On Hardy's theorem on $SU(1,1)$

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Abstract

The classical Hardy theorem asserts that f and its Fourier transform $\hat{f}$ can not both be very rapidly decreasing. This theorem was generalized on Lie groups and also for the Fourier-Jacobi transform. However, on $SU(1,1)$ there are infinitely many "good" functions in the sense that f and its spherical Fourier transform $\tilde{f}$ both have good decay. In this paper, we shall characterize such functions on $SU(1,1)$.

Keywords Heat kernel, Jacobi transform, Plancherel formula
2000 MR Subject Classification 22E30, 43A80, 43A90, 33C45

1. Introduction. The classical Hardy theorem [2] asserts that f and its Fourier transform $\tilde{f}$ can not both be "very rapidly decreasing". More precisely, suppose a measurable function f on $\mathbb{R}$ and its Fourier transform $\tilde{f}$ on $\mathbb{R}$ satisfying

$$|f(x)| \leq Ae^{-ax^2} \quad \text{and} \quad |\tilde{f}(\lambda)| \leq Be^{-b\lambda^2} \tag{1}$$

for some positive constants A, B, a and b. If $ab > 1/4$, then $f = 0$, and if $ab = 1/4$, then f is a constant multiple of e^{-ax^2}. Recently, an analogue of Hardy's theorem was established on Lie Groups by various people, where the heat kernel on Lie groups play an essential role to control the decay of f

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and, in the case of $ab = 1/4$, to express a unique function up to a constant multiplication. We refer to [9] and the references there for more information. Moreover, Hardy's theorem was generalized for the Fourier-Jacobi transform (see [1] and [3]) and, as an application, Andersen pointed out that Hardy's theorem on $SU(1,1)$ does not hold unless the K-type of f is fixed: Let $G = SU(1,1)$, and for $g \in G$ let $g = k_\phi a_x k_\psi$, $0 \le x$, $0 \le \phi, \psi \le 4\pi$, denote the Cartan decomposition of g. Let h_t denote the heat kernel on G and for integrable functions f on G let $\tilde{f}_{n,m}$, $n, m \in \frac{1}{2}\mathbb{Z}$, the spherical Fourier transform of f corresponding to the K-type (n,m) (see (12) below). We suppose that a measurable function f on G and its spherical Fourier transform $\tilde{f}_{n,m}$ on $\mathbb{R}$ satisfying

$$|f_{n,m}(g)| \le A h_{1/4a}(g) \quad \text{and} \quad |\tilde{f}_{n,m}(\lambda)| \le B e^{-b\lambda^2} \text{ for all } n, m \in \tfrac{1}{2}\mathbb{Z} \qquad (2)$$

for some positive constants A, B, a and b. Then, $f = 0$ if $ab > 1/4$, however, there are infinitely many linearly independent functions on G satisfying the above condition if $ab = 1/4$ (see Corollary 4.3).

In this paper, we restrict our attention to functions on G with K-types (n,m), $n, m = 0, 1, 2, \cdots$, and we show that the condition (2) under $ab = 1/4$ determines a function on G uniquely in the following sense: In the classical case the condition (1) under $ab = 1/4$ guarantees the limit

$$\lim_{x \to \infty} e^{ax^2} f(x) = c$$

and then f is uniquely determined as $f(x) = ce^{-ax^2}$. On $SU(1,1)$, similarly, the condition (2) under $ab = 1/4$ guarantees the limit

$$\lim_{x \to \infty} h_{1/4a}(x)^{-1} f(k_\phi a_x) = F(\phi)$$

and then f is uniquely determined by using the Fourier coefficient of F. Here $F \in H^2(\mathbb{T})$ and is real analytic. Moreover, the L^2-norm of F on $\mathbb{T}$ coincides with the L^2-norm of the principal part of f on G and the Fourier coefficients $\{d_n; n = 0, 1, 2, \cdots\}$ of F satisfy

$$\sum_{n=0}^{\infty} |d_n|^2 \Big(1 + \sum_{k=0}^{n-1} k e^{2b(2k+1)^2}\Big) < \infty.$$

In Theorem 5.1 we shall give a characterization of F.

2. Notation. Let $G = SU(1,1)$ and A, K the subgroups of G of the matrices

$$a_x = \begin{pmatrix} \mathrm{ch}x/2 & \mathrm{sh}x/2 \\ \mathrm{sh}x/2 & \mathrm{ch}x/2 \end{pmatrix}, \ x \in \mathbb{R} \quad \text{and} \quad k_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}, \ 0 \leq \phi \leq 4\pi$$

respectively. According to the Cartan decomposition of G, each $g \in G$ can be written uniquely as $g = k_\phi a_x k_\psi$ where $0 \leq x$, $0 \leq \phi, \psi \leq 4\pi$. Let $\pi_{j,\lambda}$ ($j = 0, 1/2, \lambda \in \mathbb{R}$) denote the principal series representation of G. Then the (vector-valued) spherical Fourier transform $\pi_{j,\lambda}(f)$ of f on G is defined as $\pi_{j,\lambda}(f) = \int_G f(f)\pi_{j,\lambda}(g)dg$, where dg a Haar measure on G. In the following, we shall consider functions f on G satisfying

$$f(a_x) = f(a_{-x}), \quad x \in \mathbb{R}$$

and we identify f with an even function on $\mathbb{R}$, which is denoted by the same symbol f. Under this restriction, we may suppose that $\pi_{j,\lambda}(f)$ is supported on $j = 0$ and $\lambda > 0$ and the K-types (m, n) of f is supported on $m, n \in \mathbb{Z}$ (cf. [6] and [8, §8]).

Before introducing the explicit form of the spherical Fourier transform of f on G, we shall recall the theory of the Jacobi transform on $\mathbb{R}_+$ (see [4] and [5]). Let $\alpha, \beta, \lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ and consider the differential equation;

$$(L_{\alpha,\beta} + \lambda^2 + \rho^2)f(x) = 0, \tag{3}$$

where $\rho = \alpha + \beta + 1$ and

$$L_{\alpha,\beta} = \frac{d^2}{dx^2} + ((2\alpha+1)\mathrm{cth}x + (2\beta+1)\mathrm{th}x)\frac{d}{dx}.$$

Then, for $\alpha \notin -\mathbb{N}$, the Jacobi function of the first kind with order (α, β)

$$\phi_\lambda^{\alpha,\beta}(x) = F\left(\frac{\rho + i\lambda}{2}, \frac{\rho - i\lambda}{2}; \alpha + 1; -\mathrm{sh}^2 x\right) \tag{4}$$

is a unique solution of (3) satisfying $\phi_\lambda^{\alpha,\beta}(0) = 1$ and $d\phi_\lambda^{\alpha,\beta}/dx(0) = 0$. For $\lambda \notin -i\mathbb{N}$, the Jacobi function of the second kind with order (α, β)

$$\Phi_\lambda^{\alpha,\beta}(t) = (e^t - e^{-t})^{i\lambda-\rho} F\left(\frac{\rho - 2\alpha - i\lambda}{2}, \frac{\rho - i\lambda}{2}; 1 - i\lambda; -\mathrm{sh}^{-2} t\right) \tag{5}$$

is another solution of (3). Then $\Gamma(\alpha+1)^{-1}\phi_\lambda^{\alpha,\beta}$ is entire of α, β, and for $\lambda \notin i\mathbb{Z}$, we have the identity

$$\frac{\sqrt{\pi}}{\Gamma(\alpha+1)}\phi_\lambda^{\alpha,\beta}(t) = \frac{1}{2}\Big(C_{\alpha,\beta}(\lambda)\Phi_\lambda^{\alpha,\beta}(t) + C_{\alpha,\beta}(-\lambda)\Phi_{-\lambda}^{\alpha,\beta}(t)\Big), \tag{6}$$

where $C_{\alpha,\beta}(\lambda)$ is the C-function given by

$$C_{\alpha,\beta}(\lambda) = \frac{2^\rho\Gamma(i\lambda/2)\Gamma((1+i\lambda)/2)}{\Gamma((\rho+i\lambda)/2)\Gamma((\rho-2\beta+i\lambda)/2)} \tag{7}$$

(see [4, (2.5), (2.6)]). For convenience we assume $\alpha > -1$ and $\beta \in \mathbb{R}$ in the following. Then $C_{\alpha,\beta}(-\lambda)^{-1}$ has only simple poles for $\Im\lambda \geq 0$ which lie in the finite set $D_{\alpha,\beta} = \{i(|\beta| - \alpha - 1 - 2m); m = 0, 1, 2, \cdots, |\beta| - \alpha - 1 - 2m > 0\}$. We denote the residue of $(C_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda))^{-1}$ at $\gamma \in D_{\alpha,\beta}$ by

$$d_{\alpha,\beta}(\gamma) = -i\mathrm{Res}_{\lambda=\gamma}(C_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda))^{-1}.$$

Let f be a compactly supported C^∞ even function on $\mathbb{R}$. We define the Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$ by

$$\hat{f}_{\alpha,\beta}(\lambda) = \frac{\sqrt{2}}{\Gamma(\alpha+1)}\int_0^\infty f(x)\phi_\lambda^{\alpha,\beta}(x)\Delta_{\alpha,\beta}(x)dx, \tag{8}$$

where $\Delta_{\alpha,\beta}(x) = (2\mathrm{sh}x)^{2\alpha+1}(2\mathrm{ch}x)^{2\beta+1}$ (see [4, (3.2)] and [5, (2.12)]). Then the inversion formula and the Plancherel formula respectively given as follows:

$$\begin{aligned} f(x) &= \frac{\sqrt{2}}{\Gamma(\alpha+1)}\Big(\int_0^\infty \hat{f}_{\alpha,\beta}(\lambda)\phi_\lambda^{\alpha,\beta}(x)|C_{\alpha,\beta}(\lambda)|^{-2}d\lambda \\ &+ \sum_{\gamma\in D_{\alpha,\beta}} \hat{f}_{\alpha,\beta}(\gamma)\phi_\gamma^{\alpha,\beta}(x)d_{\alpha,\beta}(\gamma)\Big) \end{aligned} \tag{9}$$

and

$$\begin{aligned} \int_0^\infty |f(x)|^2\Delta_{\alpha,\beta}(x)dx &= \int_0^\infty |\hat{f}_{\alpha,\beta}(\lambda)|^2|C_{\alpha,\beta}(\lambda)|^{-2}d\lambda \\ &+ \sum_{\gamma\in D_{\alpha,\beta}} |\hat{f}_{\alpha,\beta}(\gamma)|^2 d_{\alpha,\beta}(\gamma) \end{aligned} \tag{10}$$

(see [4, Theorem 4.2, (5.1)] and [5, Theorem 2.3 and Theorem 2.4]).

Let $h_t^{\alpha,\beta}$ denote the heat kernel for the Jacobi transform, that is, an even function on $\mathbb{R}$ satisfying

$$(h_t^{\alpha,\beta})^{\wedge}_{\alpha,\beta}(\lambda) = e^{-t(\lambda^2+\rho^2)}, \ t, \lambda \in \mathbb{R}. \tag{11}$$

We return to harmonic analysis on $SU(1,1)$. Let $n, m \in \mathbb{Z}$ and $\psi_\lambda^{n,m}(g)$ ($\lambda \in \mathbb{R}$, $g \in G$) denote the matrix coefficient of $\pi_{0,\lambda}(g)$ with K-type (n,m). Let f be a compactly supported C^∞ function on G. We define the scalar-valued spherical Fourier transform $\tilde{f}_{n,m}(\lambda)$ by

$$\tilde{f}_{n,m}(\lambda) = \int_0^\infty f(x)\psi_\lambda^{(n,m)}(x)\Delta_{0,0}(x)dx. \tag{12}$$

We recall that the explicit form of $\psi_\lambda^{n,m}(g)$ is given by using the Jacobi function (4) (cf. [5, (4.17)] and [6, (3.4.10)]): For $g = k_\phi a_x k_\psi \in G$,

$$\psi_\lambda^{n,m}(g) = (\mathrm{ch}x)^{n+m}(\mathrm{sh}x)^{|n-m|}Q_{n,m}(\lambda)\phi_\lambda^{|n-m|,n+m}(x)e^{in\phi}e^{im\psi},$$

where

$$Q_{n,m}(\lambda) = \begin{pmatrix} -1/2 - i\lambda/2 \mp m \\ |n-m| \end{pmatrix}. \tag{13}$$

and $\mp m$ is equal to $-m$ if $m \geq n$ and m if $m \leq n$. Hence from (8) and (11) it follows that

$$\begin{aligned} \tilde{f}_{n,m}(\lambda) &= 2^{-2(n+m)-2|n-m|}Q_{n,m}(\lambda) \\ &\quad \times \Big(f(x)(\mathrm{sh}x)^{-|n-m|}(\mathrm{ch}x)^{-(n+m)}\Big)^{\wedge}_{|n-m|,n+m}(\lambda). \end{aligned} \tag{14}$$

We shall consider the case of $n = m$. Let F be a compactly supported C^∞ even function on $\mathbb{R}$. We put

$$f(g) = F(x)(\mathrm{ch}x)^{2n}e^{in(\phi+\psi)}, \ g = k_\phi a_x k_\psi \in G. \tag{15}$$

Then letting $\alpha = 0, \beta = 2n$ in (10) and (13), we see that

$$\begin{aligned} \int_0^\infty |f(x)|^2\Delta_{0,0}(x)dx &= \int_0^\infty |\tilde{f}_{n,n}(\lambda)|^2|C_{0,0}(\lambda)|^{-2}d\lambda \\ &+ \sum_{k=0}^{|n|-1}(k+1/2)|\tilde{f}_{n,n}((2k+1)i)|^2. \end{aligned} \tag{16}$$

(see [6, (4.21)] and [8, Theorem 8.2]). This is nothing but the Plancherel formula for central compactly supported C^∞ functions on G . We denote by f_P and $^\circ f$ respectively the principal part and discrete part of f on G;

$$f = f_P + {}^\circ f.$$

Then (16) corresponds to the relation $\|f\|^2_{L^2(G)} = \|f_P\|^2_{L^2(G)} + \|{}^\circ f\|^2_{L^2(G)}$.

3. Asymptotic behavior of heat kernels. When $\alpha \geq \beta \geq -1/2$, the asymptotic behavior of $h_t^{\alpha,\beta}(x)$ is well-known (see [1] and [3, Theorem 3.1]), in particular,

$$h_t^{0,0}(x) \sim t^{-1} e^{-\rho^2 t} e^{-\rho x} e^{-x^2/4t} (1+t+x)^{-1/2}(1+x). \tag{17}$$

In this section we shall treat the case of $\alpha, \beta = 0, 1, 2, \cdots$, and we shall investigate a leading term of $h_t^{\alpha,\beta}(x)$ when $x \to \infty$. In the following, we fix $t > 0$ and we denote $a = 1/4t$ for simplicity.

For an even function f on $\mathbb{R}$ let $W_\mu^\sigma(f)$, $\mu \in \mathbb{C}$, $\sigma > 0$, denote the Weyl type fractional integral of f, which is defined by

$$W_\mu^\sigma(f)(y) = \Gamma(\mu)^{-1} \int_y^\infty f(x)(\mathrm{ch}\sigma x - \mathrm{ch}\sigma y)^{\mu-1} d(\mathrm{ch}\sigma x) \tag{18}$$

for $\Re\mu > 0$ and is extended to an entire function in μ (see [4, (3,10), (3.11)]). Then it is known that

$$\hat{f}_{\alpha,\beta}(\lambda) = \mathcal{F}\left(2^{3\alpha+3/2} W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(f)\right),$$

where $\mathcal{F}$ denotes the Euclidean Fourier transform (see [4, (3.7), (3.12)]). Therefore, letting $\alpha = \beta = 0$, it follows from (11) that

$$W^2_{1/2}(h_t^{0,0})(x) = \frac{1}{2^2\sqrt{t}} e^{-t} e^{-ax^2} \tag{19}$$

and moreover, letting $\alpha = m, \beta = n$, $2^{3m+3/2} e^{t(m+n+1)^2} W^1_{m-n} \circ W^2_{n+1/2}(h_t^{m,n})$ does not depend on m, n. Hence, it follows that

$$h_t^{m,n} = 2^{-3m} e^{-t((m+n+1)^2-1)} W^2_{-1/2-n} \circ W^1_{n-m} \circ W^2_{1/2}(h_t^{0,0}). \tag{20}$$

Lemma 3.1. *For* $n = 0, 1, 2, \cdots$,

$$W^2_{-n} \circ W^1_n(f)(x) = \sum_{l=0}^{n-1} c_l^n (\mathrm{ch}x)^{-(n+l)} W^1_l(f)(x), \tag{21}$$

where $4c_l^n = c_l^{n-1} - (n+l-2)c_{l-1}^{n-1}$*. In particular,* $c_0^n = 2^{-2n}$*,* $|c_{n-1}^n| = 2^{-2n}(2n-3)!!$*,* $c_l^n > 0$ *if* l *is even and* $c_l^n < 0$ *if* l *is odd, and*

$$|c_l^n| \le \frac{(2n-3)!!}{2^{2n}(n-1-l)!} \quad (0 < l \le n-1).$$

Proof. Since

$$W_{-1}^2 = \frac{1}{2\mathrm{sh}2x}\frac{d}{dx} = \frac{1}{4\mathrm{ch}x}W_{-1}^1,$$

(21) and the recursive relation $4c_l^n = c_l^{n-1} - (n+l-2)c_{l-1}^{n-1}$ follows from the induction on n. In particular, $4c_0^n = c_l^{n-1}$ and $4c_{n-1}^n = -(2n-3)c_{n-2}^{n-1}$, and thus, $c_0^n = 2^{-2n} > 0$ and $|c_{n-1}^n| = 2^{-2n}(2n-3)!!$. The signature of general c_l^n follows from the recursive relation. Since $4^{n-1}(2n-3)|c_{l-1}^{n-1}| \le 4^n|c_l^n|$, it follows that $4|c_l^n| \le |c_l^{n-1}| + 4(n+l-2)|c_l^n|/(2n-3)$ and thus, $4|c_l^n| \le (2n-3)|c_l^{n-1}|/(n-l-1)$. This means that

$$|c_l^n| \le \frac{(2n-3)!!}{2^{2(n-l-1)}(n-l-1)!(2l-1)!!}|c_l^{l+1}| = \frac{(2n-3)!!}{2^{2n}(n-1-l)!}. \ \blacksquare$$

Lemma 3.2. *Let* $p, q \ge 0$ *and suppose* $q = 0$ *if* $p = 0$*. Then there exists a positive constant* c *such that for all* $l = 0, 1, 2, \cdots$*, and* $x \ge \max(1, 1/a)$*,*

$$W_l^1(e^{-ax^2-px}x^q) \le c2^{-l}(2a)^l e^{l^2/4a} e^{-ax^2-(p-l)x} x^{q-l}.$$

Here, if $l \ge 1$*, then* $(2a)^l x^{-l}$ *can be replaced by* $\Gamma(l)^{-1}(2a)^{-1}x^{-1}$*. Moreover, if* $2^{-l}e^{l^2/4a}$ *is replaced by* 2^{-2l}*, then the lower bound follows.*

Proof. The case of $l = 0$ is obvious, so we may suppose that $l \ge 1$. Since $2\mathrm{sh}x \le e^x$, $s + x \le (1+s)(1+x) \le 2x(1+s)$ for $x \ge 1, s \ge 0$, and $e^{-px}(1+s)^q \le c$, it follows from (18) that

$$\begin{aligned}
\Gamma(l)W_l^1(e^{-ax^2-px}x^q) &= \int_x^\infty e^{-as^2-ps}s^q(\mathrm{ch}s - \mathrm{ch}x)^{l-1}\mathrm{sh}s ds \\
= \ & e^{-ax^2-px}\int_0^\infty e^{-as^2-ps-2axs}(x+s)^q\Big(2\mathrm{sh}(s/2+x)\mathrm{sh}(s/2)\Big)^{l-1}\mathrm{sh}(s+x)ds \\
\le \ & ce^{-ax^2-(p-l)x}x^q\int_0^\infty e^{-as^2+(l/2+1/2)s-2axs}(\mathrm{sh}(s/2))^{l-1}ds \qquad (22)
\end{aligned}$$

$$
\begin{aligned}
&\leq c2^{-l}e^{-ax^2-(p-l)x}x^q\int_0^\infty e^{-as^2+ls-2axs}ds \\
&= c2^{-l}e^{l^2/4a}e^{-ax^2-(p-l)x}x^q(2ax)^{-1},
\end{aligned}
$$

where we used the fact that $-as^2+ls=-a(s-l/2a)^2+l^2/4a$. Since $\mathrm{sh}x \leq xe^x$, the integral in (22) is also estimated as

$$
\leq c2^{-l}e^{l^2/4a}c\int_0^\infty e^{-2axs}s^{l-1}ds = c2^{-l}ce^{l^2/4a}\Gamma(l)(2ax)^{-l}.
$$

Therefore, we can deduce the first estimate.

We note that $\mathrm{sh}x \geq xe^x/2$ for $0 \leq x \leq 1/2$. Since $0 \leq 1/2ax \leq 1/2$ and $s+x \geq x$ for $s, x \geq 0$, it follows that

$$
\begin{aligned}
\Gamma(l)W_l^1(e^{-ax^2-px}x^q) &\geq c2^{-2l}e^{-ax^2-(p-l)x}x^q\int_0^{1/2bx} e^{-as^2-(p-l)s-2axs}s^{l-1}ds \\
&\geq c2^{-2l}e^{-ax^2-(p-l)x}(2ax)^{-l}\int_0^1 e^{-s}s^{l-1}ds.
\end{aligned}
$$

Since $\Gamma(l)^{-1}\int_0^1 e^{-s}s^{l-1}ds$ is bounded below, the lower estimate follows. ∎

Lemma 3.3. *Let $p, q \geq 0$ and suppose $q = 0$ if $p = 0$. Then there exist a positive constants c such that for all $l, n = 0, 1, 2, \cdots$, and $x \geq \max(1, 1/a)$,*

$$
\begin{aligned}
&|W_{-1/2}^2((\mathrm{ch}x)^{n+l}W_l^1(e^{-ax^2-px}x^q))| \\
\leq\ & c2^n(2a)^{-l}e^{l^2/4a}e^{-ax^2-(n+1+p)x}x^{q-l}((n+l)x^{-1/2}+x^{1/2}).
\end{aligned}
$$

Here, if $l \geq 1$, then $(2a)^{-l}x^{-l}$ can be replaced by $\Gamma(l)^{-1}(2a)^{-1}x^{-1}$. Moreover, if $2^n e^{l^2/4a}$ is replaced by 2^{n-l}, then the lower bound follows.

Proof. Since $W_{-1/2}^2 = W_{1/2}^2 \circ W_{-1}^2$, it follows that

$$
\begin{aligned}
&W_{-1/2}^2 \circ ((\mathrm{ch}x)^{-(n+l)}W_l^1) \\
=\ & \frac{1}{4}W_{1/2}^2 \circ (-(n+l)(\mathrm{ch}x)^{-(n+l+2)}W_l^1 + (\mathrm{ch}x)^{-(n+l+1)}W_{l-1}^1). \qquad (23)
\end{aligned}
$$

Therefore, we need to estimate

$$
W_{1/2}^2((\mathrm{ch}x)^{-(n+l+2)}W_l^1(e^{-ax^2-px}x^q)),\ l = -1, 0, 1, 2, \cdots.
$$

Substituting the estimate obtained in Lemma 3.2, we see that for $l \geq 0$,

$$\begin{aligned}
& W^2_{1/2}((\mathrm{ch}x)^{-(n+l+2)}W^1_l(e^{-ax^2-px}x^q)) \\
\leq \ & cc_0\int_x^\infty e^{-as^2-(n+2+p)s}s^{q-l}(\mathrm{ch}2s-\mathrm{ch}2x)^{-1/2}2\mathrm{sh}2s ds \\
\leq \ & cc_0 e^{-ax^2-(n+p)x}\int_0^\infty e^{-as^2-(n+p+2ax)s}(s+x)^{q-l}(\mathrm{ch}2(s+x)-\mathrm{ch}2x)^{-1/2}ds,
\end{aligned}$$

where $c_0 = c2^n(2a)^l e^{l^2/4a}$. We note that, if $l \geq q$, then $(s+x)^{q-l} \leq x^{q-l}$, and if $l \leq q$, then $(s+x)^{q-l} \leq (2x(1+s))^{q-l}$ and $e^{-ps}(1+s)^{q-l} \leq e^{-ps}(1+s)^q \leq c$. Therefore, applying [3, (3.1)] to $(\mathrm{ch}2(s+x)-\mathrm{ch}2x)^{-1/2}$, we have

$$\begin{aligned}
\leq \ & cc_0 e^{-ax^2-(n+1+p)x}x^{q-l}\int_0^\infty e^{-ax^2-(n+1+2ax)s}\Big(\frac{1+2(x+s)}{s(x+s)}\Big)^{1/2}ds \\
\leq \ & cc_0 e^{-ax^2-(n+1+p)x}x^{q-l}\Big(\frac{1}{x}+1\Big)^{1/2}\int_0^\infty e^{-2axs}\frac{1}{\sqrt{s}}ds \qquad (24) \\
\leq \ & cc_0 e^{-ax^2-(n+1+p)x}x^{q-l-1/2}.
\end{aligned}$$

When $l = -1$, we note that $|W^1_{-1}(e^{-ax^2-px}x^q)| \leq c(1+x)^{q+1}e^{-ax^2-px}(\mathrm{sh}x)^{-1}$. Hence, (24) is replaced by

$$\begin{aligned}
\leq \ & c2^n e^{-ax^2-(n+p)x}\int_0^\infty e^{-as^2-(n+p+2ax)s}(s+x)^{q+1}(\mathrm{ch}(s+x)-\mathrm{ch}x)^{-1/2}ds \\
\leq \ & c2^n e^{-ax^2-(n+1+p)x}x^q\int_0^\infty e^{-2axs}\frac{1}{\sqrt{s}}((x+s)(1+2(x+s))^{1/2}ds.
\end{aligned}$$

The last integral is dominated by $x^{1/2}$. Substituting these estimates to (23), we can deduce the desired upper estimate. Other desired estimates follow from Lemma 3.2 and the arguments used in [3, Theorem 3.1]. ■

When $p = q = 0$ and $l = 0$, we have the following refinement.

Lemma 3.4. *For all* $n = 0, 1, 2, \cdots$,

$$W^2_{-1/2}((\mathrm{ch}x)^{-n}e^{-ax^2}) = c_0(\mathrm{ch}x)^{-n}h_t^{0,0}(x) + O\Big(2^n ne^{-ax^2-(n+1)x}x^{-1/2}\Big)^{1)},$$

[1] $f = O(g)$ means that $|f(x)/g(x)| \leq C$ when $x \to \infty$. If C depends on some parameters γ, then we use the symbol $f = O_{(\gamma)}(g)$.

where $c_0 = 2^2\sqrt{t}e^t$.

Proof. Since $e^{-ax^2} = c_0 W^2_{1/2}(h_t^{0,0})$ (see (19)), the case of $n = 0$ is obvious and moreover, for $n \geq 1$, it follows that

$$\begin{aligned} & c_0^{-1} W^2_{-1/2}((\mathrm{ch}x)^{-n} e^{-ax^2}) \\ = & -\int_x^\infty \frac{d}{d\mathrm{ch}2s}\Big(((\mathrm{ch}x)^{-n} - (\mathrm{ch}s)^{-n}) W^2_{1/2}(h_t^{0,0})(s)\Big)(\mathrm{ch}2s - \mathrm{ch}2x)^{-1/2} d\mathrm{ch}2s \\ & +(\mathrm{ch}x)^{-n} h_t^{0,0}(x) \\ = & -\int_x^\infty ((\mathrm{ch}x)^{-n} - (\mathrm{ch}s)^{-n}) e^{-as^2} (\mathrm{ch}2s - \mathrm{ch}2x)^{-3/2} 2\mathrm{sh}2s ds + (\mathrm{ch}x)^{-n} h_t^{0,0}(x). \end{aligned}$$

We note that for $0 \leq x \leq s$,

$$(\mathrm{ch}x)^{-n} - (\mathrm{ch}s)^{-n} \leq \frac{n(\mathrm{ch}s - \mathrm{ch}x)}{\mathrm{ch}x(\mathrm{ch}x)^n}.$$

Therefore, the similar argument in the proof of Lemma 3.3 (or [3, Theorem 3.1]) yields that the last integral is dominated by $2^n n e^{-ax^2-(n+1)x} x^{-1/2}$. ■

Now we shall obtain the asymptotic behavior of $h_t^{m,n}(x)$ as $x \to \infty$. It follows from (19), (20) and (21) that

$$h_t^{m,n} = c_0^{-1} 2^{-3m} e^{-t((m+n+1)^2-1)} \sum_{l=0}^{n-1} c_l^n W^2_{-1/2}((\mathrm{ch}x)^{-(n+l)} W^1_{l-m}(e^{-ax^2})). \quad (25)$$

Since

$$W^1_{-m}(e^{-ax^2}) \sim_{(m)} (2ax)^m e^{-ax^2-mx} \quad {}^{2)},$$

Lemmas 3.4 implies that, when $x \to \infty$, the term corresponding to $l = 0$ contributes to the asymptotic behavior of $h_t^{m,n}(x)$:

Proposition 3.5. *We fix* $t > 0$ *and* $m, n = 0, 1, 2, \cdots$. *Then for* $x, ax \geq 1$

$$h_t^{m,n}(x) \sim_{(t,m,n)} e^{-\rho^2 t} e^{-\rho x} e^{-x^2/4t} (1+x)^{m+1/2}. \quad (26)$$

[2] $f \sim g$ means that there exist positive constants c_1, c_2 such that $c_1 f(x) \leq g(x) \leq c_2 f(x)$. If c_1, c_2 depend on some parameters γ, then we use the symbol $f \sim_{(\gamma)} g$.

Next we shall consider the behavior of $(\mathrm{ch}x)^n h_t^{0,n}(x)$. Let $\epsilon > 0$ and we suppose that

$$x \geq \frac{1}{2}\log\Big(\frac{1}{2^\epsilon - 1}\Big) = x(\epsilon),$$

that is, $\mathrm{ch}x \leq 2^{-1+\epsilon}e^x$ if $x \geq x(\epsilon)$ and $x \to \infty$ if $\epsilon \to 0$. Then it follows from (25), Lemmas 3.3 and 3.4 that

$$\begin{aligned}
&(\mathrm{ch}x)^n h_t^{0,n}(x)\\
= \;& 2^{n\epsilon}e^{-t((n+1)^2-1)}\Big(c_0^n h_t^{0,0}(x) + O\Big(\sum_{l=1}^{n-1}|c_l^n|e^{l^2/4a}\Gamma(l)^{-1}e^{-ax^2-x}nx^{-1/2}\Big)\Big).
\end{aligned}$$

We note that $e^{l^2/4a} \leq e^{(n-1)^2/4a}$ and

$$\sum_{l=1}^{n-1}|c_l^n|\Gamma(l)^{-1} \leq \sum_{l=1}^{n-1}\frac{(n-1)!}{2^{n+2}(n-l-1)!\Gamma(l)} = 2^{-4}(n-1).$$

Hence, it follows that

$$\begin{aligned}
&(\mathrm{ch}x)^n h_t^{0,n}(x)\\
= \;& 2^{n\epsilon}e^{-t((n+1)^2-1)}\Big(c_0^n h_t^{0,0}(x) + O\Big(e^{(n-1)^2/4a}e^{-ax^2-x}n^2x^{-1/2}\Big)\Big)\\
= \;& 2^{n\epsilon}2^{-2n}e^{-t((n+1)^2-1)}h_t^{0,0}(x)\Big(1 + O\Big(n^2 2^{2n}e^{(n-1)^2/4a}x^{-1}\Big)\Big). \qquad (27)
\end{aligned}$$

Letting $x \to \infty$, we have the following.

Proposition 3.6. *We fix $t > 0$ and $n = 0, 1, 2, \cdots$. Then,*

$$\lim_{x\to\infty}\frac{(\mathrm{ch}x)^n h_t^{0,n}(x)}{h_t^{0,0}(x)} = 2^{-2n}e^{-t((n+1)^2-1)}. \qquad (28)$$

4. Hardy's theorem. We keep the notations in the previous section. We recall the proof of Hardy's theorem for the Jacobi transform of (α, β), $\alpha \geq \beta \geq -1/2$ (see [1] and [3]). Then it is easy to see that Hardy's theorem for the Jacobi transform of (m, n), $m, n = 0, 1, 2, \cdots$, also holds:

Theorem 4.1. *Let $m, n = 0, 1, 2, \cdots$, and let f be a measurable function on $\mathbb{R}_+$ satisfying*

$$(i) \;\; g(x) = O_{(m,n)}(h_{1/4a}^{m,n}(x)),$$

$$(ii) \;\; \hat{g}_{m,n}(\lambda) = O_{(m,n)}(e^{-b\lambda^2}).$$

If $ab > 1/4$, then $g = 0$, and if $ab = 1/4$, then g is a constant multiple of $h_b^{m,n}(x)$.

Applying this theorem, we have Hardy's theorem on $SU(1,1)$ for a fixed K-type (see the example in [7]).

Theorem 4.2. *Let f be a measurable function on G of K-type (n,m), $n, m = 0, 1, 2, \cdots$, satisfying*

$$(i)\quad f(x) = O_{(n,m)}\Big(h_{1/4a}^{|n-m|,n+m}(x)(\mathrm{sh}x)^{|n-m|}(\mathrm{ch}x)^{n+m}\Big),$$
$$(ii)\ \tilde{f}_{n,m}(\lambda) = O_{(n,m)}\Big(Q_{n,m}(\lambda)e^{-b\lambda^2}\Big).$$

If $ab > 1/4$, then $f = 0$, and if $ab = 1/4$, then f is a constant multiple of $h_b^{|n-m|,n+m}(x)(\mathrm{sh}x)^{|n-m|}(\mathrm{ch}x)^{n+m}$.

Proof. Let $g(x) = f(x)(\mathrm{sh}x)^{-|n-m|}(\mathrm{ch}x)^{-(n+m)}$. Then

$$g(x) = O_{(n,m)}(h_{1/4a}^{|n-m|,n+m}(x))$$

and

$$\hat{g}_{|n-m|,n+m}(\lambda) = 2^{2(n+m)+2|n-m|}\tilde{f}_{n,m}(\lambda)Q_{n,m}^{-1}(\lambda) = O_{(n,m)}(e^{-b\lambda^2})$$

(see (14)). Then Theorem 4.1 implies that, if $ab > 1/4$, then $g = 0$, and if $ab = 1/4$, then g is a constant multiple of $h_b^{|n-m|,n+m}(x)$ and thus, f is the desired form. ■

Let $L^2_{0+}(G)$ denote the subspace of $L^2(G)$ consisting of all f of the form

$$f = \sum_{n,m=0}^{\infty} f_{n,m},$$

where $f_{n,m}$ is of K-type (n,m).

Corollary 4.3. *Let f be in $L^2_{0+}(G)$ and satisfy for all $n, m = 0, 1, 2, \cdots$,*

$$(i)\quad f_{n,m}(x) = O_{(n,m)}(h_{1/4a}^{0,0}(x)),$$
$$(ii)\ \tilde{f}_{n,m}(\lambda) = O_{(n,m)}(e^{-b\lambda^2}).$$

If $ab > 1/4$, then $f = 0$, and if $ab = 1/4$, then f is of the form

$$f(g) = \sum_{n=0}^{\infty} a_n h_b^{0,2n}(x)(\mathrm{ch}x)^{2n} e^{in(\phi+\psi)}, \tag{29}$$

where $g = k_\phi a_x k_\psi$ and $a_n \in \mathbb{C}$.

Proof. Proposition 3.5 implies that

$$h_t^{|n-m|,n+m}(x)(\mathrm{sh}x)^{|m-n|}(\mathrm{ch}x)^{m+n} \sim_{(m,n)} h_t^{0,0}(x)(1+x)^{|m-n|} \tag{30}$$

for $x, ax \geq 1$. Hence $f_{n,m}(x) = O_{(n,m)}(h_{1/4a}^{|n-m|,n+m}(x)\ (\mathrm{sh}x)^{|n-m|}(\mathrm{ch}x)^{n+m}(1+x)^{-|n-m|})$ and $\tilde{f}_{n,m}(\lambda) = O_{(n,m)}\ (e^{-b\lambda^2}) = O_{(n,m)}(Q_{n,m}(\lambda)e^{-b\lambda^2})$ (see (13)). Hence Theorem 4.2 implies that, if $ab > 1/4$, then $f_{n,m} = 0$, for all $n, m \in \mathbb{Z}$, and thus $f = 0$. If $ab = 1/4$, then $f_{n,m}$ is a constant multiple of $h_b^{|n-m|,n+m}(x)(\mathrm{sh}x)^{|n-m|}(\mathrm{ch}x)^{n+m}$. Since $f_{n,m}(x) = O_{(n,m)}(h_{1/4a}^{0,0}(x))$, it follows that $|n-m| = 0$ (see (30)). Thereby, f must be of the desired form. ■

5. Main theorem. We retain the notations and suppose that

$$f(g) = \sum_{n=0}^{\infty} a_n h_b^{0,2n}(x)(\mathrm{ch}x)^{2n} e^{in(\phi+\psi)} \in L^2_{0+}(G).$$

We recall that

$$\tilde{f}_{n,n}(\lambda) = a_n 2^{-4n} e^{-b((2n+1)^2+\lambda^2)} \tag{31}$$

(see (11) and (14)). Then letting $t = b = 1/4a$ in (16), we obtain the L^2-norm of f on G as follows.

$$\begin{aligned} \int_G |f(g)|^2 dg &= \sum_{n=0}^{\infty} |a_n|^2 \int_0^\infty |h_b^{0,2n}(x)(\mathrm{ch}x)^{2n}|^2 \Delta_{0,0}(x)dx \\ &= \sum_{n=0}^{\infty} |a_n|^2 2^{-8n} e^{-2b(2n+1)^2} \\ &\quad \times \Big(\int_0^\infty e^{-2b\lambda^2} |C_{0,0}(\lambda)|^{-2} d\lambda + \sum_{k=0}^{n-1} (k+1/2) e^{2b(2k+1)^2} \Big). \end{aligned} \tag{32}$$

We define the partial sum f_N, $N = 0, 1, 2, \cdots$, of f as

$$f_N(g) = \sum_{n=0}^{N} a_n h_b^{0,2n}(x)(\mathrm{ch}x)^{2n} e^{in(\phi+\psi)}.$$

Then Proposition 3.6 implies that

$$\begin{aligned} \lim_{N\to\infty} \lim_{x\to\infty} h_b^{0,0}(x)^{-1} f_N(k_\phi a_x) &= \sum_{n=0}^{\infty} a_n 2^{-4n} e^{-b((2n+1)^2-1)} e^{in\phi} \\ &= \sum_{n=0}^{\infty} d_n e^{in\phi} = F(\phi), \end{aligned} \tag{33}$$

where $d_n = 2^{-4n} e^{-b((2n+1)^2-1)} a_n$. Obviously, (32) implies that

$$\|F\|_{L^2(\mathbb{T})} = c\|f_P\|_{L^2(G)} \quad \text{and} \quad \sum_{n=0}^{\infty} |d_n|^2 \Big(1 + \sum_{k=0}^{n-1} k e^{2b(2k+1)^2}\Big) \sim \|f\|_{L^2(G)}^2. \tag{34}$$

Since $\sum_{n=0}^{\infty} |d_n|^2 < \infty$ and $\sum_{n=1}^{\infty} |d_n|^2 (n-1) e^{2b(2n-1)^2} < \infty$, there exists a positive constant C such that $|d_n|(1+n)^{1/2} e^{b(2n-1)^2} \le C$ for all $n = 0, 1, 2, \cdots$. This means that F is real analytic and

$$|a_n| 2^{-4n} (1+n)^{1/2} e^{-b(2n+1)^2} e^{b(2n-1)^2} \le C$$

for all $n = 0, 1, 2, \cdots$. Hence (31) implies

$$\tilde{f}_{n,n}(\lambda) = O((1+n)^{-1/2} e^{-b(2n-1)^2} e^{-b\lambda^2}).$$

We introduce a subspace $A_b^2(\mathbb{T})$ of $H^2(\mathbb{T})$ as follows:

$$\begin{aligned} A_b^2(\mathbb{T}) = \{F(\phi) = \sum_{n=0}^{\infty} d_n e^{in\phi} \in H^2(\mathbb{T}) \ ; \\ \|F\|_{A_b^2(\mathbb{T})}^2 = \sum_{n=0}^{\infty} |d_n|^2 \Big(1 + \sum_{k=0}^{n-1} k e^{2b(2k+1)^2}\Big) < \infty\}. \end{aligned}$$

For $F(\phi) = \sum_{n=0}^{\infty} d_n e^{in\phi} \in A_b^2(\mathbb{T})$, we define a function f on G as (29) with $a_n = 2^{4n} e^{b((2n+1)^2-1)} d_n$. Then (32) and (34) imply $\|f\|_{L^2(G)} \le c\|F\|_{A_b^2(\mathbb{T})}^2$.

Clearly, $|f_{n.n}(x)| = |a_n| h_b^{0,2n}(x)(\mathrm{ch}x)^{2n} = O_{(n)}(h_b^{0,0}(x))$ and (31) implies $\tilde{f}_{n,n}(\lambda) = O_{(n)}(e^{-b\lambda^2})$. Moreover, $\lim_{N\to\infty}\lim_{x\to\infty} h_b^{0,0}(x)^{-1} f_N(k_\phi a_x) = F(\phi)$.

Finally, we have the following theorem.

Theorem 5.1. *Let $ab = 1/4$. Let f be in $L^2_{0+}(G)$ and satisfy for all $n, m = 0, 1, 2, \cdots$,*

$$(i)\ \ f_{n,m}(x) = O_{(n,m)}(h^{0,0}_{1/4a}(x)),$$
$$(ii)\ \ \tilde{f}_{n,m}(\lambda) = O_{(n,m)}(e^{-b\lambda^2}).$$

Then, as an L^2-function on $\mathbb{T}$,

$$(iii)\ \ \lim_{N\to\infty}\lim_{x\to\infty} h_b^{0,0}(x)^{-1} f_N(k_\phi a_x) = F(\phi)$$

exists and $F \in A^2_b(\mathbb{T})$. Here $\|F\|_{L^2(\mathbb{T})} = c\|f_P\|_{L^2(G)}$ and $\|F\|_{A^2_b(\mathbb{T})} \sim \|f\|_{L^2(G)}$. Let $F(\phi) = \sum_{n=0}^\infty d_n e^{in\phi}$ denote the Fourier series of F. Then f is uniquely determined as a central function

$$f(g) = \sum_{n=0}^{\infty} d_n 2^{4n} e^{b((2n+1)^2-1)} h_b^{0,2n}(x)(\mathrm{ch}x)^{2n} e^{in(\phi+\psi)},$$

where $g = k_\phi a_x k_\psi$, and each $\tilde{f}_{n,n}(\lambda)$ satisfies

$$\tilde{f}_{n,n}(\lambda) = O((1+n)^{-1/2} e^{-b(2n-1)^2} e^{-b\lambda^2}).$$

Conversely, if $F \in A^2_b(\mathbb{T})$, then there exists a function $f \in L^2_{0+}(G)$ such that f satisfies (i), (ii) and (iii).

Remark 5.2. (1) We note that, if $f \in L^2_{0+}(G)$ is of the form in (29) and F is given by (iii), then $|f(g) - h_b^{0,0}(x)F(\phi)|$, $g = k_\phi a_x k_\psi$, is dominated as

$$\Big(\sum_{n=0}^{\infty} |a_n| 2^{n\epsilon} e^{-t((n+1)^2-1)} n^2 e^{(n-1)^2/4a}\Big) h_b^{0,0}(x) x^{-1}$$

(see (27)). Therefore, if this sum is finite, then we can replace (iii) by

$$(iii)'\ \ \lim_{x\to\infty} h_b^{0,0}(x)^{-1} f(k_\phi a_x) = F(\phi),$$

and deduce that $f \in L^1(G)$ and $f(x) = O(h_b^{0,0}(x))$.

(2) In Corollary 4.3 we can replace the condition (i) by

$$(i)' \quad f(x) = O(h^{0,0}_{1/4a}(x)),$$

because $(i)'$ implies (i). Also, in Theorem 5.1, it is true if we ignore the last statement of the existence of f for $F \in A^2_b(\mathbb{T})$. As remarked in (1), in order to construct $f \in L^2_{0+}(G)$ from $F \in A^2_b(\mathbb{T})$, which satisfies $(i)'$, (ii) and (iii), it is necessary to control the series in (1).

(3) In Corollary 4.3 and Theorem 5.1, if we replace the condition (i) by

$$(i)'_P \quad (f_P)_{m,n}(x) = O_{(n,m)}(h^{0,0}_{1/4a}(x)),$$

then $a_n = 0$ for $n \neq 0$, that is, f is K-biinvariant. Actually, since $\tilde{f}_{m,n} = (f_P)^{\sim}_{m,n}$, $(i)'_P$ and (ii) imply that f_P is of the form in (29). Since f_P has no discrete part, (32) implies that a_n must be 0 if $n \neq 0$.

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