathcal{A}_{\tau,\eta}^{1,N}(\mathbf{R})$ for sufficiently large positive numbers τ and N. We take $\nu > \Re(\eta+\rho) > \max\{-\Re\rho, -\Re(\alpha-\beta)\}$. Hence, Proposition 3.4, (4), (30) and the Plancherel formula for $L^2(\mathbf{R})$ yield that

$$\begin{aligned}
&\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} \\
= &\langle W_{\alpha,\beta}(f), \tilde{W}_{\overline{\alpha},\overline{\beta}}(g\Delta_{\overline{\alpha},\overline{\beta}})\rangle_{L^2(\mathbf{R}_+,dx)} \\
= &\frac{1}{2}\langle W_{\alpha,\beta}(f)e^{\nu x}, \tilde{W}_{\overline{\alpha},\overline{\beta}}(g\Delta_{\overline{\alpha},\overline{\beta}})e^{-\nu x}\rangle_{L^2(\mathbf{R},dx)} \\
= &\frac{1}{\sqrt{2\pi}}\langle \hat{f}_{\alpha,\beta}(\lambda+i\nu), \tilde{g}_{\overline{\alpha},\overline{\beta}}(\lambda-i\nu)C_{\overline{\alpha},\overline{\beta}}(-\lambda+i\nu)^{-1}\rangle_{L^2(\mathbf{R},dx)},
\end{aligned}$$

where $W_{\alpha,\beta}(f)$ and $\tilde{W}_{\overline{\alpha},\overline{\beta}}(g\Delta_{\overline{\alpha},\overline{\beta}})$ in the third line are regarded as even functions on $\mathbf{R}$. ■

Similarly, we can deduce the following.

Corollary 5.2. *Let $\alpha, \beta, \eta \in \mathbf{C}$ and $\Re\alpha > -1$. Let $\nu \geq 0$ and $\nu > \Re(\eta + \rho) > \max\{-\Re\rho, -\Re(\alpha - \beta)\}$. Then, for all $f \in C_0^\infty(\mathbf{R})$ and $g \in \mathcal{A}_{0,\eta}^{1,0}(\mathbf{R})$,*

$$\langle f, g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta})} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_{\alpha,\beta}(\lambda + i\nu)\tilde{\overline{g}}_{\alpha,\beta}(\lambda + i\nu)C_{\alpha,\beta}(-\lambda - i\nu)^{-1} d\lambda.$$

6. Analytic continuation.

6.1. We shall consider analytic continuation of the formula in Corollary 5.2. For $\theta \in \mathbf{C}$ let $W_{-\theta}^{\mathbf{R}}$ be the Weyl type fractional operator on $\mathbf{R}_0 = [0, \infty)$ (see (13)) and let $C_{[0,1]}^\theta$ (resp. $C_{[0,1)}^\theta$) denote the space of all functions H on $\mathbf{R}$ such that $\operatorname{supp}(H) \subset [0, 1]$ (resp. $[0, 1)$), $W_{-\theta}^{\mathbf{R}}(H)$ is well-defined, and

$$\sup_{0 \leq w \leq 1} |W_{-\theta}^{\mathbf{R}}(H)(w)| \leq c.$$

For $\sigma > 0, \theta, \eta \in \mathbf{C}$ let $\mathcal{B}_{0,\eta}^{\sigma,\theta}(\mathbf{R})$ denote the space consisting of all even functions f on $\mathbf{R}$ of the form

$$f(x) = (\mathrm{ch}\sigma x)^\eta H((\mathrm{ch}\sigma x)^{-1}), \quad H \in C_{[0,1]}^\theta. \tag{32}$$

We note that, if $\operatorname{supp}(H) \subset [0, 1]$, then $\operatorname{supp}(W_{-\theta}^{\mathbf{R}}(H)) \subset [0, 1]$ and thus, if $H \in C_{[0,1]}^\theta$, then $H \in C_{[0,1]}^{\theta'}$ for all θ' such that $\Re\theta' \leq \Re\theta$. When $\Re\theta \geq 0$, if we put $h(t) = H(1/t)$ as in the form of (24), we see that h satisfies (25) with $m = [\theta]$. Hence, if $\Re\theta \geq 0$, then

$$\mathcal{B}_{0,\eta}^{\sigma,\theta}(\mathbf{R}) \subset \mathcal{A}_{0,\eta}^{\sigma,[\theta]}(\mathbf{R}).$$

Let $\mathcal{B}_{\flat,\eta}^{\sigma,\theta}(\mathbf{R})$ denote the set of $f \in \mathcal{B}_{0,\eta}^{\sigma,\theta}(\mathbf{R})$ such that $0 \notin \operatorname{supp}(f)$, that is f is identically zero around 0. We may suppose that $f \in \mathcal{B}_{\flat,\eta}^{\sigma,\theta}(\mathbf{R})$ is of the form

$$f(x) = (\mathrm{ch}\sigma x)^{\eta-1}(\mathrm{sh}\sigma x)H_f((\mathrm{ch}\sigma x)^{-1}), \quad H_f \in C_{[0,1)}^\theta. \tag{33}$$

Obviously, we may suppose that $f \in \mathcal{B}_{0,\eta}^{\sigma,\theta}(\mathbf{R})$ is of the form

$$f = f_0 + f_1, \quad f_0 \in C_0^\infty(\mathbf{R}), \quad f_1 \in \mathcal{B}_{\flat,\eta}^{\sigma,\theta}(\mathbf{R}). \tag{34}$$

6.2. For $\mu = 0, -1, -2, \cdots$, it follows from the definition that

$$\tilde{W}_\mu^\sigma : \mathcal{B}_{\flat,\eta}^{\sigma,\theta}(\mathbf{R}) \to \mathcal{B}_{\flat,\eta+\mu}^{\sigma,\theta+\mu}(\mathbf{R}) \tag{35}$$

and $\mathcal{B}^{\sigma,\theta}_{0,\eta}(\mathbf{R}) \to \mathcal{B}^{\sigma,\theta+\mu}_{\mu,\eta+\mu}(\mathbf{R})$ (see Lemma 3.3 (2)). For general $\mu \in \mathbf{C}$ we have the following.

Lemma 6.1. *Let $\mu \in \mathbf{C}$ and $f \in \mathcal{B}^{\sigma,\theta}_{\flat,\eta}(\mathbf{R})$.*
(1) *If $\Re\mu > 0$ and H_f satisfies*

$$W^{\mathbf{R}}_{\mu}\left(s^{-(\eta+\mu-\theta)}W^{\mathbf{R}}_{-\theta}(H_f)\right)(w) = O(w^{-(\eta-\theta)}) \quad \text{if } \Re(\eta-\theta) \leq 0,$$

then $\tilde{W}^{\sigma}_{\mu}(f) \in \mathcal{B}^{\sigma,\theta}_{\flat,\eta+\mu}(\mathbf{R})$.
(2) *If $\Re\mu \leq 0$ and H_f satisfies the above conditions replaced μ, θ and η with $\mu-[\mu]$, $\theta+[\mu]$ and $\eta+[\mu]$ respectively, then $\tilde{W}^{\sigma}_{\mu}(f) \in \mathcal{B}^{\sigma,\theta+[\mu]}_{\flat,\eta+\mu}(\mathbf{R})$.*

Proof. Let $\Re\mu > 0$ and f be of the form (33). Clearly, if f is identically zero around 0, then $\tilde{W}^{\sigma}_{\mu}(f)$ is also identically zero around 0. Letting $\mathrm{ch}\sigma x = t$ in (20), we see that

$$\begin{aligned}
\tilde{W}^{\sigma}_{\mu}(f)(x) &= c\int_1^t s^{\eta-1}H_f(s^{-1})(t-s)^{\mu-1}ds \cdot \sqrt{t^2-1} \\
&= ct^{\eta+\mu-1}\sqrt{t^2-1}\int_{1/t}^1 s^{\eta-1}H_f(1/ts)(1-s)^{\mu-1}ds \\
&= ct^{\eta+\mu-1}\sqrt{t^2-1}H_1(t^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
H_1(w) &= \int_w^1 s^{\eta-1}H_f(w/s)(1-s)^{\mu-1}ds \\
&= \int_1^\infty s^{-(\eta+\mu)}H_f(ws)(s-1)^{\mu-1}ds \\
&= w^\eta\int_w^\infty s^{-(\eta+\mu)}H_f(s)(s-w)^{\mu-1}ds \\
&= w^\eta W^{\mathbf{R}}_{\mu}(s^{-(\eta+\mu)}H_f)(w).
\end{aligned}$$

Since $H_f \in C^{\theta}_{[0,1)}$ and $W^{\mathbf{R}}_{-\gamma}(H_f(ws)) = s^\gamma W^{\mathbf{R}}_{-\gamma}(H_f)(ws)$, $\Re\gamma \leq \Re\theta$, as a function of w, it follows that

$$\begin{aligned}
W^{\mathbf{R}}_{-\theta}(H_1)(w) &= \int_1^\infty s^{-(\eta+\mu-\theta)}W^{\mathbf{R}}_{-\theta}(H_f)(ws)(s-1)^{\mu-1}ds \\
&= w^{\eta-\theta}W^{\mathbf{R}}_{\mu}\left(s^{-(\eta+\mu-\theta)}W^{\mathbf{R}}_{-\theta}(H_f)\right)(w). \qquad (36)
\end{aligned}$$

Therefore, if $\Re(\eta-\theta) > 0$ and $W^{\mathbf{R}}_{-\theta}(H_f)$ is bounded, then Lemma 3.1 (1) yields that $W^{\mathbf{R}}_{-\theta}(H_1)$ is bounded. On the other hand, if $\Re(\eta-\theta) \leq 0$, then the assumption on H_f also yields that $W^{\mathbf{R}}_{-\theta}(H_1)$ is bounded. Hence, $H_1 \in C^{\theta}_{[0,1)}$

and the desired result follows. Let $\Re\mu \le 0$. When $\mu = 0, -1, -2, \cdots$, the assertion is nothing but (35). Otherwise, since $\tilde{W}^\sigma_\mu = \tilde{W}^\sigma_{\mu-[\mu]} \circ \tilde{W}^\sigma_{[\mu]}$, the desired result follows from (35) and the first case. ■

Corollary 6.2. *Let* $\mu \in \mathbf{C}$ *and* $f \in \mathcal{B}^{\sigma,\eta}_{\flat,\eta}(\mathbf{R})$.

(1) *If* $\Re\mu < 0$, *then* $\tilde{W}^\sigma_\mu(f) \in \mathcal{B}^{\sigma,\eta+\mu}_{\flat,\eta+\mu}(\mathbf{R})$.

(2) *If* $\Re\mu = 0$, *then* $\tilde{W}^\sigma_\mu(f) \in \mathcal{B}^{\sigma,\eta+\mu-\delta}_{\flat,\eta+\mu}(\mathbf{R})$ *for* $\delta > 0$.

(3) *If* $\Re\mu > 0$ *and* $W^{\mathbf{R}}_{-\eta}(H_f)(w) = O(w^{\mu-\delta_1})$, *then* $\tilde{W}^\sigma_\mu(f) \in \mathcal{B}^{\sigma,\eta+\mu-\delta_2}_{\flat,\eta+\mu}(\mathbf{R})$, *where* $\delta_2 > \delta_1 \ge 0$ *or* $\delta_1 = \delta_2 = 0$.

Proof. (3) Let $\Re\mu > 0$ and suppose that $W^{\mathbf{R}}_{-\eta}(H_f)(w) = O(w^\mu)$. Then, letteing $\theta = \eta$ in (36), it follows that

$$W^{\mathbf{R}}_{-\eta}(H_1) = W^{\mathbf{R}}_\mu(s^{-\mu}W^{\mathbf{R}}_{-\eta}(H_f))$$

and thus,

$$W^{\mathbf{R}}_{-(\eta+\mu)}(H_1)(w) = w^{-\mu}W^{\mathbf{R}}_{-\eta}(H_f)(w). \tag{37}$$

Hence $H_1 \in C^{\eta+\mu}_{[0,1)}$ follows. When $W^{\mathbf{R}}_{-\eta}(H_f)(w) = O(w^{\mu-\delta_1})$, (37) is replaced with $W^{\mathbf{R}}_{-(\eta+\mu-\delta_2)}(H_1) = W^{\mathbf{R}}_{\delta_2}(s^{-\mu}W^{\mathbf{R}}_{-\eta}(H_f))$. Therefore, $H_1 \in C^{\eta+\mu-\delta_2}_{[0,1)}$ provided $\delta_2 > \delta_1$. (1) Let $\Re\mu < 0$. When $\mu = -1, -2, \cdots$, the assertion is obvious from (35). We may suppose that $\mu \ne -1, -2, \cdots$. Because of (35) and $\tilde{W}^\sigma_\mu = \tilde{W}^\sigma_{\mu-[\mu]-1} \circ \tilde{W}^\sigma_{[\mu]+1}$, we may suppose that $-1 < \Re\mu < 0$. Then it is easy to see that

$$\begin{aligned}\tilde{W}^\sigma_\mu(f)(x) &= c\frac{d}{dx}\left(t^{\eta+\mu}G_{\eta,\mu}(t^{-1})\right) \cdot \sqrt{t^2-1} \\ &= ct^{\eta+\mu-1}\sqrt{t^2-1}((\eta+\mu)G_{\eta,\mu}(t^{-1}) - t^{-1}G'_{\eta,\mu}(t^{-1})),\end{aligned}$$

where $\mathrm{ch}\sigma x = t$ and

$$G_{\eta,\mu}(w) = w^\eta W^{\mathbf{R}}_{\mu+1}(s^{-(\eta+\mu+1)}H_f))(w).$$

Therefore,

$$H_1(w) = (\eta+\mu)G_{\eta,\mu}(w) - wG'_{\eta,\mu}(w) = \mu G_{\eta,\mu}(w) - G_{\eta+1,\mu-1}(w). \tag{38}$$

Let $G = G_{\eta,\mu}$. As before, $W^{\mathbf{R}}_{-\eta}(G) = W^{\mathbf{R}}_{\mu+1}(s^{-(\mu+1)}W^{\mathbf{R}}_{-\eta}(H_f))$ and thus, $W^{\mathbf{R}}_{-(\eta+\mu)}(G) = W^{\mathbf{R}}_1(s^{-(\mu+1)}W^{\mathbf{R}}_{-\eta}(H_f))$. Since $\Re(\mu+1) > 0$, $\mathrm{supp}(H_f) \subset [0,1)$ and $W^{\mathbf{R}}_{-\eta}(H_f)$ is bounded, Lemma 3.1 means that $W^{\mathbf{R}}_{-(\eta+\mu)}(G)$ is bounded. Let $G = G_{\eta+1,\mu-1}$. Then the same process yields that $W^{\mathbf{R}}_{-(\eta+\mu)}(G)$

$= W_1^{\mathbf{R}}(s^{-\mu} W_{-1}^{\mathbf{R}} \circ W_{-\eta}^{\mathbf{R}}(H_f))$. This function is again bounded. Hence, $H_1 \in C_{[0,1)}^{\eta+\mu}$. (2) The case of $\Re\mu = 0$ follows from the same process in (1) replaced $\eta+\mu$ with $\eta+\mu-\delta$ and $W_1^{\mathbf{R}}$ with $W_{1+\delta}^{\mathbf{R}}$ respectively. ■

6.3. Now we shall consider analytic continuation of $\tilde{g}_{\alpha,\beta}(\lambda)$ in Corollary 5.2 provided $g \in \mathcal{B}_{0,\eta/2}^{2,\theta}(\mathbf{R})$ where θ will be suitably determined. We recall (34). When $g \in C_0^\infty(\mathbf{R})$, Lemma 2.1 (2) and the fact that

$$\Delta_{\alpha,\beta}(x) = (\mathrm{ch}x)^{2\rho}(\mathrm{th}x)^{2\alpha+1}$$

has zero of order $2\alpha+1$ at $x=0$ mean that, if $\Re\alpha > -1$, $\tilde{g}_{\alpha,\beta}(\lambda)$ is a holomorphic function on $\Im\lambda \geq 0$ of exponential type (see §5). Therefore, it is enough to consider the analytic continuation of $\tilde{g}_{\alpha,\beta}(\lambda)$ for $g \in \mathcal{B}_{\flat,\eta/2}^{2,\theta}(\mathbf{R})$. Since g is identically zero around 0, Corollary 4.3 yields that, if $\Re\alpha > -1$, then $\tilde{g}_{\alpha,\beta}(\lambda)$ is holomorphic on $\Im\lambda > \Re(\eta+\rho)$ and $2^{3(\alpha+1/2)}\tilde{g}_{\alpha,\beta}(\lambda)\, C_{\alpha,\beta}(-\lambda)^{-1}$ is the Euclidean Fourier transform of

$$\tilde{W}^1_{-(\alpha-\beta)}(\tilde{W}^2_{-(\beta+1/2)}(g\Delta_{\alpha,\beta})).$$

In the following, let $\Re\alpha > -1, \epsilon > 0$ and

$$f = g\Delta_{\alpha,\beta}.$$

Obviously, $f \in \mathcal{B}_{\flat,\eta/2+\rho}^{2,\theta}(\mathbf{R})$ and is of the form

$$f(x) = (\mathrm{ch}2x)^{\eta/2+\rho-1}(\mathrm{sh}2x)H_f((\mathrm{ch}2x)^{-1}), \quad H_f \in C_{[0,1)}^{\theta}. \tag{39}$$

We here take θ as

$$\theta_{\alpha,\beta}^{\eta} = \frac{\eta}{2} + \rho \tag{40}$$

and assume that,

$$\text{if } -\Re(\beta+1/2) > 0, \text{ then } W_{-(\eta/2+\rho)}^{\mathbf{R}}(H_f)(w) = O(w^{-(\beta+1/2)}). \tag{41}$$

Then, by taking η and μ in Corollary 6.2 as $\theta_{\alpha,\beta}^{\eta} = \eta/2+\rho$ and $-(\beta+1/2)$ respectively, it follows that for a sufficiently small $\epsilon_1 > 0$

$$\tilde{W}^2_{-(\beta+1/2)}(f) \in \mathcal{B}_{\flat,\eta_1}^{2,\eta_1-\epsilon_1}(\mathbf{R}),$$

where $\eta_1 = (\eta+\rho+(\alpha-\beta))/2$. This means that $\tilde{W}^2_{-(\beta+1/2)}(f)$ is of the form

$$\tilde{W}^2_{-(\beta+1/2)}(f)(x) = (\mathrm{ch}2x)^{\eta_1-1}(\mathrm{sh}2x)H_f^1(\mathrm{ch}^{-1}2x), \quad H_f^1 \in C_{[0,1)}^{\eta_1-\epsilon_1}.$$

We here rewrite this function as

$$\tilde{W}^2_{-(\beta+1/2)}(f)(x) = (\mathrm{ch}x)^{2\eta_1-1}(\mathrm{sh}x)H^2_f(\mathrm{ch}^{-2}x), \quad H^2_f \in C^{\eta_1-\epsilon_1}_{[0,1)}, \tag{42}$$

where

$$H^2_f(w) = 2(2-w)^{\eta_1}H^1_f(w/(2-w)). \tag{43}$$

Before applying $\tilde{W}^1_{-(\alpha-\beta)}$ to $\tilde{W}^2_{-(\beta+1/2)}(f)$, we prepare the following lemma.

Lemma 6.3. *Let $\mu \in \mathbf{C}, \epsilon > 0$ and f be of the form*

$$f(x) = (\mathrm{ch}x)^{2\eta-1}(\mathrm{sh}x)H(\mathrm{ch}^{-2}x), \quad H \in C^{\eta-\epsilon}_{[0,1)}. \tag{44}$$

(1) *If $\Re\mu < 0$, then $\tilde{W}^1_\mu(f)$ is of the form*

$$\tilde{W}^1_\mu(f)(x) = (\mathrm{ch}x)^{2\eta+\mu-1}(\mathrm{sh}x)H_1(\mathrm{ch}^{-2}x), \quad H_1 \in C^{\eta+\mu/2-\epsilon}_{[0,1)}. \tag{45}$$

(2) *If $\Re\mu = 0$, then (45) holds with $H_1 \in C^{\eta+\mu/2-\delta}_{[0,1)}$ for $\delta > \epsilon$.*

(3) *If $\Re\mu > 0$ and $W^{\mathbf{R}}_\mu(W^{\mathbf{R}}_{-(\mu/2+\eta)}(H)(s^2))(w) = O(w^{\mu-\delta_1})$, then (45) holds with $H_1 \in C^{\eta+\mu/2-\delta_2}_{[0,1)}$, where $\delta_2 > \delta_1 \geq 0$ or $\delta_1 = \delta_2 = 0$.*

Proof. We repeat the similar arguments in the proof of Corollary 6.2. (3) Let $\Re\mu > 0$ and suppose that $W^{\mathbf{R}}_\mu(W^{\mathbf{R}}_{-(\eta+\mu/2)}(H)(s^2))(w) = O(w^\mu)$. From the proof of Lemma 6.1, letting $\mathrm{ch}^{-2}x = w$, it follows that

$$\begin{aligned} H_1(w) &= c\int_{\sqrt{w}}^1 s^{2\eta-1}H(w/s^2)(1-s)^{\mu-1}ds \\ &= c\int_0^1 s^{2\eta-1}H(w/s^2)(1-s)^{\mu-1}ds \\ &= cw^\eta W^{\mathbf{R}}_\mu(s^{-(2\eta+\mu)}\tilde{H}))(\sqrt{w}), \end{aligned}$$

where $\tilde{H}(w) = H(w^2)$. Since $W^{\mathbf{R}}_{-\gamma}(H(w/s^2))(w) = cs^{-2\gamma}W^{\mathbf{R}}_{-\gamma}(H)(w/s^2)$, $\Re\gamma \leq \Re\eta$, as a function of w, we see that

$$\begin{aligned} W^{\mathbf{R}}_{-(\eta+\mu/2)}(H_1)(w) &= c\int_0^1 s^{-\mu}W^{\mathbf{R}}_{-(\eta+\mu/2)}(H)(w/s^2)(1-s)^{\mu-1}ds \\ &= cw^{-\mu/2}W^{\mathbf{R}}_\mu((W^{\mathbf{R}}_{-(\eta+\mu/2)}(H))^\sim)(\sqrt{w}). \end{aligned} \tag{46}$$

Hence $H_1 \in C^{\eta+\mu/2}_{[0,1)}$ follows. The case of $\delta_2 > \delta_1 \geq 0$ also follows as in (3) of Corollary 6.2. (1) Let $\Re\mu < 0$. As before, we may assume that $-1 < \Re\mu < 0$.

Then, using (45) replaced μ with $\mu+1$, we can repeat the proof of Corollary 6.2. Actually, $G_{\eta,\mu}$ is replaced by

$$w^{\eta}W^{\mathbf{R}}_{\mu+1}\left(s^{-(2\eta+\mu+1)}\tilde{H}\right)(\sqrt{w})$$

and $G_{\eta+1,\mu-1}$ by $G_{\eta+1/2,\mu-1}$. Hence, applying $W^{\mathbf{R}}_{-(\eta+\mu/2-\epsilon)} = W^{\mathbf{R}}_{1/2}\circ W^{\mathbf{R}}_{-(\mu+1)/2}\circ W^{\mathbf{R}}_{-(\eta-\epsilon)}$ to these functions, the desired result similarly follows as in Corollary 6.2. (2) The case of $\Re\mu=0$ also follows from the above argument. ∎

We apply $W^{\mathbf{R}}_{-(\alpha-\beta)}$ to $\tilde{W}^{2}_{-(\beta+1/2)}(f)$ (see (42)) under the assumption that,

$$\text{if } -\Re(\alpha-\beta)>0,\text{ then } W^{\mathbf{R}}_{-(\alpha-\beta)}(W^{\mathbf{R}}_{-(\eta+\rho)/2}(H^2_f)(s^2))(w) = O(w^{-(\alpha-\beta)}). \quad (47)$$

Then, by taking η,μ and H in Lemma 6.3 as $(\eta+\rho+(\alpha-\beta))/2$, $-(\alpha-\beta)$ and H^2_f respectively, it follows that for a sufficiently small $\epsilon_2>0$,

$$\tilde{W}^{1}_{-(\alpha-\beta)}(\tilde{W}^{2}_{-(\beta+1/2)}(f))(x) = (\mathrm{ch}x)^{\eta_2-1}(\mathrm{sh}x)H^3_f(w^2), \quad H^3_f\in C^{\eta_2/2-\epsilon_2}_{[0,1)}, \quad (48)$$

where $w=\mathrm{ch}^{-1}x$ and $\eta_2=\eta+\rho$.

Remark 6.4. (1) If $-\Re(\beta+1/2)\le 0$ and $-\Re(\alpha-\beta)\le 0$, then no extra conditions on zero of H_f and H^2_f (see (41) and (47)) are required.

(2) If $-\Re(\beta+1/2)>0$ and $-\Re(\alpha-\beta)>0$, then the both extra conditions on zero of H_f and H^2_f are required. However, the extra condition on zero of H^2_f means the one of H_f. First we note that (37) implies that

$$W^{\mathbf{R}}_{-(\eta/2+\rho)}(H_f)(w) = w^{-(\beta+1/2)}W^{\mathbf{R}}_{\theta}(H^1_f)(w),$$

where $\theta=-(\eta+\rho+(\alpha-\beta))/2$. Thereby, if $W^{\mathbf{R}}_{\theta}(H^1_f)$ is bounded, then $W^{\mathbf{R}}_{-(\eta/2+\rho)}(H_f)$ has zero of order $-(\beta+1/2)$ at $w=0$ and thus, the extra condition on zero of H_f follows. Now, let us suppose the extra condition on zero of H^2_f: $W^{\mathbf{R}}_{-(\alpha-\beta)}(W^{\mathbf{R}}_{-(\eta+\rho)/2}(H^2_f)(s^2))(w) = O(w^{-(\alpha-\beta)})$. We denote this function by $h(w)$. Then $H^2_f = W^{\mathbf{R}}_{(\eta+\rho)/2}(W^{\mathbf{R}}_{(\alpha-\beta)}(h)(\sqrt{s}))$ and thus

$$W^{\mathbf{R}}_{\theta}(H^2_f) = W^{\mathbf{R}}_{-(\alpha-\beta)/2}(W^{\mathbf{R}}_{(\alpha-\beta)}(h)(\sqrt{s})).$$

Hence, from Lemma 3.1 (1) it is easy to see that the right hand side is bounded. Similarly, $W^{\mathbf{R}}_{\theta-\delta}(H^2_f)$ is bounded for $\delta>0$ such that $-\Re(\alpha-\beta)/2-\delta>0$. Let $\theta=-n+\gamma$, $n=0,1,2,\cdots$ and $0<\Re\gamma\le 1$. Since $H^1_f(w)=G(w)H^2_f(w/(2-w))$ for $G\in C^\infty_c$ (see (43)) and $W^{\mathbf{R}}_{\theta}=W^{\mathbf{R}}_{\gamma}\circ W^{\mathbf{R}}_{-n}$, it follows that

$$\begin{aligned} W_\theta(H^1_f) &\sim \sum_{k=0}^{n} W^{\mathbf{R}}_{\gamma}(G_k W^{\mathbf{R}}_{-k}(H^2_f)) \\ &= \sum_{k=0}^{n} W^{\mathbf{R}}_{\gamma}(G_k W^{\mathbf{R}}_{-\gamma+\delta+(n-k)}\circ W^{\mathbf{R}}_{\theta-\delta}(H^2_f)), \end{aligned}$$

where $G_k \in C_c^\infty$. Therefore, since $W^{\mathbf{R}}_{\theta-\delta}(H_f^2)$ is bounded, Lemma 3.1 (1) implies that $W_\theta(H_f^1)$ is bounded as desired.

(3) From (39) and (42) H_f and H_f^2 can be written as

$$
\begin{aligned}
H_f(w) &= f((\text{arccosh } w^{-1})/2)\frac{w^{\eta/2+\rho}}{\sqrt{1-w^2}},\\
H_f^2(w) &= \tilde{W}^2_{-(\beta+1/2)}(f)(\text{arccosh } w^{-1/2})\frac{w^{(\eta+\rho+(\alpha-\beta))/2}}{\sqrt{1-w}}.
\end{aligned}
$$

Since $2^{3(\alpha+1/2)}\tilde{g}_{\alpha,\beta}(\lambda)\ C_{\alpha,\beta}(-\lambda)^{-1}$ is the Euclidean Fourier transform of (48), in order to carry out the analytic continuation of $\tilde{g}_{\alpha,\beta}(\lambda)\ C_{\alpha,\beta}(-\lambda)^{-1}$, it is enough to consider $\hat{F}(\lambda)$ of

$$
F(x) = (\text{ch}x)^{\gamma-1}(\text{sh}x)H(w^2), \quad H \in C^{\gamma/2-\epsilon}_{[0,1)},\ \Re\gamma > 0.
$$

For simplicity, put $\theta = \gamma/2-\epsilon$. Since $e^{i\lambda x} = (\text{ch}x)^{i\lambda}(1+\text{th}x)^{i\lambda}$, by changing the variable as $w = \text{ch}^{-1}x$, it follows that for $\Im\lambda > \Re\gamma$,

$$
\begin{aligned}
\hat{F}(\lambda) &= \int_0^1 H(w^2)\left(1+\sqrt{1-w^2}\right)^{i\lambda} w^{-i(\lambda-i\gamma)-1}dw,\\
&= \int_0^1 I(w)w^{-i(\lambda-i\gamma)/2-1}dw.
\end{aligned}
$$

Here, $I \in C^\theta_{[0,1)}$. Then, applying Proposition 3.4 with $\alpha=\beta=-1/2$ and (14), we see that

$$
\begin{aligned}
\hat{F}(\lambda) &= \int_0^1 W^{\mathbf{R}}_{-\theta}(I)(w)\tilde{W}^{\mathbf{R}}_{\theta}(w^{-i(\lambda-i\gamma)/2-1})dw\\
&= \frac{\Gamma(-i(\lambda-i\gamma)/2)}{\Gamma(-i(\lambda-i\gamma)/2)+\theta+1)}\int_0^1 W^{\mathbf{R}}_{-\theta}(I)(w)w^{-i(\lambda-i\gamma)/2+\theta}dw
\end{aligned}
$$

Since $I \in C^\theta_{[0,1)}$ and $-i(\lambda-i\gamma)/2+\theta = -i\lambda-\epsilon$, this integral is bounded if $\Im\lambda > -1+\epsilon$. Therefore, $\hat{F}(\lambda)$ has a meromorphic extension in $\Im\lambda \geq 0$ with simple poles lie in

$$
F_\gamma = \{\xi_m = i(\gamma-2m)\ ;\ m = 0,1,2,\cdots,\Im\xi_m \geq 0\}
$$

and

$$
\text{Res}_{\lambda=\xi_m}(\hat{F}(\lambda)) = \frac{(-1)^m}{m!\Gamma(-m+\theta+1)}\int_0^1 W^{\mathbf{R}}_{-\theta}(I)(w)w^{-m+\theta}dw.
$$

Finally, noting (41), (47) and Remark 6.4 (2), we have the following.

Theorem 6.5. *Let* $\alpha,\beta,\eta\in\mathbf{C}$*,* $\Re\alpha>-1$ *and* $g\in\mathcal{B}^{2,\eta/2+\rho}_{0,\eta/2}(\mathbf{R})$*. We suppose that there exists a decomposition* $g=g_0+g_1$*,* $g_0\in C_0^\infty(\mathbf{R})$ *and* $g_1\in\mathcal{B}^{2,\eta/2+\rho}_{\flat,\eta/2}(\mathbf{R})$*, such that* $f=g_1\Delta_{\alpha,\beta}$ *satisfies that, if* $-\Re(\alpha-\beta)>0$*, then* $W^{\mathbf{R}}_{-(\alpha-\beta)}(W^{\mathbf{R}}_{-(\eta+\rho)/2}(H_f^2)(s^2))(w)=O(w^{-(\alpha-\beta)})$ *and, if* $-\Re(\alpha-\beta)\le 0$ *and* $-\Re(\beta+1/2)>0$*, then* $W^{\mathbf{R}}_{-(\eta/2+\rho)}(H_f)\ (w)=O(w^{-(\beta+1/2)})$*. Then, for all* $\phi\in C_0^\infty(\mathbf{R})$*,*

$$\begin{aligned}\langle\phi,g\rangle_{L^2(\mathbf{R}_+,\Delta_{\alpha,\beta}dx)} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\phi}_{\alpha,\beta}(\lambda)\tilde{\overline{g}}_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-(\lambda))^{-1}d\lambda\\ &\quad -\sqrt{2\pi}i\sum_{D_{\alpha,\beta}\cup F_{\eta+\rho}}\hat{\phi}_{\alpha,\beta}(\gamma)\mathrm{Res}_{\lambda=\gamma}\left(\tilde{\overline{g}}_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda)^{-1}\right),\end{aligned}$$

where we supposed that $(D_{\alpha,\beta}\cup F_{\eta+\rho})\cap\mathbf{R}=\emptyset$*. All poles appered in the second sum are simple. If* $g_0=0$ *(resp.* $g_1=0$*), then the second sum corresponding to* $D_{\alpha,\beta}$ *(resp.* $F_{\eta+\rho}$*) vanishes.*

Remark 6.6. (1) If $-\Re(\alpha-\beta)\le 0$ and $-\Re(\beta+1/2)\le 0$, then there are no assumptions on f and $D_{\alpha,\beta}=\emptyset$. This case perfectly coincides with (3) of Theorem 5.1.

(2) In [5] the analytic continuation of $\tilde{g}_{\alpha,\beta}(\lambda)$ is also calculated directly; the poles of $\tilde{g}_{\alpha,\beta}(\lambda)$ lie in $F_{\eta+\rho}$ and if $D_{\alpha,\beta}\cap F_{\eta+\rho}\neq\emptyset$, then $\tilde{g}_{\alpha,\beta}(\lambda)C(-\lambda)^{-1}$ has double poles. However, in Theorem 6.5, no double poles appear, because we use the reduction formula in Corollary 4.3 and we assume the extra conditions on zero of H_f and H_f^2.

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