On a weak L^1 property of maximal operators on non-compact semisimple Lie groups

Takeshi Kawazoe [1]
Keio University at Fujisawa
kawazoe@sfc.keio.ac.jp

Liu Jianming [2]
Peking University
liujm@math.pku.edu.cn

Abstract. We shall give a simple proof of the weak type L^1 inequality for the K-bi-invariant Hardy-Littlewood maximal functions on non-compact real rank one semisimple Lie groups. For higher rank groups we do under an assumption which holds for the most parts. And on $SU(n, n+k)$ we introduce a maximal operator defined by the characteristic function supported on a cube, and show that the operator also satisfies the weak L^1 property.

§1. Introduction.

The maximal theorem, the strong type L^p $(p > 1)$ and the weak type L^1 inequalities for the Hardy-Littlewood maximal functions, was first obtained in the Euclidean space, and then generalized to various spaces. For example, homogeneous groups and semisimple Lie groups.

On homogeneous groups, an appropriate family of dilations is equipped, and the Hardy-Littlewood maximal operator is defined by $\sup_{r>0} |f| * \chi_{B,r}$, where $\chi_{B,r}$ is a dilation of the characteristic function χ_B of the unit ball B. Since the covering lemma, based on the so-called doubling condition, holds on the group, we can prove the maximal theorem by using analogous arguments in the Euclidean space. In this process the shape of the domain on which the characteristic function is supported is not essential, and the fact that dilations of the domain satisfies the doubling property is essential (see [4, Chap.2]).

Non-compact semisimple Lie groups are not homogeneous groups. The L^p inequality was first proved by Clerc and Stein [1] for $p > 1$ and the

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weak type L^1 inequality by Strömberg [10]. In his proof, Strömberg obtained deep estimates of the convolution structure, which are based on the Iwasawa decomposition of G, and carried out a little bit complicated argument. Therefore, it is worth to simplify his proof or to find a new approach to the maximal theorem. Recently, the second author [9] gave a simple proof of the maximal theorem for the Hardy-Littlewood maximal functions associated to the Jacobi transform. By specializing the Jacobi transform, his result gives a simple proof in the case of K-bi-invariant functions on real rank one semisimple Lie groups and his estimates are based on the Cartan decomposition of G.

In this paper, we shall give a further simplified proof of the maximal theorem for K-bi-invariant functions on semisimple Lie groups. The key is, since we assume K-bi-invariance, that the pointwise estimate remarked in [10, Remark 2] can be replaced by the estimate of an integral over K (see ($\star\star$) in Proposition 3.1), which essentially comes from an estimate of the kernel appeared in the integral formula of the product of zonal spherical functions (see ($\star$) in Definition 2.1). By this process we can simplify the arguments in [10] which yield the estimate of σ, however, we still apply [10, Lemma 2] for our conclusion.

This inequality ($\star\star$) was used by the first author in [7,8] for real rank one case. For higher rank case we notice that most of all semisimple Lie groups satisfy the estimate ($\star$) and thereby ($\star\star$), so we can give a simplified proof of the maximal theorem for these higher rank semisimple Lie groups. Actually, except three simple Lie groups; $SL(3, \mathbf{R})$, $SL(4, \mathbf{R})$, and $SO(3, 2)$, all simple Lie groups satisfy ($\star\star$) (see Definition 2.1, Remark 2.2, and Proposition 3.1).

The organization of this paper is the following. In §2 we shall recall some basic facts on the kernel form and we define a class of semisimple Lie groups satisfying the estimate ($\star$), which includes most of all semisimple Lie groups (see Remark 2.2). Then, combining ($\star$) and a sharp estimate of the volume of the ball, we deduce the key inequality ($\star\star$) in §3. The weak type L^1 inequality for the Hardy-Littlewood maximal functions easily follows from this estimate and [10, Lemma 2] in §4 (see Theorem 4.1). In §5 we treat the case of $SU(n, n+k)$ and we introduce a cubic maximal operator, which is defined by using the characteristic function supported on a cube, instead of the unit ball. This operator is one of generalized maximal operators remarked in [10, Remark 2] and thus, it also satisfies the weak type L^1 inequality. In 5.2 we shall give an another approach. We obtain the corresponding estimate

($\star$) inductively and thereby, without using [10, Lemma 2], we prove the weak type L^1 estimate simply and directly (see Theorem 5.4). In this sense this operator is a little better than the Hardy-Littlewood maximal operator.

§2. Kernel form.

Let G be a non-compact connected semisimple Lie group with finite center and $G = KAN$ an Iwasawa decomposition of G. Let Σ^+ denote the set of positive roots for (G, A), A_+ the positive Weyl chamber of A, and $G = KCL(A_+)K$ the Cartan decomposition of G. In what follows we identify A with $\mathbf{R}^n$ and we denote the image of A_+ under the identification by $\mathbf{R}^n_W$. We denote the dual space of the Lie algebra of A by $\mathcal{F}$ and we also identify $\mathcal{F}$ with $\mathbf{R}^n$. Each K-bi-invariant function f on G is determined by its restriction on A as a W-invariant function on A, and thus, as one on A_+. We abuse the following notation:

$$f(g) = f(a_x) = f(x) \quad (g \in Ka_xK, a_x \in A_+, x \in \mathbf{R}^n_W).$$

Especially, the invariant integral on G can be written as

$$\int_G f(g)dg = \int_{\mathbf{R}^n_W} f(x)\Delta(x)dx,$$

where $\Delta(x) = \prod_{\alpha\in\Sigma^+} (e^{\alpha(x)} - e^{-\alpha(x)})$.

Let ϕ_λ, $\lambda \in \mathcal{F}$, be the zonal spherical function on G. The kernel form of the product of two spherical functions is given as

$$\phi_\lambda(x)\phi_\lambda(y) = \int_{\mathbf{R}^n_W} \phi_\lambda(z)K(x,y,z)\Delta(z)dz, \quad x, y \in \mathbf{R}^n_W.$$

Then the Plancherel formula yields that, for all $f \in C_c^\infty(K\backslash G/K)$

$$\begin{aligned}\int_{\mathbf{R}^n_W} f(z)K(x,y,z)\Delta(z)dz &= \int_{\mathcal{F}} \hat{f}(\lambda)\phi_\lambda(x)\phi_\lambda(y)|C(\lambda)|^{-2}d\lambda \\ &= \int_{\mathcal{F}} \hat{f}(\lambda)\int_K \phi_\lambda(a_xka_y)dk|C(\lambda)|^{-2}d\lambda \\ &= \int_K f(a_xka_y)dk. \qquad (1)\end{aligned}$$

Therefore, as a distribution sense, it follows that

$$K(x,y,z) = \int_{\mathcal{F}} \phi_\lambda(x)\phi_\lambda(y)\phi_\lambda(z)|C(\lambda)|^{-2}d\lambda \tag{2}$$

and thereby

$$\sigma(a_x a_y^{-1}) \le \sigma(a_z) \le \sigma(a_x a_y) \quad \text{if } a_x, a_y \in A_+ \text{ and } K(x,y,z) \ne 0, \tag{3}$$

where σ is the distance function on G/K (see [5]).

Definition 2.1. We say that G has a fine kernel if the kernel $K(x,y,z)$ satisfies

$$(\star) \quad K(x,y,z) \le ce^{-(\rho(x)+\rho(y)+\rho(z))}(1+\sigma(z))^{n-1}, \quad x,y,z \in \mathbf{R}^n_W,$$

if $\sigma(a_x), \sigma(a_y), \sigma(a_z) \ge 1$ and $\sigma(a_x a_y^{-1}) \ge 1$.

Remark 2.2. When G is of real rank one ($n = 1$) and not $SU(1,1)$, the desired estimate follows from the explicit form of $K(x,y,z)$ obtained by [3, (4.19)], so except $SU(1,1)$ the real rank one semisimple Lie gorups have fine kernels. For higher rank case, we recall the following Harish-Chandra expansion of $\phi_\lambda(x)$:

$$\phi_\lambda(x) = e^{-\rho(x)} \sum_{w\in W} e^{is\lambda}\Phi(s\lambda, x)C(s\lambda). \tag{4}$$

Especially, if $\sigma(x) \ge 1$, it follows from [5] that

$$|\phi_\lambda(x)||C(\lambda)|^{-1} \le ce^{-\rho(x)}, \quad \lambda \in \mathcal{F}.$$

Here we assume that the C-function is integrable far from the wall. On the wall the C-function has the singularities corresponding to short simple roots. However, in (4) these singularities are canceled by taking the sum over W, because the left hand side has no singularities. Especially, if $\sigma(x) \ge 1$, noting that $\Phi(s\lambda, x)$ is uniformly bounded and its l-th derivative on λ has a polynomial growth of order l on $\sigma(x)$, we see that $\sum_{w\in W} e^{is\lambda}\Phi(s\lambda,x)C(s\lambda)$ is integrable on $\mathcal{F}$ and has a polynomial growth of oredr $n-1$ on $\sigma(x)$. This means that if the C-fucntion is integrable far from the wall, then $e^{\rho(x)}\phi_\lambda(x)$, as a function of λ, is integrable on $\mathcal{F}$ and has a polynomial growth of oredr $n-1$ on $\sigma(x)\geq 1$. Therefore, $(\star)$ follows from (2).

Let d be the dimension of G/K and n the real rank of G/K. When λ is far from the wall, the order of the C-function is gives as

$$|C(\lambda)| \sim (1+\|\lambda\|)^{-\frac{1}{2}(d-n)}, \quad \|\lambda\|\geq 1.$$

Hence, if $d>3n$, the C-function is integrable far from the wall. Therefore, combining the previous observation for real rank one case, except $SU(1,1)$, $SL(3,\mathbf{R})$, $SL(4,\mathbf{R})$, and $SO(3,2)$, all simple Lie groups satisfy the estimate $(\star)$.

§3. A key estimate.

We choose a coordinate of $\mathbf{R}^n$ so that ρ is identified with $(\rho_1,\rho_2,\cdots,\rho_n)$ in $\mathbf{R}^n_W$:

$$\rho(x)=\rho_1x_1+\rho_2x_2+\cdots+\rho_nx_n=\langle\rho,x\rangle, \quad \rho\in\mathbf{R}^n_W,$$

and moreover, we denote $x\in\mathbf{R}^n$ by

$$x=x_0\frac{\rho}{\|\rho\|}+\tilde{x}, \quad x_0\in\mathbf{R},\ \tilde{x}\in\mathbf{R}^{n-1}, \tag{5}$$

where $\langle\rho,\tilde{x}\rangle=0$. Clearly,

$$\Delta(x)\leq e^{2\langle\rho,x\rangle}=e^{2\|\rho\|x_0}, \quad x\in\mathbf{R}^n_W, \tag{6}$$

and, if x is far from the boundaries of $\mathbf{R}^n_W$, then

$$\Delta(x)\sim e^{2\langle\rho,x\rangle}, \tag{7}$$

where the symbol " $\sim$ " means that the ratio of the left hand side to the right hand side is bounded above and below by a positive constant.

Let $B(r)$, $r > 0$, denote the ball with radius r centered at the origin:

$$B(r) = \{g \in G \ ; \ \sigma(g) \leq r\}$$

and $|B(r)|$ the volume of the ball. We define a K-bi-invarinat function τ on G by

$$\tau(g) = \frac{1}{1 + |B(\sigma(g)|} \quad (g \in G). \tag{8}$$

Lemma 3.1. Let S_{n-1} be the unit sphere in $\mathbf{R}^n$ and $d\omega$ the surface measure on S_{n-1}. We fix $c > 0$. Then, for $r > 0$

$$\int_{S_{n-1} \cap \mathbf{R}^n_W} e^{-c(\|\rho\| r - \langle \rho, r\omega \rangle)} d\omega \sim (1+r)^{-(n-1)/2}.$$

Proof. Let $\delta > 0$ be a sufficiently small constant and $S^{\rho,\delta}_{n-1}$ the intersection of S_{n-1} and the cone defined by $C_{\rho,\delta} = \{x \in \mathbf{R}^n_W; \|\rho\|\|x\| - \langle \rho, x \rangle \leq \delta \|\rho\|\|x\|\}$. Then the integration outside of $S^{\rho,\delta}_{n-1}$ is dominated below and above by $e^{-c\delta\|\rho\| r}$. As for the integration over $S^{\rho,\delta}_{n-1}$, we see that

$$\begin{aligned} \int_{S^{\rho,\delta}_{n-1}} e^{-c(\|\rho\| r - \langle \rho, r\omega \rangle)} d\omega &\sim \int_{S^{\rho,\delta}_{n-1}} e^{-c\|\rho\| r(1-\cos\theta_\omega)} d\omega \\ &\sim \int_{S^{\rho,\delta}_{n-1}} e^{-c\|\rho\| r \theta_\omega^2} d\omega \\ &\sim (1+r)^{-(n-1)/2}. \quad \square \end{aligned}$$

Lemma 3.2. When $r > 1$,

$$|B(r)| = \int_{\sigma(a_z) \leq r} \Delta(z) dz \sim e^{2\|\rho\| r} r^{(n-1)/2}.$$

Proof. When the real rank of G is one, the estimate is obvious from (6). We suppose that the real rank of G is greater than one.

$$
\begin{aligned}
|B(r)| &= \int_{\sigma(a_z)\le r} \Delta(z)dz \\
&\le \int_{\|z\|\le r} e^{2\langle\rho,z\rangle}dz \\
&= \int_{\|z\|\le r} e^{-2(\|\rho\|\|z\|-\langle\rho,z\rangle)}e^{2\|\rho\|\|z\|}dz \\
&= \int_0^r \left(\int_{S_{n-1}\cap\mathbf{R}^n_W} e^{-2(\|\rho\|t-\langle\rho,t\omega\rangle)}d\omega\right) e^{2\|\rho\|t}t^{(n-1)}dt \\
&\sim \int_0^r e^{2\|\rho\|t}t^{(n-1)/2}dt \\
&\sim e^{2\|\rho\|r}r^{(n-1)/2}.
\end{aligned}
$$

Let $\delta > 0$ be a sufficiently small constant and $S^{\rho,\delta}_{n-1}$ the surface domain defined in the proof of Lemma 3.1. We put $D^{\rho,\delta}_r = \{z = t\omega\ ;\ r - 1/2 \le t \le r,\ \omega \in S^{\rho,\delta}_{n-1}\}$. Since $D^{\rho,\delta}_r \subset B(r)$ and $D^{\rho,\delta}_r$ is far from the boundaries of $\mathbf{R}^n_W$, it follows from (7) that

$$
\begin{aligned}
|B(r)| &= \int_{\sigma(a_z)\le r} \Delta(z)dz \\
&\ge c\int_{D^{\rho,\delta}_r} e^{2\langle\rho,x\rangle}dz \\
&= c\int_{r-1/2}^r \left(\int_{S^{\rho,\delta}_{n-1}} e^{-2(\|\rho\|t-\langle\rho,t\omega\rangle)}d\omega\right) e^{2\|\rho\|t}t^{(n-1)}dt \\
&\ge c\int_{r-1/2}^r e^{2\|\rho\|t}t^{(n-1)/2}dt \\
&\sim e^{2\|\rho\|r}r^{(n-1)/2}. \quad \square
\end{aligned}
$$

Proposition 3.3. We suppose that $G = SU(1,1)$ or G has a fine kernel and $0 < \gamma < 1$. Then, for all $a_x, a_y \in A_+, \sigma(a_x a_y^{-1}) \ge 2$

$$
(\star\star) \quad \int_K |B(\sigma(a_x k a_y))|^{-1}dk \le ce^{-2\rho(x)}e^{-(\|\rho\|\|x-y\|-\langle\rho,x-y\rangle)}\|x-y\|^{n-1}.
$$

In particular, for all $a_x, a_y \in A_+$

$$\int_K \tau(\sigma(a_x k a_y))dk \leq ce^{-2\rho(x)} e^{-(\|\rho\|\|x-y\|-\langle\rho,x-y\rangle)}(1+\|x-y\|)^{n-1}.$$

Proof. Since $\sigma(a_x a_y^{-1}) \geq 2$, if $\sigma(a_y) \leq 1$, then $\sigma(a_x) > 1$ and

$$\begin{aligned}|B(\sigma(a_x k a_y))|^{-1} &\leq ce^{-2\|\rho\|\|x-y\|}\|x-y\|^{-(n-1)/2} \\ &\leq ce^{-2\|\rho\|\|x\|} \leq ce^{-2\rho(x)}e^{-(\|\rho\|\|x-y\|-\langle\rho,x-y\rangle)}.\end{aligned}$$

On the other hand, if $\sigma(a_x) \leq 1$, the same argument yields ($\star\star$). Thereby, we may assume that $\sigma(a_x), \sigma(a_y) \geq 1$.

When $G = SU(1,1)$, we can obtain ($\star\star$) from a direct calculation (cf. [7, Lemma 2.5]). Hence, we may assume that G has a fine kernel and thus, we can apply ($\star$) to prove ($\star\star$):

$$\begin{aligned}&\int_K |B(a_x k a_y)|^{-1}dk \\ =\ &\int_{A^+} |B(\sigma(a_z))|^{-1}K(x,y,z)\Delta(z)dz \\ \leq\ &ce^{-\langle\rho,x+y\rangle}\int_{\sigma(a_x a_y^{-1})\leq\sigma(z)\leq\sigma(a_x a_y)} e^{-2\|\rho\|\|z\|}\|z\|^{-(n-1)/2}\|z\|^{n-1}e^{\langle\rho,z\rangle}dz \\ =\ &e^{-\langle\rho,x+y\rangle}\int_{\|x-y\|}^{\|x+y\|} e^{-\|\rho\|t}t^{3(n-1)/2}\left(\int_{S_{n-1}\cap\mathbf{R}^n_W} e^{-2(\|\rho\|t-\langle\rho,t\omega\rangle)}d\omega\right)dt \\ \sim\ &ce^{-\langle\rho,x+y\rangle}\int_{\|x-y\|}^{\|x+y\|} e^{-\|\rho\|t}t^{n-1}dt \\ \leq\ &ce^{-\langle\rho,x+y\rangle}e^{-\|\rho\|\|x-y\|}\|x-y\|^{n-1} \\ =\ &ce^{-2\langle\rho,x\rangle}e^{-(\|\rho\|\|x-y\|-\langle\rho,x-y\rangle)}\|x-y\|^{n-1}.\end{aligned}$$

Next we shall estimate τ. When $\sigma(a_x a_y^{-1}) \geq 2$, the assertion follows from ($\star\star$). Hence, we may assume that $\sigma(a_x a_y^{-1}) \leq 2$. If $\sigma(a_x) \leq 1$, then the right hand side is bounded. Since $\tau \leq 1$, the desired estimate is clear. Therefore, we shall consider the case that $\sigma(a_x a_y^{-1}) \leq 2$ and $\sigma(a_x) \geq 1$.

We fix an element a_0 in A^+ such that $\sigma(a_0) \geq 4$. Since τ is continuous as a function of x, $\sigma(a_x a_y^{-1}) \leq 2$, and $\sigma(a_0 a_x a_y^{-1}) \geq 4-2 = 2$, it follows that

$$\begin{aligned}
&\int_K \tau(a_x k a_y^{-1})dk \sim \int_K \tau(a_0 a_x k a_y^{-1})dk \\
\leq\ & ce^{-2\rho(x+\log a_0)} e^{-(\|\rho\|\|x+\log a_0 - y\| - \langle \rho, x+\log a_0 - y\rangle)} \|x + \log a_0 - y\|^{n-1} \\
\sim\ & ce^{-2\rho(x)} e^{-(\|\rho\|\|x-y\| - \langle \rho, x-y\rangle)} (1 + \|x-y\|)^{n-1}. \quad \square
\end{aligned}$$

For the real rank one groups we have a little bit better estimate, which will be used in §5.

Corollary 3.4. We suppose that G is of real rank one. Then for any $\epsilon > 0$

$$\int_K e^{-(2+\epsilon)\rho\sigma(a_x k a_y)} dk \leq ce^{-2\rho x} e^{-\epsilon\rho|x-y|} \quad \text{if } |x-y| > 2.$$

Proof. Since $\tau(g) \sim e^{2\rho\sigma(g)}$, it follows that

$$\int_K e^{-(2+\epsilon)\rho\sigma(a_x k a_y)} dk \leq \int_K \tau(\sigma(a_x k a_y)) dk e^{-\epsilon\rho\sigma(a_x a_y^{-1})} \leq ce^{-2\rho x} e^{-\epsilon\rho|x-y|}. \quad \square$$

Conjecture. The estimate $(\star\star)$ in Proposition 3.3 holds for any semisimple Lie groups.

§4. Weak type L^1 inequality.

The Hardy-Littlewood maximal operator on G is defined by

$$M_{HL}f(g) = \sup_{r>0} \frac{1}{|B(r)|} \int_{B(r)} |f(gh)| dh, \quad g \in G$$

for $f \in L^1(K\backslash G/K)$. We define the local (resp. global) maximal operator M_{HL}^0 (resp. M_{HL}^1) with supremum restricted to the balls of redius $0 < r \leq 1$ (resp. $r > 1$). Then we easily see that

$$M_{HL}f(g) \leq M_{HL}^0 f(g) + M_{HL}^1 f(g), \quad g \in G$$

and

$$M^1_{HL}f(g) \leq \tau * |f|(g),$$

where τ is defined by (8).

Theorem 4.1. We suppose that $G = SU(1,1)$ or G has a fine kernal. Then the maximal operator M_{HL} is of strong (L^p, L^p), $1 < p \leq \infty$, and satisfies the weak type L^1 inequality: for any $\epsilon > 0$ and f in $L^1(K\backslash G/K)$

$$\int_{\{x \in \mathbf{R}^n_W \ ; \ M_{HL}f(x) > \epsilon\}} \Delta(x)dx \leq c\frac{\|f\|_1}{\epsilon}.$$

Proof. Clearly, M_{HL} is of strong (L^∞, L^∞), so we may suppose that $p < \infty$. We shall prove the theorem for M^0_{HL} and M^1_{HL} respectively. As for the local maximal operator M^0_{HL}, we can apply the same argument used in the Euclidean case, and we can deduce that M^0_{HL} is of strong (L^p, L^p), $1 < p < \infty$, and it satisfies the weak type L^1 inequality.

As for M^1_{HL}, since τ belongs to L^{p_0} for all $p_0 > 1$, [1, Lemma 2] yields that M^1_{HL} is of strong (L^p, L^p), $1 < p < \infty$. When $p = 1$, it follows from (1) that

$$M^1_{HL}f(x) \leq c\tau * |f|(x) \leq \int_G \left(\int_K \tau(a_x k a_y) dk \right) |f(y)| \Delta(y) dy.$$

Here we take a sufficiently small $\delta > 0$, and we devide the domain of integration as $\mathbf{R}^n_W = D_1 \cup D_2$, where

$$D_1 = \{y \in \mathbf{R}^n_W \ ; \ \|\tilde{x} - \tilde{y}\| \leq \delta(x_0 - y_0)\}$$

and $D_2 = \mathbf{R}^n_W - D_1$. On D_1, we note that

$$\|x - y\| = |x_0 - y_0| \left(1 + \frac{\|\tilde{x} - \tilde{y}\|^2}{|x_0 - y_0|^2}\right)^{1/2} \sim |x_0 - y_0| + \frac{\|\tilde{x} - \tilde{y}\|^2}{2|x_0 - y_0|}.$$

Therefore, since $\tau(a_x k a_y) \leq \tau(a_x a_y^{-1})$, it follows from Lemma 3.2 that

$$
\begin{aligned}
& M^1_{HL}f(x) \\
\leq\ & c\int_{D_1} e^{-2\|\rho\|\|x-y\|}\|x-y\|^{-(n-1)/2}|f(y)|\Delta(y)dy \\
\leq\ & c\int_{D_1} e^{-2\|\rho\|(|x_0-y_0|+\|\tilde{x}-\tilde{y}\|^2/2|x_0-y_0|)}|x_0-y_0|^{-(n-1)/2}|f(y)|\Delta(y)e^{-2\langle\rho,y\rangle}dy \\
=\ & H_1\tilde{*}F_1(x),
\end{aligned}
$$

where $\tilde{*}$ is the convolution on $\mathbf{R}^n$ and

$$
H_1(x)=e^{-2\|\rho\|(|x_0|+\|\tilde{x}\|^2/2|x_0|)}|x_0|^{-(n-1)/2},\quad F_1(x)=|f(y)|\Delta(y)e^{-2\langle\rho,y\rangle}.
$$

On D_2 we recall that $\|\tilde{x}-\tilde{y}\|>\delta(x_0-y_0)$. If $x_0-y_0\leq 0$, then $\|\rho\|\|x-y\|-\langle\rho,x-y\rangle>\|\rho\|\|x-y\|$, and if $x_0-y_0>0$, then

$$
\begin{aligned}
\|\rho\|\|x-y\|-\langle\rho,x-y\rangle &> \|\rho\|\|x-y\|\left(1-1/\sqrt{1+\frac{\|\tilde{x}-\tilde{y}\|^2}{|x_0-y_0|^2}}\right) \\
&> \|\rho\|\|x-y\|\left(1-1/\sqrt{1+\delta}\right) \\
&= \delta'\|\rho\|\|x-y\|,\quad 0<\delta'<1.
\end{aligned}
$$

Therefore, by using the ineqaulity in Proposition 3.3, we see that

$$
\begin{aligned}
& ce^{-2\langle\rho,x\rangle}\int_{D_2} e^{-(\|\rho\|\|x-y\|-\langle\rho,x-y\rangle)}\|x-y\|^{n-1}|f(y)|\Delta(y)dy \\
\leq\ & ce^{-2\langle\rho,x\rangle}\int_{D_2} e^{-\delta'\|\rho\|\|x-y\|}\|x-y\|^{n-1}|f(y)|\Delta(y)dy \\
=\ & ce^{-2\langle\rho,x\rangle}H_2\tilde{*}F_2(x),
\end{aligned}
$$

where

$$
H_2(x)=e^{-\delta'\|\rho\|\|x\|}\|x\|^{n-1},\quad F_2(x)=|f(y)|\Delta(y).
$$

Since $\int_{\mathbf{R}^n} e^{-\delta'\|\rho\|\|x\|}\|x\|^{n-1}dx<\infty$, it follows that

$$\int_{\mathbf{R}^n} H_2 \tilde{*} F_2(x)dx \leq \int_{\mathbf{R}^n_W} |f(y)|\Delta(y)dy = c\|f\|_1.$$

Hence, we have deduced that

$$\tau * |f|(x) \leq cH_1 \tilde{*} F_1(x) + ce^{-2\langle\rho,x\rangle} H_2 \tilde{*} F_2(x) = J_1(x) + J_2(x).$$

In order to obtain the weak L^1 estimate for M^1_{HL} it is enough to show the estimate for each J_1 and J_2 respectively. As for J_1 we can apply Lemma 2 in [10] and as for J_1, since

$$\|J_2\|_1 = c\int_{\mathbf{R}^n_W} e^{-2\langle\rho,x\rangle} H_2 \tilde{*} F_2(x)\Delta(x)dx \leq c\|f\|_1,$$

it satisfies the weak L^1 inequality.

This completes the proof of the theorem. □

§5. A cubic maximal operator.

In this section we shall define a cubic maximal operator M_C in the case of $SU(n, n+k)$ and we shall prove that M_C satisfies the weak type L^1 inequality as in Theorem 4.1. This result follows from Strömberg's criteria in [10, Remark 2], however, we shall give an inductive and simple proof. We retain the notations used in [5].

5.1. Let $G = G_n = SU(n, n+k)$ $(n \in \mathbf{N}, k \in \mathbf{N} \cup \{0\})$ and $G_n = K_n A_n N_n$ the Iwasawa decomposition of G_n: A_n is the set of all matrices of the form

$$a_t = \exp H_t, \quad H_t = \begin{pmatrix} O_{n,n} & \mathrm{diag}(t_1, t_2, \cdots, t_n) & O_{n,k} \\ \mathrm{diag}(t_1, t_2, \cdots, t_n) & O_{n,n} & O_{n,k} \\ O_{k,n} & O_{k,n} & O_{k,k} \end{pmatrix}$$

for any $t = (t_1, t_2, \cdots, t_n) \in \mathbf{R}^n$ and $K_n = S(U(n) \times U(n+k))$. As in §2, we identify A_n and $\mathcal{F}_n$, the dual space of the Lie algebra of A_n, with $\mathbf{R}^n$. If we define $\alpha_i \in \mathcal{F}_n$, $1 \leq i \leq n$, by $\alpha_i(H_t) = t_i$, then

$$\Sigma^+ = \{\alpha_i, 2\alpha_i \ (1 \leq i \leq n), \alpha_i \pm \alpha_j \ (1 \leq i < j \leq n)\}$$

and

$$m_\alpha = \begin{cases} 2k & \alpha = \alpha_i, \\ 1 & \alpha = 2\alpha_i, \\ 2 & \alpha = \alpha_i \pm \alpha_j. \end{cases}$$

where m_α is the multiplicity of α. The weight $\Delta = \Delta_n$ and $\rho = \rho_n$, half the sum of the positive roots, are respectively given as follows:

$$\Delta_n(t) = \prod_{\alpha \in \Sigma^+} \left(e^{\alpha(t)} - e^{-\alpha(t)} \right) = \sigma(t)\omega(t)^2,$$

where

$$\begin{aligned} \sigma(t) &= 2^{n(2k+1)} \prod_{i=1}^{n} (\sinh 2t_i)^{2k} \sinh 2t_i = 2^{n(2k+1)} \prod_{i=1}^{n} \Delta_1(t_i), \\ \omega(t) &= 2^{n(n-1)/2} \prod_{i<j} (\cosh 2t_i - \cosh 2t_j) = 2^{n(n-1)/2} \det \left(\cosh t_i^{2(j-1)} \right), \end{aligned}$$

and

$$\begin{aligned} \rho_n(t) &= (k+1+2(n-1))t_1 + (k+1+2(n-2))t_2 + \cdots + (k+1)t_n \\ &= \rho_{n,1}t_1 + \rho_{n,2}t_2 + \cdots + \rho_{n,n}t_n. \end{aligned}$$

We note

$$\rho_1 = \rho_{1,1} = k+1.$$

We denote the zonal spherical function and Harish-Chandra's C-function of G_n by $\phi_s^n(g)$ and $C^n(s)$ $(s \in \mathbf{R}^n)$ respectively. For their explicit forms we refer to [3] and [6]. Then Hoogenboom [6] deduced the following reduction formulas: For $t = (t_1, t_2, \cdots, t_n)$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbf{R}_W^n$

$$\phi_\lambda^n(t) = \frac{A}{\prod_{i<j}(\lambda_i^2 - \lambda_j^2)} \cdot \frac{\det \left(\phi_{\lambda_i}^1(t_j) \right)}{\omega(t)}$$

and

$$C^n(\lambda) = A\frac{C^1(\lambda_1)C^1(\lambda_2)\cdots C^1(\lambda_n)}{(-1)^{n(n-1)/2}\det(\lambda_i^{2(j-1)})},$$

where

$$A = (-1)^{n(n-1)/2}2^{2n(n-1)}\prod_{j=1}^{n-1}\left((k+j)^{n-j}j!\right).$$

We now introduce a cubic maximal operator M_C on G. Let $D(r)$ $(r>0)$ denote the domain in $\mathbf{R}_W^n$ defined by

$$D(r) = \{t = (t_1, t_2, \cdots, t_n) \in \mathbf{R}_W^n\ ;\ t_1 + t_2 + \cdots + t_n \leq r\}$$

and χ_r the characteristic function of $D(r)$. We regard χ_r as a K-bi-invariant function on G. Then the maximal operator M_C is defined by

$$M_C f(g) = \sup_{r>0}\frac{1}{|D(r)|}\chi_r * |f|(g), \quad g \in G$$

for $f \in L^1(K\backslash G/K)$.

5.2. In order to obtain the weak L^1 inequality for M_C, we shall apply the same process used in §4 (see 5.3 below). In this process we need to estimate the following integral: For $s \in \mathbf{R}, x, y \in \mathbf{R}_W^n$

$$\int_K \cosh_n(a_x k a_y)^{-2s}dk,$$

where $\cosh_n$ is the K-bi-invariant function on G_n defined by

$$\cosh_n(a_t) = \cosh(t_1)\cosh(t_2)\cdots\cosh(t_n), \quad t = (t_1, t_1, \cdots, t_n) \in \mathbf{R}_W^n.$$

In this subsection, applying a result in [2], we shall estimate the integral. We first calculate the spherical Fourier transform of $\cosh_n^{-2s}$.

Lemma 5.1. For any $\epsilon > 0$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbf{R}_W^n$

$$\left(\cosh_n^{-2(\rho_1+(n-1)+\epsilon)}\right)^\wedge(\lambda) = \left(\cosh_1^{-2(\rho_1+\epsilon)}\right)^\wedge(\lambda_1)\cdots\left(\cosh_1^{-2(\rho_1+\epsilon)}\right)^\wedge(\lambda_n).$$

Proof. For $\gamma_1, \gamma_2, \cdots, \gamma_n > n+k$, we put

$$F(t) = (\cosh t_1)^{-2\gamma_1} (\cosh t_2)^{-2\gamma_2} \cdots (\cosh t_n)^{-2\gamma_n}.$$

By definition, the spherical Fourier transform $\hat{F}$ of F is given by

$$\begin{aligned} \hat{F}(\lambda) &= \int_G f(g) \phi_\lambda^n(g) dg = \int_{\mathbf{R}_W^n} F(t) \phi_\lambda^n(t) \Delta_n(t) dt \\ &= \frac{A}{\prod_{i<j}(\lambda_i^2 - \lambda_j^2)} \\ &\times \int (\cosh t_1)^{-2\gamma_1} \cdots (\cosh t_n)^{-2\gamma_n} \det\left(\phi_{\lambda_i}(t_j)\right) \sigma(t) \omega(t) dt. \end{aligned}$$

We here note that

$$\begin{aligned} & \det\left(\phi_{\lambda_i}(t_j)\right) \omega(t) (\cosh t_1)^{-2\gamma_1} (\cosh t_2)^{-2\gamma_2} \cdots (\cosh t_n)^{-2\gamma_n} \\ = & \det\left(\phi_{\lambda_i}(t_j)\right) \cdot \det\left(\cosh t_i^{2(j-1)}\right) (\cosh t_1)^{-2\gamma_1} \cdots (\cosh t_n)^{-2\gamma_n} \\ = & \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \phi_{\lambda_{\sigma(1)}}(t_1) \phi_{\lambda_{\sigma(2)}}(t_2) \cdots \phi_{\lambda_{\sigma(n)}}(t_n) \\ \times & \sum_{\sigma' \in S_n} \operatorname{sgn}(\sigma') (\cosh t_1)^{2(\sigma'(1)-1)-2\gamma_1} \cdots (\cosh t_n)^{2(\sigma'(n)-1)-2\gamma_n}. \end{aligned}$$

When the real rank of G_n is one ($n=1$), the spherical Fourier transform $\hat{F}_s(\lambda),\ \lambda \in \mathbf{R}$, of $F_s = \cosh^{-2s}$ is given by

$$\hat{F}_s(\lambda) = \left(\cosh^{-2s}\right)^\wedge(\lambda) = \frac{\Gamma(s + (i\lambda - \rho_1)/2)\Gamma(s + (-i\lambda - \rho_1)/2)}{\Gamma(s)^2},$$

where $s > \rho_1 = k+1$ (see [2, p.120]). Especially, if $\gamma - \beta > \rho_1$, then

$$\begin{aligned} & \hat{F}_{\gamma-\beta}(\lambda) \\ = & \int \phi_\lambda^1(t) (\cosh t)^{2\beta} (\cosh t)^{-2\gamma} \Delta_1(t) dt \\ = & \frac{\Gamma(\gamma - \beta + (i\lambda - \rho_1)/2)\Gamma(\gamma - \beta + (-i\lambda - \rho_1)/2)}{\Gamma(\gamma-\beta)^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(\gamma-(n-1)+(i\lambda-\rho_1)/2)\Gamma(\gamma-(n-1)+(-i\lambda-\rho_1)/2)}{\Gamma(\gamma-(n-1))^2} \\
&\quad \times \frac{1}{\prod_{k=1}^{n-1-\beta}(\gamma-\beta-k)^2} \\
&\quad \times 2^{-2(n-1-\beta)} \prod_{k=1}^{n-1-\beta} \left(4(\gamma-\beta-k-\rho_1/2)^2+\lambda^2\right) \\
&= B\hat{F}_{\gamma-(n-1)}(\lambda) \prod_{k=1}^{n-1-\beta} \left(4(\gamma-\beta-k-\rho_1/2)^2+\lambda^2\right),
\end{aligned}
$$

where

$$
B = \frac{2^{-2(n-1-\beta)}}{\prod_{k=1}^{n-1-\beta}(\gamma-\beta-k)^2}.
$$

Therefore, since $\sigma(t) = c\prod_{i=1}^{n}\Delta_1(t_i)$, $\hat{F}(\lambda)$ can be written as

$$
\begin{aligned}
&\frac{B}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)} \\
\times\ & \sum_{\sigma} \operatorname{sgn}(\sigma) \left(\cosh^{-2(\gamma_1-(n-1))}\right)^{\wedge}(\lambda_{\sigma(1)}) \cdots \left(\cosh^{-2(\gamma_n-(n-1))}\right)^{\wedge}(\lambda_{\sigma(n)}) \\
\times\ & \sum_{\sigma'} \operatorname{sgn}(\sigma') \prod_{k=1}^{n-\sigma'(1)} \left(4(\gamma_1-\sigma'(1)+1-k-\rho_1/2)^2+\lambda_{\sigma(1)}^2\right) \\
\times\ & \cdots \prod_{k=1}^{n-\sigma'(n)} \left(4(\gamma_n-\sigma'(n)+1-k-\rho_1/2)^2+\lambda_{\sigma(n)}^2\right).
\end{aligned}
$$

If $\gamma_1 = \gamma_2 = \cdots = \gamma_n = \gamma$, then the last sum is equal to

$$
\begin{aligned}
&\sum_{\sigma'} \operatorname{sgn}(\sigma') \prod_{k=1}^{n-\sigma'(1)} \left(4(\gamma-\sigma'(1)+1-k-\rho_1/2)^2+\lambda_{\sigma(1)}^2\right) \\
&\times \cdots \prod_{k=1}^{n-\sigma'(n)} \left(4(\gamma-\sigma'(n)+1-k-\rho_1/2)^2+\lambda_{\sigma(n)}^2\right) \\
=\ & \sum_{\tau} \operatorname{sgn}(\tau) \prod_{k=1}^{n-1} \left(4(\gamma-1+1-k-\rho_1/2)^2+\lambda_{\tau(1)}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{k=1}^{n-2} \left(4(\gamma - 2 + 1 - k - \rho_1/2)^2 + \lambda_{\tau(2)}^2\right) \\
&\times \cdots \prod_{k=1}^{1} \left(4(\gamma - (n-1) + 1 - k - \rho_1/2)^2 + \lambda_{\tau(n-1)}^2\right) \cdot 1 \\
= \;& \det\left(\prod_{k=1}^{i} \left(4(\gamma - (n-i) + 1 - k - \rho_1/2)^2 + \lambda_j^2\right)\right) \\
= \;& \prod_{i<j} (\lambda_i^2 - \lambda_j^2).
\end{aligned}
$$

Finally, letting $\gamma = n + k + \epsilon = \rho_1 + (n-1) + \epsilon$, $\epsilon > 0$, we obtain the desired result. □

As an application of this lemma, we can deduce the following inequality.

Proposition 5.2. Let notation be as above. Then, for $x, y \in \mathbf{R}_W^n$

$$
\int_K \cosh_n(a_x k a_y)^{-2\rho_{n,1}} dk \le c e^{-2\rho(x)} e^{-2(|x_2-y_2|+2|x_3-y_3|+\cdots+(n-1)|x_n-y_n|)}.
$$

Proof. We easily see that, if $\|z\| \le 1$, then $\cosh_n(a_z a_x k a_y) \sim \cosh_n(a_x k a_y)$ and $e^{-2\rho(z+x)} \sim e^{-2\rho(x)}$. Thereby, we may assume that x, y are far from the boundaries of $\mathbf{R}_W^n$. It follows from (1), Lemma 5.1 with $\epsilon = n-1$, and the Fourier inversion formula for G_1 that

$$
\begin{aligned}
&\int_K \cosh_n(a_x k a_y)^{-2\rho_{n,1}} dk \\
= \;& \int \phi_\lambda(x)\phi_\lambda(y) \left(\cosh_n^{-2(\rho_1 + 2(n-1))}\right)^\wedge (\lambda) |C(\lambda)|^{-2} d\lambda \\
= \;& c \frac{1}{\omega(x)} \frac{1}{\omega(y)} \int \cdots \int \frac{\det(\phi_{\lambda_i}(x_j))}{\prod_{i<j}(\lambda_i^2 - \lambda_j^2)} \frac{\det(\phi_{\lambda_i}(y_j))}{\prod_{i<j}(\lambda_i^2 - \lambda_j^2)} \\
& \times \left(\cosh_1^{-2(\rho_1 + (n-1))}\right)^\wedge (\lambda_1) \cdots \left(\cosh_1^{-2(\rho_1 + (n-1))}\right)^\wedge (\lambda_n) \\
& \times \left| \frac{C_1(\lambda_1) \cdots C_1(\lambda_n)}{\prod_{i<j}(\lambda_i^2 - \lambda_j^2)} \right|^{-2} d\lambda_1 \cdots d\lambda_n
\end{aligned}
$$

$$
\begin{aligned}
&= c\frac{1}{\omega(x)}\frac{1}{\omega(y)}\sum_{\sigma}\sum_{\sigma'}\mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma') \\
&\quad \times\int\cdots\int \phi_{\lambda_1}(x_{\sigma(1)})\cdots\phi_{\lambda_n}(x_{\sigma(n)})\phi_{\lambda_1}(y_{\sigma'(1)})\cdots\phi_{\lambda_n}(y_{\sigma'(n)}) \\
&\quad \times\left(\cosh_1^{-2(\rho_1+(n-1))}\right)^{\wedge}(\lambda_1)\cdots\left(\cosh_1^{-2(\rho_1+(n-1))}\right)^{\wedge}(\lambda_n) \\
&\quad \times|C_1(\lambda_1)|^{-2}\cdots|C_1(\lambda_n)|^{-2}d\lambda_1\cdots d\lambda_n \\
&= c\frac{1}{\omega(x)}\frac{1}{\omega(y)}\sum_{\sigma}\sum_{\sigma'}\mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma')\int_{K_1}\cosh_1(a_{x_{\sigma(1)}}k_1a_{y_{\sigma'(1)}})^{-2(\rho_1+\epsilon)}dk_1 \\
&\quad \times\cdots\int_{K_1}\cosh_1(a_{x_{\sigma(n)}}k_1a_{y_{\sigma'(n)}}))^{-2(\rho_1+\epsilon)}dk_1.
\end{aligned}
$$

For $x=(x_1,x_2,\cdots,x_n)$ and $y=(y_1,y_2,\cdots,y_n)$, we put

$$
u_i=\max(x_i,y_i), \quad v_i=\min(x_i,y_i).
$$

Then, Corollary 3.4 yields that

$$
\begin{aligned}
&\int_K \cosh_n(a_xka_y)^{-2\rho_{n,1}}dk \\
&\le c\frac{1}{\omega(x)}\frac{1}{\omega(y)}(\cosh_n u)^{-2\rho_1}(\cosh_n u)^{-2(n-1)}(\cosh_n v)^{2(n-1)}.
\end{aligned}
$$

Here, we recall that x,y are far from the boundaries of $\in \mathbf{R}_W^n$, and thereby

$$
\begin{aligned}
&\frac{1}{\omega(x)}\frac{1}{\omega(y)}(\cosh_n u)^{-2(n-1)}(\cosh_n v)^{2(n-1)} \\
&\le ce^{-2((n-1)x_1+(n-2)x_2+\cdots+x_{n-1})}e^{-2((n-1)y_1+(n-2)y_2+\cdots+y_{n-1})} \\
&\quad \times e^{-2(n-1)(u_1+u_2+\cdots+u_n)}e^{2(n-1)(v_1+v_2+\cdots+v_n)} \\
&\le ce^{-4((n-1)u_1+(n-2)u_2+\cdots+u_{n-1})}e^{-2(|x_2-y_2|+2|x_3-y_3|+\cdots+(n-1)|x_n-y_n|)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_K \cosh_n(a_xka_y)^{-2\rho_{n,1}}dk \\
&\le c(\cosh_n u)^{-2\rho_1}e^{-4((n-1)u_1+(n-2)u_2+\cdots+u_{n-1})}e^{-2(|x_2-y_2|+\cdots+(n-1)|x_n-y_n|)} \\
&\le ce^{-2\rho(u)}e^{-2(|x_2-y_2|+2|x_3-y_3|+\cdots+(n-1)|x_n-y_n|)}.
\end{aligned}
$$

This completes the proof of Proposition 5.2. □.

5.3. Now we shall obtain the weak type L^1 inequality of the cubic maximal operator M_C. First we note the following.

Lemma 5.3. Let notation be as above. Then

$$\frac{1}{|D(r)|}\chi_r(t) \leq c(\cosh_n t)^{-2\rho_{n,1}} \quad (r \geq n^2).$$

Proof. Let $D'(r)$ be the domain in $\mathbf{R}^n_W$ defined by

$$\begin{array}{ccccc} 2(n-1)+1 & \leq & t_1 & \leq & r-n(n-1), \\ 2(n-2)+1 & \leq & t_2 & \leq & 2(n-1), \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 3 & \leq & t_{n-1} & \leq & 4, \\ 1 & \leq & t_n & \leq & 2. \end{array}$$

Clearly, $D'(r) \subset D(r)$ and $|t_i - t_j| \geq 1$ if $t \in D'(r)$ and $i \neq j$. Hence,

$$\begin{aligned} |D(r)| &\geq \int_{D'(r)} \Delta(t)dt \\ &= c\int_{2(n-1)+1}^{r-n(n-1)} e^{2\rho_{n,1}t_1}dt_1 \int_{2(n-2)+1}^{2(n-1)} e^{2\rho_{n,2}t_2}dt_2 \cdots \int_1^2 e^{2\rho_{n,n}t_n}dt_n \\ &= ce^{2\rho_{n,1}r}. \end{aligned}$$

Since $e^{-sr}\chi_r(t)$, $s > 0$, is dominated by $\cosh_n^{-s}(t)$, the lemma follows. □

We note that

$$(\cosh_n t)^{-2\rho_{n,1}} \leq ce^{-2\rho(t)}e^{-2(t_2+2t_3+\cdots+(n-1)t_n)}$$

and $\omega(t) = e^{-2(t_2+2t_3+\cdots+(n-1)t_n)}$ satisfies the contitions (I) and (II) in [10, Remark 2]. Therefore, M_C satisfies the weak type L^1 inequality as remarked in [10]. Here, we shall give a simple and direct proof based on Proposition 5.2, and we don't use Lemma 2 in [10].

Theorem 5.4. The maximal operator M_C is of strong (L^p, L^p), $1 < p \leq \infty$, and satisfies the weak type L^1 inequality: for any $\epsilon > 0$ and f in $L^1(K\backslash G/K)$

$$\int_{\{x\in\mathbf{R}^n_W;\ M_Cf(x)\geq\epsilon\}}\Delta(x)dx\leq c\frac{\|f\|_1}{\epsilon}.$$

Proof. We note that the L^1 norm of $\chi_r/|D(r)|$ equals 1 and $\cosh_n^{-2(\rho_1+2(n-1))}$ belongs to L^{p_0} for all $p_0>1$. Hence, as in the proof of Theorem 4.1, it is enough to show that a global part of M_C satisfies the weak type L^1 estimate. Proposition 5.2 and Lemma 5.3 yield that

$$\begin{aligned}
&\sup_{n^2\leq r}\frac{1}{|D(r)|}\chi_r*|f|(x)\\
\leq\ & c\cosh_n^{-2\rho_{n,1}}*|f|(x)\\
\leq\ & ce^{-2\rho(x)}\int_{\mathbf{R}^n_W}e^{-2(|x_2-y_2|+2|x_3-y_3|+\cdots+(n-1)|x_n-y_n|)}|f(y)|\Delta(y)dy\\
\leq\ & ce^{-2\rho(x)}H(x').
\end{aligned}$$

Here $x'=(x_2,x_3,\ldots,x_n)$ and, as a function on $\mathbf{R}^{n-1}$,

$$H(x')=\int_{\mathbf{R}}E*'|f\Delta|(y_1,x')dy_1,$$

where $E(x')=e^{-2(|x_2|+2|x_3|+\cdots+(n-1)|x_n|)}$ and $*'$ is the convolution on $\mathbf{R}^{n-1}$. Clearly, $\|H\|_{L^1(\mathbf{R}^{n-1})}\leq\|E\|_{L^1(\mathbf{R}^{n-1})}\|f\|_1$. We denote $\rho=\rho_0+\rho'$, where $\rho_0(x)=\rho_{n,1}x_1$, and thus, $\rho(x)=\rho_0(x_1)+\rho'(x')$. Since

$$\begin{aligned}
&\int_{\{x\in\mathbf{R}^n_W\ ;\ e^{-2\rho(x)}H(x')>\lambda\}}\Delta(x)dx\\
\leq\ & \int_{\mathbf{R}^{n-1}}\left(\int_{\{x_1>0\ ;\ e^{-2\rho_0(x_1)}>\lambda/e^{-2\rho'(x')}H(x')\}}e^{2\rho_0(x_1)}dx_1\right)e^{2\rho'(x')}dx'\\
=\ & \int_{\mathbf{R}^{n-1}}\frac{H(x')}{\lambda}dx'\leq\frac{\|f\|_1}{\lambda},
\end{aligned}$$

the weak type L^1 estimate follows. □

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