HEAT KERNEL AND HARDY'S THEOREM FOR JACOBI TRANSFORM

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Abstract. In the first part of this paper, we obtain sharp upper and lower bounds for the heat kernel associated with Jacobi transform. In the second part, by using the sharp estimate of the heat kernel, we get some analogues of Hardy' Theorem for Jacobi transform, which asserts that a even function f on $\mathbb{R}$ and its Jacobi transform can not both be "very rapidly decreasing".

1. Introduction

Jacobi functions $\phi_\lambda(t)$ of order $(\alpha, \beta)(\alpha \neq -1, -2, \dots)$ is the even C^∞-function on $\mathbb{R}$ which equals 1 at 0 and which satisfies the differential equation

$$(L + \lambda^2 + (\alpha + \beta + 1)^2)\phi_\lambda(t) = 0, \tag{1.1}$$

where

$$L = \frac{d^2}{dt^2} + ((2\alpha + 1)\coth t + (2\beta + 1)\tanh t)\frac{d}{dt} \tag{1.2}$$

The functions $\phi_\lambda(t)$ can be expressed by using Gaussian hypergeometric function as

$$\phi_\lambda(t) = F(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda); \alpha + 1; -\sinh^2 t),$$

where $\rho = \alpha + \beta + 1$.

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In this paper, we obtain the sharp upper and lower bounds for the corresponding heat kernel, i.e. the kernel of e^{-tL}, when $\alpha \geq \beta \geq -\frac{1}{2}$. For certain discrete values of α, β, L has an interpretation as the radial part of the Laplace operator on rank one symmetric space, the sharp estimates of the corresponding heat kernel was obtain in [4], [7] and [1].

In the second part we discuss the analogue of Hardy' Theorem for Jacobi transform. The Hardy's theorem[8] asserts f and its Fourier transform $\mathcal{F}f$ can not both be "very rapidly decreasing". More precisely, if $|f(x)| \leq A\exp(-\alpha|x|^2)$ and $|\hat{f}(y)| \leq B\exp(-\beta|y|^2)$ and $\alpha\beta > \frac{1}{4}$, then $f \equiv 0$. Here we use the Fourier transform defined by

$$\mathcal{F}f(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}dx.$$

M. G. Cowling and J. F. Price[3] get an L^p-version of Hary' theorem which says: suppose that $1 \leq p, q \leq \infty$ and one of them is finite. If a measurable function f on $\mathbb{R}$ satisfies $\|e^{ax}f(x)\|_{L^p(\mathbb{R})} < \infty$ and $\|e^{bx}f(x)\|_{L^p(\mathbb{R})} < \infty$ and $ab \geq \frac{1}{4}$ then $f = 0$ almost everywhere.

Recently, an analogue of Hardy's theorem and Cowling and Price' theorem were established for some Lie Groups (see [2], [5], [10] and [11]).

In this paper, we will establish the corresponding theorem for Jacobi transform. By using the sharp estimate of the heat kernel, we we get a more precise result (Theorem 4.3) than what obtained for some Lie Groups.

2. Notations and Preliminaries

We have the following elementary estimates(see [9]):

Lemma 2.1. *Assume that* $\alpha \geq \beta \geq -\frac{1}{2}$, $\lambda = \mu + i\nu$, $\mu, \nu \in \mathbb{R}$. *Then*

$$|\phi_{\mu+i\nu}| \leq |\phi_{i\nu}|$$
$$|\phi_\lambda| \leq C(1+t)e^{t(|\nu|-\rho)}, \qquad \text{for all } t \geq 0, \lambda \in \mathbb{C}.$$

The Jacobi transform is defined by

$$\hat{f}(\lambda) = \int_0^\infty f(t)\phi_\lambda(t)\Delta(t)\,dt, \tag{2.1}$$

where

$$\Delta(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}. \tag{2.2}$$

Let $\mathcal{S}$ denotes the set of even rapidly decreasing function on $\mathbb{R}$. We have the inversion formula(See [6] and [9]):

Lemma 2.2. *For $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha \neq -\frac{1}{2}$, $f \in (\cosh t)^{-r}\mathcal{S}(r > \rho)$, $t \in \mathbb{R}$, then*

$$f(t) = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(t)|c(\lambda)|^{-2}d\lambda,$$

where $c-$function $c(\lambda)$ is defined by

$$c(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\rho))\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))}. \tag{2.3}$$

Let $L^p(\mathbb{R}^+, \Delta(t)dt)$ $(1 \leq p < \infty)$ be the set of even functions on $\mathbb{R}$ such that

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p\Delta(t)\,dt\right)^{\frac{1}{p}} < \infty.$$

The space $L^\infty(\mathbb{R}^+, \Delta(t)dt)$ is defined in the obvious way.

For $f \in (\cosh t)^{-r}\mathcal{S}(r > \rho)$, the Jacobi transform $f \to \hat{f}$ has a factorization

$$\hat{f} = \mathcal{F}(F_f), \tag{2.4}$$

where $\mathcal{F}$ is the classical Fourier transform and $f \to F_f$ is the Abel transform. The fractional integral operator is used to give the inversion of the Abel transform. Let $\Re\mu > 0$, $\tau > 0$, the integral operator $f \to W_\mu^\tau f$ is defined by

$$W_\mu^\tau f(s) = \frac{1}{\Gamma(\mu)}\int_s^\infty f(t)(\cosh\tau t - \cosh\tau s)^{\mu-1}d\cosh\tau t. \tag{2.5}$$

Then W_μ^τ has analytic continuation to all complex μ: if $n = 0, 1, 2, \ldots,$, $\Re\mu > -n$, then

$$W_\mu^\tau f(s) = \frac{(-1)^n}{\Gamma(\mu+n)}\int_s^\infty \frac{d^n}{d(\cosh\tau t)^n}f(t)(\cosh\tau t - \cosh\tau s)^{\mu+n-1}d\cosh\tau t. \tag{2.6}$$

We have the following facts

$$W_\mu^\tau \cdot W_\nu^\tau = W_{\mu+\nu}^\tau, \tag{2.7}$$

$$W_{-n}^\tau = (-1)^n \frac{d^n}{d(\cosh \tau t)^n}. \tag{2.8}$$

The Abel transform F_f can be expressed by

$$F_f = 2^{3\alpha+\frac{1}{2}} \pi^{-\frac{1}{2}} \Gamma(\alpha+1) W_{\alpha-\beta}^1 W_{\beta+\frac{1}{2}}^2 f, \tag{2.9}$$

and it can be inverted by

$$f = 2^{-3\alpha-\frac{1}{2}} \pi^{\frac{1}{2}} (\Gamma(\alpha+1))^{-1} W_{-\beta-\frac{1}{2}}^2 W_{-\alpha+\beta}^1 (F_f), \tag{2.10}$$

3. Heat Kernel

By the use of Jacobi transform, one has

$$e^{-tL} f = h_t * f$$

for all $t > 0$ and $f \in L^2(\Delta)$, where $*$ denote the convolution associated with Jacobi transform and h_t is given by

$$h_t(x) = \frac{1}{2\pi} \int_0^\infty e^{-t(\lambda^2+\rho^2)} \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda. \tag{3.1}$$

Let

$$g_t(x) = \mathcal{F}^{-1}(e^{-t(\lambda^2+\rho^2)})(x) = \frac{1}{\sqrt{4\pi t}} e^{-\rho^2 t - \frac{x^2}{4t}}, \tag{3.2}$$

then by (2.4) and the inversion formula (2.10) of Abel transform, we have

$$h_t = 2^{-3\alpha-\frac{1}{2}} \pi^{\frac{1}{2}} (\Gamma(\alpha+1))^{-1} W_{-\beta-\frac{1}{2}}^2 W_{-\alpha+\beta}^1 (g_t). \tag{3.3}$$

In the following we will use the following simple facts: there are positive constants C_1 and C_2 such that for all $x, h > 0$ we have

$$C_1 \frac{h(x+h)}{1+x+h} e^{x+\frac{1}{2}h} \leq \cosh(x+h) - \cosh x \leq C_2 \frac{h(x+h)}{1+x+h} e^{x+h}. \tag{3.4}$$

Now will prove the following sharp estimate for the heat kernel $h_t(x)$.

Theorem 3.1. *For $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$, and $t > 0$, we have*

$$h_t(x) \sim t^{-\alpha-1} e^{-\rho^2 t - \rho x - \frac{x^2}{4t}} (1+t+x)^{\alpha-\frac{1}{2}} (1+x). \tag{3.5}$$

Here $f \sim g$ means there exist positive constants C_1 and C_2 such that $C_1 g \leq f \leq C_2 g$.

We divide the proof in four steps:

Step 1. Let $K_n(t,x) = W^1_{-n}(g_t)(x)$, then we have

Lemma 3.2. *For $n = 0, 1, 2, \ldots$, $t > 0$,*

$$K_n(t,y) \sim t^{-n-\frac{1}{2}} e^{-\rho^2 t - ny - \frac{y^2}{4t}} (1+t+y)^{n-1} (1+y). \tag{3.6}$$

Proof. By (2.8),

$$K_n(t,y) = (-1)^n \frac{d^n}{d(\cosh y)^n} \left(\frac{1}{\sqrt{4\pi t}} e^{-\rho^2 t - \frac{y^2}{4t}}\right).$$

Let

$$K_n(t,y) = \frac{1}{\sqrt{4\pi t}} \left(\frac{1}{2t}\right)^n e^{-\rho^2 t - \frac{y^2}{4t}} \alpha_n(t,y).$$

It is easy to see that

$$\alpha_{n+1}(t,y) = \frac{y}{\sinh y} \alpha_n(t,y) - \frac{2t}{\sinh y} \frac{d\alpha_n}{dy}(t,y)$$

with $\alpha_0(t,y) = 0$, $\alpha_1(t,y) = y/\sinh y$. Then just as in [4], we have

$$\alpha_n(t,y) \sim e^{-ny}(1+y)(1+y+t)^{n-1}.$$

This completes the proof of (3.6).

Step 2. We have

Lemma 3.3. *For $\alpha \geq \beta \geq \frac{1}{2}$, $t > 0$,*

$$W^1_{-\alpha+\beta}(g_t)(y) \sim t^{-(\alpha-\beta+\frac{1}{2})} e^{-\rho^2 t - \frac{y^2}{4t} - (\alpha-\beta) y} (1+t+y)^{\alpha-\beta-1} (1+y). \tag{3.7}$$

Proof. Since the proof is similar to that in [4], we only give a sketch. The case of $\alpha-\beta$ be an integer has been proved in Step. 1. Now we assume that $\alpha-\beta=n-a$, where $0<a<1$, then

$$W^1_{-\alpha+\beta}(g_t)=W^1_aW^1_{-n}(g_t)=W^1_a(K_n)$$

then by (3.6), and the fact $\sinh y\sim y(1+y)^{-1}e^y$, we have

$$\begin{aligned}W^1_{-\alpha+\beta}(g_t)(x)&=\frac{1}{\Gamma(a)}\int_x^\infty K_n(t,y)\frac{d\cosh y}{(\cosh y-\cosh x)^{1-a}}\\&\sim t^{-n-\frac{1}{2}}e^{-\rho^2t-(n-1)x-\frac{x^2}{4t}}I_1\end{aligned}\tag{3.8}$$

where

$$I_1=\int_0^\infty e^{-(n-1)h-\frac{2xh+h^2}{4t}}(1+t+x+h)^{n-1}(x+h)\frac{dh}{(\cosh(x+h)-\cosh x)^{1-a}}\tag{3.9}$$

Now we only need to prove

$$I_1\sim e^{(-1+a)x}t^a(1+t+x)^{n-a-1}(1+x)\tag{3.10}$$

First by (3.4), and use the simple inequalities $1+x+h+t\le(1+x+t)(1+h)$ and $1+x+h\le(1+x)(1+h)$, we have

$$I_1\le Ce^{(-1+a)x}(1+x+t)^{n-1}(1+x)I_2$$

where

$$I_2=\int_0^\infty e^{-(n-\frac{a+1}{2})h-\frac{2xh+h^2}{4t}}(1+h)^n\frac{1}{h^{1-a}}(\frac{x+h}{1+x+h})^adh$$

If $t\ge 1+x$,

$$I_2\le\int_0^\infty e^{-(n-\frac{a+1}{2})h}(1+h)^n\frac{1}{h^{1-a}}dh\le C\sim t^a(1+x+t)^{-a}.$$

If $t\le 1+x$, and $x\ge 1$,

$$I_3\le C\int_0^\infty e^{-\frac{2xh}{4t}}\frac{1}{h^{1-a}}dh\sim Ct^a(1+x+t)^{-a}$$

If $t \le 1+x$ and $x \le 1$,

$$\begin{aligned} I_2 &\le C\int_0^\infty e^{-\frac{2xh}{4t}} x^a \frac{1}{h^{1-a}} dh + C\int_0^\infty e^{-\frac{h^2}{4t}} h^a \frac{1}{h^{1-a}} dh \\ &\le C(\frac{2t}{x})^a \int_0^\infty e^{-h} x^a \frac{1}{h^{1-a}} dh + C(\sqrt{t})^{2a} \int_0^\infty e^{-\frac{h^2}{4}} h^{2a-1} dh \\ &= Ct^a \sim Ct^a(1+x+t)^{-a} \end{aligned}$$

Now we have proved that

$$I_1 \le Ce^{(-1+a)x} t^a (1+t+x)^{n-a-1}(1+x) \tag{3.11}$$

On the other hand, by (3.4), it is easy to see that

$$I_1 \ge Ce^{(-1+a)x}(1+x+t)^{n-1}(1+x)^{1-a} I_2' \tag{3.12}$$

where

$$I_2' = \int_0^\infty e^{-(n-a)h - \frac{2xh+h^2}{4t}} \frac{1}{h^{1-a}} (x+h)^a dh.$$

We have the following estimates for I_2':

$$\begin{aligned} I_2' &\ge \int_0^\infty e^{-(n-a)h - \frac{2xh+h^2}{4t}} h^{2a-1} dh \\ &\sim t^a(1+x/\sqrt{t}+2\sqrt{t}(n-a))^{-2a} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} I_2' &\ge \int_0^\infty e^{-(n-a)h - \frac{2xh+h^2}{4t}} x^a h^{a-1} dh \\ &\ge \int_0^{\sqrt{t}} e^{-(n-a)h - \frac{2xh}{4t}} x^a h^{a-1} dh \\ &\sim x^a t^{a/2}(1+\sqrt{t}(\frac{x}{2t}+n-a))^{-a}. \end{aligned} \tag{3.14}$$

Just as in [4], we let $A_1 = \{(x,t) : 0 \le x \le 1 \text{ and } t \ge x^2\}$, $A_2 = \{(x,t) : 1 \le x \le t$, $A_3 = \{(x,t) : 0 \le t \le x \text{ and } t \le x^2\}$. Then $\{(x,t) : x, t \ge 0\} = A_1 \cup A_2 \cup A_3$. For $(x,t) \in A_1$,

$$t^a(1+x/\sqrt{t}+2\sqrt{t}(n-a))^{-2a} \sim t^a(1+t)^{-a} \sim (1+x)^a t^a (1+x+t)^{-a}.$$

And for $(x,t) \in A_2 \cup A_3$,

$$x^a t^{a/2}(1+\sqrt{t}(\frac{x}{2t}+n-a))^{-a} \sim (1+x)^a t^a (1+x+t)^{-a}.$$

Then from (3.12), (3.13) and (3.14) we get

$$I_1 \geq Ce^{(-1+a)x} t^a (1+t+x)^{n-a-1}(1+x), \tag{3.15}$$

And from (3.11) and (3.15) we get (3.10).

Step 3. Let $J_n(t,y) = W^2_{-n} W^1_{-\alpha+\beta}(F_{h_t})(y)$, then

Lemma 3.4.

$$J_n(t,y) \sim \frac{1}{\sqrt{4\pi t}}(\frac{1}{2t})^{\alpha-\beta+n} e^{-\rho^2 t - \frac{y^2}{4t} - (\alpha-\beta+2n)y}(1+t+y)^{\alpha-\beta+n-1}(1+y). \tag{3.16}$$

Proof. Since

$$W^2_{-n} = (-\frac{d}{d\cosh 2y})^n = (-\frac{1}{2\cosh t}\frac{d}{d\cosh y})^n = (\frac{1}{2\cosh t} W^1_{-1})^n$$

It is easy to see that

$$\begin{aligned} J_n(t,y) = \frac{a_1}{(\cosh y)^{2n-1}} W^1_{-\alpha+\beta-1}(F_{h_t})(y) + \frac{a_2}{(\cosh y)^{2n-2}} W^1_{-\alpha+\beta-2}(F_{h_t})(y) \\ + \cdots + \frac{a_n}{(\cosh y)^n} W^1_{-\alpha+\beta-n}(F_{h_t})(y) \end{aligned} \tag{3.17}$$

for some positive constants $a_1, a_2, \ldots, a_n$. Since

$$\begin{aligned} &\frac{a_k}{(\cosh y)^{2n-k}} W^1_{-\alpha+\beta-k}(F_{h_t})(y) \\ &\sim a_k(\frac{2t}{(1+t+y)})^{n-k} \cdot \frac{1}{\sqrt{4\pi t}}(\frac{1}{2t})^{\alpha-\beta+n} e^{-\rho^2 t - \frac{y^2}{4t} - (\alpha-\beta+2n)y}(1+t+y)^{\alpha-\beta+n-1}(1+y). \end{aligned}$$

So from (3.17) we have

$$\begin{aligned} J_n(t,y) \sim (a_1(\frac{2t}{(1+t+y)})^{n-1} + a_2(\frac{2t}{(1+t+y)})^{n-2} + \cdots + a_n) \\ \cdot \frac{1}{\sqrt{4\pi t}}(\frac{1}{2t})^{\alpha-\beta+n} e^{-\rho^2 t - \frac{y^2}{4t} - (\alpha-\beta+2n)y}(1+t+y)^{\alpha-\beta+n-1}(1+y). \end{aligned}$$

Then the (3.16) is followed from the fact that

$$a_1(\frac{2t}{(1+t+y)})^{n-1} + a_2(\frac{2t}{(1+t+y)})^{n-2} + \cdots + a_n \sim C$$

Step 4. Now we can prove the final result (3.5). Let $\beta + \frac{1}{2} = n - a$, with $0 < a < 1$, then

$$h_t(x) = W_a^2 W_{-n}^2 W_{-\alpha+\beta}^1 (F_{h_t})(x) = W_a^2 (J_n(t,\cdot))(x)$$

Using Lemma 3.4, similar as in Step 2, we have

$$\begin{aligned} h_t(x) &= \frac{1}{\Gamma(a)} \int_x^\infty J_n(t,y) \frac{d\cosh 2y}{(\cosh 2y - \cosh 2x)^{1-a}} \\ &\sim t^{-\alpha+\beta-n-1/2} e^{-\rho^2 t} e^{-(\alpha-\beta+2n-2)x - \frac{x^2}{4t}} I_1' \end{aligned}$$

where

$$I_1' = \int_0^\infty e^{-(\alpha-\beta+2n-2)h - \frac{2xh+h^2}{4t}} (1+t+x+h)^{\alpha-\beta+n-1} \frac{(x+h)dh}{(\cosh 2(x+h) - \cosh 2x)^{1-a}}$$

Since

$$\frac{1}{\cosh 2(x+h) - \cosh 2x} \leq C \frac{1+x+h}{2h(x+h)} e^{-2x-h}$$

we have

$$I_1' \leq C e^{(-1+a)2x} (1+x+t)^{\alpha-\beta+n-1} (1+x) I_2'$$

where

$$I_2' = \int_0^\infty e^{-(\alpha-\beta+2n-a-1)h - \frac{2xh+h^2}{4t}} (1+h)^{\alpha-\beta+n} \frac{1}{h^{1-a}} (\frac{x+h}{1+x+h})^a dh$$

Just the same as in Step 2, we can prove that

$$I_2' \leq C t^a (1+t+x)^{-a}.$$

So we have

$$h_t(x) \leq C t^{-\alpha-1} e^{-\rho^2 t - \rho x - \frac{x^2}{4t}} (1+t+x)^{\alpha-\frac{1}{2}} (1+x).$$

Similarly we can prove that

$$h_t(x) \geq C t^{-\alpha-1} e^{-\rho^2 t - \rho x - \frac{x^2}{4t}} (1+t+x)^{\alpha-\frac{1}{2}} (1+x)$$

This completes the prove of Step 4, also completes the proof of the Theorem.

4. Hardy's Theorem for Jacobi Transform

We need the following lemma of [11]

Lemma 4.1. *Let h be an entire function on $\mathbb{C}$ such that*

$$|h(z)| \leq Ce^{a|z|^2}, \qquad z \in \mathbb{C}$$
$$|h(t)| \leq Ce^{-a|t|^2}, \qquad t \in \mathbb{R}$$

for some positive constants a and C. Then $h(z) = Const.e^{-az^2}$, $z \in \mathbb{C}$

let H_t be the even function on $\mathbb{R}$ which satisfies

$$\hat{H}_t(\lambda) = e^{-t\lambda^2},$$

then from Theorem 3.1, we have

$$H_t(x) = e^{-\rho^2 t} h_t(x) \sim e^{-\rho x - \frac{x^2}{4t}} (1+t+x)^{\alpha - \frac{1}{2}} (1+x) \tag{4.1}$$

we have

Theorem 4.2. *Suppose f is a measurable function on $[0, \infty]$ satisfing*

$$|f(x)| \leq AH_{\frac{1}{4a}}(x) \tag{4.2}$$
$$|\hat{f}(\lambda)| \leq B \exp(-b|\lambda|^2), \qquad \lambda \in \mathbb{R} \tag{4.3}$$

for some positive constants A, B, a and b. If $ab > \frac{1}{4}$, then $f = 0$ a.e. If $ab = \frac{1}{4}$, then $f = CH_{\frac{1}{4a}}$ a.e.

Proof. First since by the condition (4.2), $\hat{f}(\lambda)$ is well defined for all $\lambda \in \mathbb{C}$, and $\hat{f}$ is an entire function on $\mathbb{C}$. Moreover,

$$\begin{aligned} |\hat{f}(\lambda)| &\leq \int_0^\infty |f(x)||\phi_\lambda(x)|\Delta(x)dx \\ &\leq c\int_0^\infty H_{\frac{1}{4a}}(x)\phi_{i\Im\lambda}(x)\Delta(x)dx \\ &= C\hat{H}_{\frac{1}{4a}}(i\Im\lambda) = Ce^{\frac{1}{4a}(\Im\lambda)^2} \leq Ce^{\frac{1}{4a}|\lambda|^2} \end{aligned}$$

since $ab \geq \frac{1}{4}$, we have

$$|\hat{f}(\lambda)| \leq Ce^{\frac{1}{4a}|\lambda|^2} \leq Ce^{b|\lambda|^2}, \text{ for } \lambda \in \mathbb{C}$$
$$|\hat{f}(\lambda)| \leq e^{-b|\lambda|^2} \leq e^{-\frac{1}{4a}|\lambda|^2}, \text{ for } \lambda \in \mathbb{R},$$

So by Lemma 4.1, we have $\hat{f}(\lambda) = Ce^{-b\lambda^2} = C'e^{-\frac{1}{4a}\lambda^2}$ for $\lambda \in \mathbb{C}$. If $4ab > 1$, then $\hat{f} = 0$, hence $f = 0$, a.e. If $4ab = 1$, then $f = CH_{\frac{1}{4a}}$ a.e.

From this theorem, by using the estimate in Theorem 3.1, we have

Theorem 4.3. *Suppose f is a measurable function on $[0, \infty]$, for some $a_i (i = 1, 2, 3)$, $b > 0$ it satisfies*

$$|f(t)| \leq Ae^{-a_2 x - a_1 x^2}(1+x)^{a_3} \tag{4.4}$$
$$|\hat{f}(\lambda)| \leq B\exp(-b|\lambda|^2), \qquad \lambda \in \mathbb{R} \tag{4.5}$$

for some positive constants A, B. If $a_1 b > \frac{1}{4}$ or if $a_1 b = \frac{1}{4}$ and $a_2 > \rho$, or if $a_1 b = \frac{1}{4}$, $a_2 = \rho$ and $a_3 < \alpha + \frac{1}{2}$, then $f = 0$ a.e.. If $a_1 b = \frac{1}{4}$, $a_2 = \rho$ and $a_3 = \alpha + \frac{1}{2}$, then $f = CH_b(t)$.

Now we will give an L^p-version of the Hardy's theorem. We still need the following lemma(see[5]),

Lemma 4.4. *Let h be an entire function on $\mathbb{C}$ such that*

$$|h(z)| \leq Ce^{a(\Re z)^2}, \qquad z \in \mathbb{C}$$
$$\|h(t)\|_{L^p(\mathbb{R})} \leq C,$$

for some positive constants a and C. Then $h(z)$ is a constant on $\mathbb{C}$. Moreover if $p < \infty$, then $h(z) \equiv 0$.

Now we have

Theorem 4.5. *Let $1 \le p, q \le \infty$, and f is a function on $[0,\infty)$ such that*

$$\|e^{at^2} f(t)\|_{L^p(\Delta)} \le C, \|e^{b|\lambda|^2} \hat{f}(\lambda)\|_{L^q(|c(\lambda)|^{-2})} \le C$$

for some positive constants C, a and b such that $ab > \frac{1}{4}$, then $f = 0$ a.e. Moreover, if $ab = \frac{1}{4}$, $p < 2$ we also have $f = 0$ a.e.

Proof. Let $ab \ge \frac{1}{4}$, using the estimates of ϕ_λ in Lemma 2.1, we have

$$\begin{aligned}
|\hat{f}(\lambda)| &\le \int_0^\infty |f(t)||\phi_\lambda(t)|\Delta(t)dt \\
&\le c\int_0^\infty |f(t)|e^{\alpha t^2} e^{-\alpha t^2} e^{t(|\Im\lambda|-\rho)}(1+t)\Delta(t)dt \\
&\le \|f(t)e^{\alpha t^2}\|_p c\Big[\int_0^\infty (e^{-\alpha t^2 + t(|\Im\lambda|-\rho)}(1+t))^{p'}\Delta(t)dt\Big]^{\frac{1}{p'}}.
\end{aligned}$$

If $ab > \frac{1}{4}$, we can choose $a' \in (0,a)$ such that $a'b > \frac{1}{4}$ and

$$(e^{-(a-a')t^2 - t(\rho)}(1+t))^{p'}\Delta(t) \le C,$$

then

$$\begin{aligned}
|\hat{f}(\lambda)| &\le c\Big[\int_0^\infty (e^{-a't^2 + t|\Im\lambda|})^{p'}dt\Big]^{\frac{1}{p'}} \\
&= e^{\frac{1}{4a'}|\Im\lambda|^2}\Big[\int_0^\infty (e^{-a'(t-\frac{|\Im\lambda|}{2a'})^2})^{p'}dt\Big]^{\frac{1}{p'}} \\
&\le e^{\frac{1}{4a'}|\Im\lambda|^2}\Big[\int_{-\infty}^\infty (e^{-a'(t-\frac{|\Im\lambda|}{2a'})^2})^{p'}dt\Big]^{\frac{1}{p'}} \\
&= Ce^{\frac{1}{4a'}|\Im\lambda|^2} \qquad (4.6)
\end{aligned}$$

If $ab = \frac{1}{4}$ and $p < 2$, i.e. $p' > 2$, then

$$e^{-\rho t}(1+t))^{p'}\Delta(t) \le C,$$

so we also have (4.6) with $a' = a$.

Let $h(\lambda) = e^{\frac{1}{4a'}\lambda^2}\hat{f}(\lambda)$, then

$$|h(\lambda)| \le Ce^{\frac{1}{4a'}(\Re\lambda)^2}$$

By the condition (4.4), since $e^{\frac{1}{4a'}|\lambda|^2} \leq e^{b|\lambda|^2}$

$$\|h(\lambda)\|_q \leq \|e^{b|\lambda|^2}\hat{f}(\lambda)\|_q \leq C$$

If $q < \infty$, by Lemma 4.4, we have $h(\lambda) = 0$ hence $\hat{h} = 0$ and $f = 0$ a.e. If $q = \infty$, we get $h(\lambda) = C$, $\hat{f}(\lambda) = Ce^{-b\lambda^2}$, then $f(x) = CH_b(x)$. If $4ab > 1$ of if $4ab = 1$ and $p < 2$, by (4.1),

$$H_b(x)e^{ax^2} \sim e^{(a-\frac{1}{4b})x^2}e^{-\rho x}(1+b+x)^{\alpha-\frac{1}{2}}(1+x) \notin L^p(\Delta),$$

so we also have $f(x) = 0$. This proves the Theorem.

For the proof of Theorem 4.5, it is easy to see that we have the following corollary

Corollary 4.6. *Let $1 \leq p, q \leq \infty$, and f is a function on $[0, \infty)$ such that*

$$\|e^{\frac{1}{4a}t^2+bt}f(t)\|_{L^p(\Delta)} \leq C, \qquad \|e^{a|\lambda|^2}\hat{f}(\lambda)\|_{L^q(|c(\lambda)|^{-2})} \leq C$$

for some positive constants a and b such that $b > \rho(1-\frac{2}{p})$, then $f = 0$ a.e.

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