L^1 ESTIMATES FOR MAXIMAL FUNCTIONS AND RIESZ TRANSFORM ON REAL RANK 1 SEMISIMPLE LIE GROUPS

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Abstract. Let G be a real rank one semisimple Lie group and K a maximal compact subgroup of G. Radial maximal operators for suitable dilations, the heat and Poisson maximal operators, and the Riesz transform, which act on K-biinvariant functions on G, satisfy the L^p-norm inequalities for $p > 1$ and a weak type L^1 estimate. In this paper, through the Fourier theories on $\mathbf{R}$ and G we shall duplicate the Hardy space $H^1(\mathbf{R})$ to a subspace $H^1_s(G)$ $(s \geq 0)$ of $L^1(G)$ and show that these operators are bounded from $H^1_s(G)$ to $L^1(G)$.

1. Introduction. Let G be a real rank one connected semisimple Lie group with finite center, $G = KAN$ an Iwasawa decomposition of G, and $dg = dkdadn$ a corresponding decomposition of a Haar measure dg on G. Let $L^1_{loc}(G//K)$ denote the space of locally integrable, K-biinvariant functions on G and $L^p(G//K)$ $(0 < p \leq \infty)$ the subspace of $L^1_{loc}(G//K)$ consisting of functions with finite L^p-norm on G. In the following, if we say that T is an operator on G, it means that T is an operator acting on these spaces. The first problem we shall treat in this paper is concerned with the Abel transform on G. For $f \in L^1(G//K)$ the Abel transform F_f of f is defined by

$$F_f(x) = e^{\rho x} \int_N f(a_x n) dn \quad (x \in \mathbf{R}),$$

where A is parametrized as $\{a_x; x \in \mathbf{R}\}$ (for the definition of ρ see (3) below). We let $F^1_f(x) = e^{\rho x} F_f(x)$. Then the integral formula for the Iwasawa decomposition of G (cf. [**6**, p.373]) yields that $f \in L^1(G)$ if and only if $F^1_{|f|} \in L^1(\mathbf{R})$, and thus,

Proposition 1.1. *If $f \in L^1(G)$, then $F^1_f \in L^1(\mathbf{R})$.*

The reverse of this proposition is not true: even if F^1_f is well-defined for $f \in L^1_{loc}(G//K)$ and it belongs to $L^1(\mathbf{R})$, f does not always belong to $L^1(G//K)$. The first problem we offer is the following,

Problem A. *For $f \in L^1_{loc}(G//K)$ find a condition on F^1_f under which f belongs to $L^1(G//K)$.*

We next consider a problem concerning with a maximal operator on G. Let $\sigma : G \to \mathbf{R}_+$ denote the K-biinvariant function on G induced by the distance function on $X = G/K$ (cf. [**13**, p.320]). For each $r \in \mathbf{R}_+$ we let $B(r) = \{g \in G; \sigma(g) \leq r\}$ and $\chi_{B(r)}$

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the characteristic function of $B(r)$. The Hardy-Littlewood maximal operator M_{HL}^G on G is defined as follows: for $f \in L^1_{loc}(G//K)$

$$(M_{HL}^G f)(g) = \sup_{0<r<\infty} |B(r)|^{-1}(|f| * \chi_{B(r)})(g) \quad (g \in G).$$

This definition makes sense for $f \in L^p(X)$ $(p \geq 1)$. Clerc and Stein [**3**] and Strömberg [**11**] have shown the following,

Theorem 1.2. *M_{HL}^G satisfies the L^p-norm inequalities for $p > 1$, and a weak type L^1 estimate, that is, for all $\alpha \in \mathbf{R}_+$ and all $f \in L^1(X)$*

$$|\{g \in G; (M_{HL}^G f)(g) > \alpha\}| \leq \frac{c}{\alpha}\|f\|_{L^1(G)},$$

where c is independent of α and f.

We here define a radial maximal operator on G as an analogue of the one on $\mathbf{R}$. We fix $\phi \in C_c^\infty(G//K)$, the space of C^∞, compactly supported K-biinvariant functions on G, and suppose that ϕ is sufficiently zero at the origin of G. The radial maximal operator $M_{\phi,\epsilon}^{G,\sharp}$ $(\epsilon \geq 0)$ on G is defined as follows: for $f \in L^1_{loc}(G//K)$

$$(M_{\phi,\epsilon}^{G,\sharp} f)(g) = \sup_{0<r<\infty} (1+r)^{-\epsilon}|(f * \phi_r^\sharp)(g)| \quad (g \in G),$$

where the dilation $\phi_r^\sharp$ $(r \in \mathbf{R}_+)$ of ϕ is given by

$$\phi_r^\sharp(g) = |B(r)|^{-1}\phi(a_{\sigma(g)/r}) \quad (g \in G).$$

This dilation is different from the one used in [**7**] and see §6 for other dilations. Since $M_{\phi,\epsilon}^{G,\sharp} f \leq c M_{HL}^G f$ pointwisely, $M_{\phi,\epsilon}^{G,\sharp}$ also satisfies the property stated in Theorem 1.2. As Folland and Stein have shown in their book [**5**], when a Lie group $\mathcal{G}$ is of homogeneous type, a radial maximal operator $M_\phi^{\mathcal{G}}$ on $\mathcal{G}$ is bounded from the atomic Hardy space $H^1_{\infty,0}(\mathcal{G})$ to $L^1(\mathcal{G})$. More precisely, integrability of the maximal function $M^{\mathcal{G}} f$ $= \sup_\phi M_\phi^{\mathcal{G}} f$ where the supremum is taken over ϕ in a suitable class, is equivalent to that f belongs to $H^1_{\infty,0}(\mathcal{G})$ (see [**5**, Chapter 3]). At present, we have no definition of the (atomic) Hardy space on G on which $M_{\phi,\epsilon}^{G,\sharp}$ is bounded. The second problem, which was also treated in [**7**], is the following,

Problem B. *Find a subspace of $L^1(G//K)$ on which $M_{\phi,\epsilon}^{G,\sharp}$ is bounded to $L^1(G//K)$.*

These two problems look unrelated each other. However, the answers obtained in this paper indicate that they are deeply related. Let $C(\lambda)$ $(\lambda \in \mathbf{R})$ be the Harish-Chandra's C-function. Since $C(\lambda)$ has a meromorphic extension on $\mathbf{C}$, the following definition of the Fourier multiplier $\mathcal{C}_+^1$ on $\mathbf{R}$ makes sense:

$$(\mathcal{C}_+^1 F)^\sim(\lambda) = C(-\lambda - i\rho)^{-1} F^\sim(\lambda) \quad (\lambda \in \mathbf{R}),$$

where $F^\sim$ denotes the Euclidean Fourier transform of F. Let $H^1(\mathbf{R})$ be the H^1-Hardy space on $\mathbf{R}$ (cf. [**10**, Chap.3]). Then our answer of Problem A can be stated as

Answer A. *If $\mathcal{C}_+^1 F_f^1 \in H^1(\mathbf{R})$, then $f \in L^1(G//K)$.*

Let $\mathcal{Q}_s$ $(s \geq 0)$ denote the Fourier multiplier on $\mathbf{R}$ defined by

$$(\mathcal{Q}_s F)^\sim(\lambda) = \left(\frac{1+|\lambda|}{|\lambda|}\right)^s F^\sim(\lambda) \quad (\lambda \in \mathbf{R}).$$

We set

$$H_s^1(G//K) = \{f \in L_{loc}^1(G//K); \mathcal{Q}_s \mathcal{C}_+^1 F_f^1 \in H^1(\mathbf{R})\}$$

and $\|f\|_{H_s^1(G)} = \|\mathcal{Q}_s \mathcal{C}_+^1 F_f^1\|_{H^1(\mathbf{R})}$. Answer A means that $H_0^1(G//K)$ is a subspace of $L^1(G//K)$ and moreover, the fact that $(|\lambda|/1+|\lambda|)^s$ satisfies the Hörmander condition on Fourier multiplier yields that $H_s^1(G//K)$ $(s > 0)$ is a subspace of $H_0^1(G//K)$ (see §4). Our answer of Problem B is given as follows.

Answer B. *For $\delta > 0$, $M_{\phi,0}^{G,\sharp}$ is a bounded operator of $H_{2+\delta}^1(G//K)$ to $L^1(G//K)$ and $M_{\phi,1+\delta}^{G,\sharp}$ is one of $H_0^1(G//K)$ to $L^1(G//K)$.*

In order to understand a true character of $\mathcal{C}_+^1 F_f^1$ we need the Fourier analysis on G, so we refer to the Warner's book [**14**]. Let $\phi_\lambda(g)$ $(g \in G, \lambda \in \mathbf{R})$ be the zonal spherical function on G where the dual space of the Lie algebra of A is identified with $\mathbf{R}$. Then, for $f \in L^1(G//K)$ the Fourier transform $\hat{f}(\lambda)$ of f is defined by $\hat{f}(\lambda) = \int_G f(g)\phi_\lambda(g)dg$. The inversion formula is of the following form: for $f \in C_c^\infty(G//K)$

$$f(g) = \int_{\mathbf{R}} \hat{f}(\lambda)\phi_\lambda(g)|C(\lambda)|^{-2} d\lambda, \tag{1}$$

in which case, $\hat{f}(\lambda)$ is a holomorphic function of exponential type. Hence, by regarding each K-biinvariant function on G as an even function on $\mathbf{R}$, by substituting the expansion of ϕ_λ (see (5) and (6) below) into (1), and then, by shifting the integral line $\mathbf{R}$ to $\mathbf{R} + i\rho$, we can rewrite (1) as follows ([**14**, p.356]):

$$\begin{aligned} f(x) &= e^{-\rho x} \int_{\mathbf{R}} \hat{f}(\lambda)\Phi(\lambda, x)C(-\lambda)^{-1} e^{i\lambda x} d\lambda \quad (x \in \mathbf{R}_+) \\ &= e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} \int_{\mathbf{R}} C(-\lambda - i\rho)^{-1} \Gamma_m(\lambda + i\rho) \hat{f}(\lambda + i\rho) e^{i\lambda x} d\lambda. \end{aligned} \tag{2}$$

Since $(F_f)^\sim(\lambda) = \hat{f}(\lambda)$ and $\Gamma_0 \equiv 1$ ([**14**, Proposition 9.2.2.3 and 9.1.5]), the leading term corresponding to $m = 0$ in the right side of (2) is nothing but $e^{-2\rho x}(\mathcal{C}_+^1 F_f^1)(x)$. Roughly speaking, Answers A and B can be restated as follows: if the leading term of $e^{2\rho x} f(x) (x \in \mathbf{R}_+)$ belongs to $H^1(\mathbf{R})$, then f and $M_{\phi,1+\delta}^{G,\sharp} f$ belong to $L^1(G//K)$.

We state the organization of this paper. In §3 we shall obtain a key lemma for the integrability of f satisfying $\mathcal{C}_+^1 F_f^1 \in H^1(\mathbf{R})$. Then the proof of Answer A and some basic properties of $H_s^1(G//K)$ are given in §4. In §5 we shall obtain some criteria by which we can judge whether a radial maximal operator is bounded from $H_s^1(G//K)$ to $L^1(G//K)$. Actually, we apply a criterion to $M_{\phi,\epsilon}^{G,\sharp}$ and obtain the proof of Answer B in

§6. Moreover, we consider the same problem for the heat and Poisson maximal operators $M^G_{H,\epsilon}$ and $M^G_{P,\epsilon}$ ($\epsilon \geq 0$) on G, which are defined as follows: for $f \in L^1_{loc}(G//K)$

$$(M^G_{H,\epsilon}f)(g) = \sup_{0<r<\infty} (1+r)^{-\epsilon}|(f * h_r)(g)| \quad (g \in G),$$
$$(M^G_{P,\epsilon}f)(g) = \sup_{0<r<\infty} (1+r)^{-\epsilon}|(f * p_r)(g)| \quad (g \in G),$$

where $\hat{h}_r(\lambda) = e^{-(\lambda^2+\rho^2)r}$ and $\hat{p}_r(\lambda) = e^{-(\lambda^2+\rho^2)^{1/2}r}$. As shown by Stein [**9**] in great generality these operators also satisfy the property stated in Theorem 1.2. In §6 we shall show that $M^G_{H,0}$ (resp. $M^G_{P,0}$) is a bounded sublinear operator of $H^1_{1+\delta}(G//K)$ ($\delta > 0$) (resp. $H^1_{1/2}(G//K)$) to $L^1(G//K)$ and moreover, $M^G_{H,1}$ and $M^G_{P,2}$ are ones of $H^1_0(G//K)$ to $L^1(G//K)$. In §7 we shall treat the Riesz transform $\mathcal{R}^G$ on G which are defined as follows: for $f \in L^1_{loc}(G//K)$

$$(R^G f)(g) = (|\nabla|(-\Delta)^{-1/2})(f)(g) \quad (g \in G),$$

where Δ is the Laplacian on G and $|\nabla|^2(h) = \Delta(h^2) - 2\Delta h \cdot h$ for $h \in C^\infty(G)$. As studied by Anker [**1**] and Lohoué [**8**] $\mathcal{R}^G f$ is deeply related with $M^G_{P,\epsilon}f$, especially, the L^1-norm of $M^G_{P,\epsilon}f$ is controled by the one of $\mathcal{R}^G f$. Therefore, we can expect that the (Hardy) space $H^1_s(G//K)$ might be useful to obtain an L^1 estimate for $\mathcal{R}^G$. Indeed, we shall prove that $\mathcal{R}^G$ is bounded from $H^1_{1/2}(G//K)$ to $L^1(G//K)$.

2. Notations. Let G be a real rank one connected semisimple Lie group with finite center and $G = KAN$ an Iwasawa decomposition of G. Let $\mathfrak{a}$ be the Lie algebra of A and $\mathcal{F} = \mathfrak{a}^*$ the dual space of $\mathfrak{a}$. Let H be the unique element in $\mathfrak{a}$ satisfying $\gamma(H) = 1$ where γ is the positive simple root of (G, A) determined by N. We parametrize each element in A, $\mathfrak{a}$, and $\mathcal{F}$ as $a_x = \exp(xH)$, xH, and $x\gamma$ ($x \in \mathbf{R}$) respectively. In what follows we often identify these spaces with $\mathbf{R}$ and also $\mathcal{F}_c$, the complexification of $\mathcal{F}$, with $\mathbf{C}$ without making mention of the identification. We put $\mathcal{F}(s) = \{\lambda \in \mathcal{F}_c; |\Im(\lambda)| < s\}$ ($s \in \mathbf{R}_+$) and $A_+ = \{a_x; x \in \mathbf{R}_+\}$. Then, according to the Cartan decomposition $G = KCL(A_+)K$ of G and the action of the Weyl group of (G, A) on A, every K-biinvariant functions f on G are determined by their restriction to $CL(A_+)$ and hence, they are identified with even functions on $\mathbf{R}$. We denote them by the same letter, that is, if $g \in Ka_{x(g)}K$ and $x(g) \in \mathbf{R}_+$,

$$f(g) = f(a_{x(g)}) = f(x(g)) = f(-x(g)).$$

Let dg (resp. dk and dn) denote the Haar measure on G (resp. K and N), normalized as $\int_K dk = 1$ and the following integral formula holds for all integrable, K-biinvariant functions f on G:

$$\int_G f(g)dg = \int_0^\infty f(x)D(x)dx,$$

where $D(x) = (\sinh x)^{m_1}(\sinh 2x)^{m_2}$ ($x \in \mathbf{R}_+$), m_1 and m_2 are the multiplicities of γ and 2γ respectively. We put

(3) $$\alpha = \frac{m_1 + m_2 - 1}{2} \quad \text{and} \quad \rho = \frac{m_1 + 2m_2}{2}.$$

Then the order of $D(x)$ is given by

$$
(4) \qquad D(x) \sim \begin{cases} x^{2\alpha+1} & (0 < x \leq 1) \\ e^{2\rho x} & (1 < x < \infty), \end{cases}
$$

where the symbol " $\sim$ " means that the ratio of the left side and the right side is bounded below and above by positive constants. Let $L^1_{loc}(G//K)$ denote the space of locally integrable, K-biinvariant functions on G. Let $L^p(G//K)$ $(0 < p \leq \infty)$ and $C_c^\infty(G//K)$ denote the subspaces of $L^1_{loc}(G//K)$ consisting of, respectively, functions with finite L^p-norm on G and C^∞, compactly supported functions on G. Henceforth, for each normed space V we denote the norm of $v \in V$ as $\|v\|_V$, excepting that $G//K$ is abbreviated by G and the L^∞-norm is denoted by $\|v\|_\infty$.

We recall the bases of the Fourier analysis on G and refer to [**4**] and [**14**]. Let $\phi_\lambda(g)$ $(\lambda \in \mathcal{F}, g \in G)$ be the zonal spherical function of G. The Harish-Chandra expansion of ϕ_λ is given as

$$
(5) \qquad \phi_\lambda(x) = e^{-\rho x}\left(\Phi(\lambda, x)C(\lambda)e^{i\lambda x} + \Phi(-\lambda, x)C(-\lambda)e^{-i\lambda x}\right) \quad (x \in \mathbf{R}_+)
$$

and furthermore, $\Phi(\lambda, x)$ has the so-called Gangolli expansion:

$$
(6) \qquad \Phi(\lambda, x) = \sum_{m=0}^{\infty} \Gamma_m(\lambda)e^{-2mx} \quad (\lambda \in \mathcal{F}, x \in \mathbf{R}_+)
$$

([**14**, 9.1.4 and 9.1.5]). Their explicit forms and some basic properties of $C(\lambda)$, $\Phi(\lambda, x)$, and $\Gamma_m(\lambda)$, which will be used in the following arguments, are summarized in [**4**, §2 and §3], and a sharp estimate for the derivatives of $\Gamma_m(\lambda)$ will be obtained in the appendix of this paper (see §8). For $f \in L^1(G//K)$ the Fourier transform $\hat{f}(\lambda)$ $(\lambda \in \mathcal{F})$ of f is defined by

$$
\hat{f}(\lambda) = \int_G f(g)\phi_\lambda(g)dg \quad (\lambda \in \mathcal{F}).
$$

From the Riemann-Lebesgue's lemma on G ([**4**, Lemma 11]) it follows that $\hat{f}(\lambda)$ is an even holomorphic function on $\mathcal{F}(\rho)$ satisfying $|\hat{f}(\lambda)| \to 0$ as $|\lambda| \to \infty$ in $CL(\mathcal{F}(\rho))$ and hence,

$$
\sup_{\lambda \in CL(\mathcal{F}(\rho))} |\hat{f}(\lambda)| \leq \|f\|_{L^1(G)}.
$$

When f is in $C_c^\infty(G//K)$, the Paley-Wiener theorem on G ([**14**, 9.2.3]) implies that $\hat{f}(\lambda)$ is an even holomorphic function on $\mathcal{F}_c$ of exponential type, in which case, the Fourier inversion formula:

$$
(7) \qquad f(g) = \int_{\mathbf{R}} \hat{f}(\lambda)\phi_\lambda(g)|C(\lambda)|^{-2}d\lambda \quad (g \in G)
$$

and the Plancherel formula:

$$
(8) \qquad \int_0^\infty |f(x)|^2 D(x)dx = \int_0^\infty |\hat{f}(\lambda)|^2 |C(\lambda)|^{-2}d\lambda
$$

hold. Thereby, the Fourier transform $f \mapsto \hat{f}$ of $C_c^\infty(G//K)$ is uniquely extended to an isometry between $L^2(G//K)$ and $L^2(\mathbf{R}, |C(\lambda)|^{-2}d\lambda)$ ([**14**, Theorem 9.2.2.13]).

We now introduce some operators on G and $\mathbf{R}$. In what follows most of operators on G and $\mathbf{R}$ are denoted by scripts: $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$, excepting M and T. Especially, $\mathcal{A}\mathcal{B}\ldots\mathcal{C}f$ means that $\mathcal{A}(\mathcal{B}(\ldots(\mathcal{C}(f))))$ and $\mathcal{A}^G$ an operator on G.

For $s \in \mathbf{R}$ we define the Abel transform F_f^s of $f \in L^1_{loc}(G//K)$ as

$$F_f^s(x) = e^{s\rho x} F_f(x) = e^{(s+1)\rho x} \int_N f(a_x n) dn \quad (x \in \mathbf{R})$$

and the Fourier multipliers $\mathcal{C}_+^s$ and $\mathit{\Gamma}_m^s$ $(m \in \mathbf{N})$ on $\mathbf{R}$ as

$$(\mathcal{C}_+^s F)(x) = \int_{\mathbf{R}} C(-\lambda - is\rho)^{-1} F^\sim(\lambda) e^{i\lambda x} d\lambda \quad (x \in \mathbf{R}),$$

$$(\mathit{\Gamma}_m^s F)(x) = \int_{\mathbf{R}} \Gamma_m(\lambda + is\rho) F^\sim(\lambda) e^{i\lambda x} d\lambda \quad (x \in \mathbf{R}),$$

where $F^\sim$ denotes the Euclidean Fourier transform of F. Of course, these definitions make sense if the integrals of the right sides exist. Since $\hat{f}(\lambda) = (F_f)^\sim(\lambda)$ $(\lambda \in \mathcal{F}_c)$ for $f \in C_c^\infty(G//K)$ ([**14**, Proposition 9.2.2.3]), the relation

$$(F_f^s)^\sim(\lambda) = \hat{f}(\lambda + is\rho) \tag{9}$$

holds if the both sides exist for $\lambda \in \mathcal{F}_c$. We here suppose $f \in C_c^\infty(G//K)$ and we observe that on $\mathcal{F}(\rho)$, $\hat{f}(\lambda)$ is holomorphic and rapidly decreasing, $C(-\lambda)^{-1}$ is holomorphic and tempered, and $\Gamma_m(\lambda)$ is holomorphic and uniformly dominated by a polynomial of m ([**14**, Proposition 9.1.7.2 and p.334]). Hence, by substituting the expansions in (5) and (6) into (7), by changing the order of integration and summation, and by shifting the integral line from $\mathbf{R}$ to $\mathbf{R} + i\rho$, we can deduce that

$$\begin{aligned} f(x) &= e^{-\rho x} \int_{\mathbf{R}} \hat{f}(\lambda) \Phi(\lambda, x) C(-\lambda)^{-1} e^{i\lambda x} d\lambda \quad (x \in \mathbf{R}_+) \\ &= e^{-\rho x} \sum_{m=0}^{\infty} e^{-2mx} (\mathit{\Gamma}_m^0 \mathcal{C}_+^0 F_f^0)(x) \\ &= e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} (\mathit{\Gamma}_m^1 \mathcal{C}_+^1 F_f^1)(x). \end{aligned} \tag{10}$$

This manipulation is valid provided that $\hat{f}(\lambda)$ is holomorphic on $\mathcal{F}(\rho)$ and satisfies $\sup_{0 \le \xi \le \rho} \int_{\mathbf{R}+i\xi} |\hat{f}(\lambda)\, C(-\lambda)^{-1}| d\lambda < \infty$. In this case, since $|\Gamma_m(\lambda)| \le cm^{2\alpha}$ $(m \in \mathbf{N})$ if $\Im(\lambda) \ge \rho$ (see Proposition 8.3 below) and $\sum_{m=0}^\infty e^{-2mx}\, m^{2\alpha} \sim x^{-(2\alpha+1)}$ as $x \to 0$ and ~ 1 as $x \to \infty$, the right side of (10) is dominated by $e^{-2\rho x}\, (1 + D(x))/D(x)$ (see (4)). We then define

$$\mathcal{L}(G//K) = \{f \in L^1_{loc}(G//K); \hat{f} \text{ is holomorphic on } \mathcal{F}(\rho) \text{ and}$$
$$\|f\|_{\mathcal{L}(G)} = \sup_{0 \le \xi \le \rho} \int_{\mathbf{R}+i\xi} |\hat{f}(\lambda) C(-\lambda)^{-1}| d\lambda < \infty\}.$$

3. A key lemma.

Lemma 3.1. *Let T be a sublinear operator of $\mathbf{R}$ satisfying the following properties: there exists $\epsilon > 0$ such that for each (1,2,0)-atom a with $supp(a) \subset [x_0 - \ell, x_0 + \ell]$*

$$\begin{aligned} &(a) \quad \|Ta\|_{L^2(\mathbf{R})} \le \|a\|_{L^2(\mathbf{R})}, \\ &(b) \quad |(Ta)(x)| \le \ell^\epsilon |x - x_0|^{-(1+\epsilon)} \quad \text{if } |x - x_0| \ge 2\ell. \end{aligned}$$

Then there exists a positive constant $c = c_\epsilon$ such that for any $F \in H^1(\mathbf{R})$

$$\|e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} (T\Gamma_m^1 F) \cdot D\|_{L^1(\mathbf{R}_+)} \le c\|F\|_{H^1(\mathbf{R})}. \tag{11}$$

Proof. We abbreviate $\|\cdot\|_{L^2(\mathbf{R})}$ as $\|\cdot\|_2$. We first observe that $\|Ta\|_{L^1(\mathbf{R})} \le \int_{|x-x_0|\le 2\ell} |(Ta)(x)|dx + \int_{|x-x_0|>2\ell} |(Ta)(x)|dx \le (2\ell)^{1/2}\|a\|_2 + \ell^\epsilon \int_{|x-x_0|>2\ell} |x - x_0|^{-(1+\epsilon)}dx \le c$, because a is a (1,2,0)-atom. Hence T is a bounded sublinear operator of $H^1(\mathbf{R})$ to $L^1(\mathbf{R})$. Let $F \in H^1(\mathbf{R})$. Since $\Gamma_0 \equiv 1$ (cf. [**14**, 9.1.5]), the term corresponding to $m = 0$ in the left side of (11) is dominated as

$$\|e^{-2\rho x}(TF) \cdot D\|_{L^1(\mathbf{R}_+)} \le c\|TF\|_{L^1(\mathbf{R})} \le c\|F\|_{H^1(\mathbf{R})}. \tag{12}$$

In estimating the rest terms, by substituting a (1,2,0)-atomic decomposition of F (cf. [**5**, Theorem 3.30]) it suffices to show that for any (1,2,0)-atom a on $\mathbf{R}$,

$$\|e^{-2\rho x} \sum_{m=1}^{\infty} e^{-2mx} (T\Gamma_m^1 a) \cdot D\|_{L^1(\mathbf{R}_+)} \le c. \tag{13}$$

We notice that $\Gamma_m(\lambda + i\rho)$ $(\lambda \in \mathcal{F})$ satisfies the Hörmander condition with the constant $cm^{4\alpha}$ (see Corollary 8.4 below) and therefore, the multiplier theorem obtained by Taibleson and Weiss [**12**, Theorem 4.2] asserts that $\Gamma_m^1 a$ is a (1,2,0,1/2)-molecule with $\|\Gamma_m^1 a\|_{H^1(\mathbf{R})} \le cm^{2\alpha}$. Moreover, the proof of Theorem 2.9 in [**12**] yields that $\Gamma_m^1 a$ has a (1,2,0)-atomic decomposition such that

$$\Gamma_m^1 a = \sum_{k=0}^{\infty} d_k^m b_k^m,$$

where $|d_k^m| = c2^{-k/2}m^{2\alpha}$ and, if we denote by x_0 the center of the support of a and we put $\sigma_m = \|\Gamma_m^1 a/m^{2\alpha}\|_2^{-2}$, b_k^m is a (1,2,0)-atom on $\mathbf{R}$ supported on the interval $I_k^m = \{x; |x - x_0| \le 2^k\sigma_m\}$. Let $\ell_k^m = |I_k^m|$. Since a and $\Gamma_m a$ are $(1,2,0,1/2)$-molecules on $\mathbf{R}$, it follows from the proof of Theorem 4.2 in [**12**] that $\sigma_m = m^{4\alpha}\ \|\Gamma_m a\|_2^{-2} \ge c\|a\|_2^{-2}$ and $\sigma_m = m^{-4\alpha}\||x - x_0|\Gamma_m a\|_2^2 \le \|a\|_2^{-2}$, that is,

$$\sigma_m \sim \|a\|_2^{-2} \quad \text{and} \quad \ell_k^m \sim 2^k\|a\|_2^{-2}. \tag{14}$$

We now split the region of integration in (13) into $(0,1]$ and $(1,\infty)$. The integral over $(1,\infty)$ is estimated as

$$\begin{aligned}&\int_1^\infty \sum_{m=1}^\infty e^{-2mx}|(T\Gamma_m^1 a)(x)|e^{-2\rho x}D(x)dx\\ \leq& c\sum_{m=1}^\infty e^{-2m}\|T\Gamma_m^1 a\|_{L^1(\mathbf{R})}\\ \leq& c\sum_{m=1}^\infty e^{-2m}m^{2\alpha}<\infty.\end{aligned}$$

On the other hand, the estimate for $(0,1]$ is obtained as follows:

$$\begin{aligned}&\int_0^1 \sum_{m=1}^\infty e^{-2mx}|(T\Gamma_m^1 a)(x)|e^{-2\rho x}D(x)dx\\ \leq&\ c\int_0^1 \sum_{m=1}^\infty e^{-2mx}\sum_{k=0}^\infty |d_k^m||(Tb_k^m)(x)|D(x)dx\\ \leq&\ c\sum_{k=0}^\infty\sum_{m=1}^\infty \int_0^1 e^{-2mx}2^{-k/2}m^{2\alpha}|(Tb_k^m)(x)|x^{2\alpha+1}dx\\ =&\ I_1+I_2,\end{aligned}$$

where I_1 is the integral over the region D_1: $0<x\leq 1$, $|x-x_0|\geq 2\ell_k^m$ and I_2 is the integral over the region D_2: $0<x\leq 1$, $|x-x_0|<2\ell_k^m$. Since $\ell_k^m\sim 2^k\|a\|_2^{-2}$ by (14) and $\sum_{m=1}^\infty e^{-2mx}m^{2\alpha}\sim x^{-(2\alpha+1)}$ $(0<x\leq 1)$, the property (b) of T yields that

$$\begin{aligned}I_1\leq&\ c\sum_{k=0}^\infty 2^{-k/2}\sum_{m=1}^\infty 2^{k\epsilon}\|a\|_2^{-2\epsilon}\int_{D_1} e^{-2mx}m^{2\alpha}x^{2\alpha+1}|x-x_0|^{-(1+\epsilon)}dx\\ \leq&\ c\sum_{k=0}^\infty 2^{-k/2}2^{k\epsilon}\|a\|_2^{-2\epsilon}\int_{|x-x_0|>c2^k\|a\|_2^{-2}}|x-x_0|^{-(1+\epsilon)}dx\\ \leq&\ c\sum_{k=0}^\infty 2^{-k/2}<\infty.\end{aligned}$$

In obtaining the estimate of I_2 we break the integral over D_2 as

$$\int_{D_2}=\sum_{j=0}^\infty\int_{\substack{2^{-j-1}<x\leq 2^{-j}\\ |x-x_0|\leq 2\ell_k^m,x_0\leq 3\ell_k^m}}+\int_{\substack{0<x\leq 1\\ |x-x_0|\leq 2\ell_k^m,x_0>3\ell_k^m}}$$

and we denote the corresponding terms by I_{21} and I_{22} respectively. As for I_{21}, if the integral does not vanish, we see that $2^{-j-1}\leq x_0+2\ell_k^m\leq 5\ell_m^k\leq c2^k\|a\|_2^{-2}$ by (14) and moreover, $\|Tb_m^k\|_2\leq\|b_m^k\|_2\leq c(\ell_m^k)^{-1/2}\sim 2^{-k/2}\|a\|_2$ by the property (a) of T.

Therefore, taking the maximal values of e^{-2mx} and $x^{2\alpha+1}$ in the integrand and applying the Schwarz inequality to the integral, we can estimate I_{21} as

$$\begin{aligned}
&c\sum_{k=0}^{\infty}2^{-k/2}\sum_{2^{-j}\leq c2^k\|a\|_2^{-2}}\left(\sum_{m=1}^{\infty}e^{-m2^{-j}}m^{2\alpha}2^{-j(2\alpha+1)}\right)2^{-j/2}2^{-k/2}\|a\|_2\\
\leq& c\sum_{k=0}^{\infty}2^{-k/2}\left(\sum_{2^{-j}\leq c2^k\|a\|_2^{-2}}2^{-j/2}\right)2^{-k/2}\|a\|_2\\
\leq& c\sum_{k=0}^{\infty}2^{-k/2}<\infty.
\end{aligned}$$

As for I_{22}, if the integral does not vanish, we see that $x \leq x_0 + 2\ell_k^m \leq 5x_0/3$ and $x \geq x_0 - 2\ell_k^m > x_0/3$. Therefore, we can deduce that

$$\begin{aligned}
I_{22} &\leq c\sum_{k=0}^{\infty}2^{-k/2}\left(\sum_{m=1}^{\infty}e^{-2mx_0/3}m^{2\alpha}\right)(5x_0/3)^{2\alpha+1}\|Tb_k^m\|_{L^1(\mathbf{R})}\\
&\leq c\sum_{k=0}^{\infty}2^{-k/2}<\infty.
\end{aligned}$$

This completes the proof of the lemma. □

Remark 3.2. Let T be as in Lemma 3.1. Let $T_{\Phi,D}$ denote a pseudo-differential operator with the symbol $\sigma_{\Phi,D}(x,\ \lambda)$ defined by $e^{-2\rho x}\ \Phi(\lambda+i\rho,x)\ D(x)$ if $x\in\mathbf{R}_+$ and 0 otherwise. Then from (6) the conclusion of Lemma 3.1 can be restated as $\|T_{\Phi,D}\ TF\|_{L^1(\mathbf{R})}\leq c\|F\|_{H^1(\mathbf{R})}$ for all $F\in H^1(\mathbf{R})$. Especially, taking the identity operator as T, we see that $T_{\Phi,D}$ is a bounded linear operator of $H^1(\mathbf{R})$ to $L^1(\mathbf{R})$. On the other hand, it follows from Proposition 8.3 below that

$$|(\frac{d}{d\lambda})^M\sigma_{\Phi,D}(x,\lambda)|\leq c(1+|\lambda|)^{-M}$$

for $M=0,1$. If the same estimate were also true for the derivative of x, or a modulus continuity of x, we would obtain the boundedness of $T_{\Phi,D}$ directly from Théorème 9 in Coifman and Meyer [**2**].

4. L^1-condition. As stated in Theorem 1.1, F_f^1 is integrable on $\mathbf{R}$ if f is integrable on G. However, the reverse is not true. In this section we obtain a condition of F_f^1 under which f is integrable on G.

Theorem 4.1. *Let us suppose that $\mathcal{C}_+^1 F_f^1$ is well-defined for $f \in L_{loc}^1(G//K)$ and it belongs to $H^1(\mathbf{R})$. Then f belongs to $L^1(G//K)$. In particular,*

$$\|f\|_{L^1(G)} \le c\|\mathcal{C}_+^1 F_f^1\|_{H^1(\mathbf{R})},$$

where c is independent of f.

Proof. We first prove the inequality for $f \in \mathcal{L}(G//K)$ with $\|\mathcal{C}_+^1 F_f^1\|_{H^1(\mathbf{R})} < \infty$, in which case we see from (10) that

$$f(x) = e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} (\Gamma_m^1 \mathcal{C}_+^1 F_f^1)(x) \quad (x \in \mathbf{R}_+).$$

By taking the identity operator as T in Lemma 3.1 we can deduce that

$$\|f\|_{L^1(G)} = \|fD\|_{L^1(\mathbf{R}_+)} \le c\|\mathcal{C}_+^1 F_f^1\|_{H^1(\mathbf{R})}. \tag{15}$$

For a general $f \in L_{loc}^1(G//K)$ with $\|\mathcal{C}_+^1 F_f^1\|_{H^1(\mathbf{R})} < \infty$ we approximate f by functions in $\mathcal{L}(G//K)$. Let ψ_0 be an even C^∞ function on $\mathbf{R}$ with $\int_{\mathbf{R}} \psi_0(x)dx = 0$ and put $\psi^{(\epsilon)}(x) = \psi_0(x)e^{\epsilon\rho x}$ for $\epsilon \in \mathbf{R}_+$. Since $\psi_0^{\sim}(\epsilon\lambda)$ is an even holomorphic function of exponential type, the Paley-Wiener theorem on G yields that there exists $\phi^{(\epsilon)} \in C_c^\infty(G//K)$ such that $(\phi^{(\epsilon)})^\wedge(\lambda) = \psi_0^{\sim}(\epsilon\lambda)$. Let $\psi_\delta^{(\epsilon)}(x) = \delta^{-1}\ \psi^{(\epsilon)}(\delta^{-1}x)$ for $\delta \in \mathbf{R}_+$. Then $F_{\phi^{(\epsilon)}}^1(x) = \psi_\epsilon^{(\epsilon)}(x)$ because $(F_{\phi^{(\epsilon)}}^1)^{\sim}(\lambda) = (\phi^{(\epsilon)})^\wedge(\lambda + i\rho) = \psi_0^{\sim}(\epsilon(\lambda + i\rho)) = (\psi_\epsilon^{(\epsilon)})^{\sim}(\lambda)$, and moreover, $f*\phi^{(\epsilon)} \in \mathcal{L}(G//K)$ and $\mathcal{C}_+^1 F_{f*\phi^{(\epsilon)}}^1 = (\mathcal{C}_+^1 F_f^1)*\psi_\epsilon^{(\epsilon)}$. We here note the following,

Lemma 4.2. *Let F be in $H^1(\mathbf{R})$. Then $F * \psi_\epsilon^{(\epsilon)} \in H^1(\mathbf{R})$ for $0 < \epsilon < 1$ and*

$$\|F - F * \psi_\epsilon^{(\epsilon)}\|_{H^1(\mathbf{R})} \to 0 \quad (\epsilon \to 0).$$

Proof. We refer to the notations and the results in Folland and Stein [**5**, §2]. Since $\|\psi^{(\epsilon)}\|_{(1)} \le c$ for all $0 < \epsilon < 1$, it follows from Lemma 3.31 in [**5**] that there exists $N \ge 1$ such that $\sup_{0<\delta<\infty} (M_{(N)}(F*\psi_\delta^{(\epsilon)}))(x) \le c(M_{(1)}F)(x)$. Especially, $(M_{(N)}(F*\psi_\epsilon^{(\epsilon)}))(x) \le c(M_{(1)}F)(x)$ and thus, $F * \psi_\epsilon^{(\epsilon)} \in H^1(\mathbf{R})$ by Theorem 3.30 in [**5**]. The rest of the proof follows from the same argument as in the proof of Theorem 3.33 in [**5**]. □

Hence, $\mathcal{C}_+^1\ F_{f*\phi^{(\epsilon)}}^1 = (\mathcal{C}_+^1 F_f^1) * \psi_\epsilon^{(\epsilon)} \in H^1(\mathbf{R})$ and $\|\mathcal{C}_+^1\ F_f^1 - \mathcal{C}_+^1\ F_{f*\phi^{(\epsilon)}}^1\|_{H^1(\mathbf{R})} \to 0$ as $\epsilon \to 0$. On the other hand, since $f*\ \phi^{(\epsilon)} \in \mathcal{L}(G//K)$, it follows from (15) that $\|f*\ \phi^{(\epsilon)}\|_{L^1(G)} \le c\ \|\mathcal{C}_+^1\ F_{f*\phi^{(\epsilon)}}^1\|_{H^1(\mathbf{R})}$. Therefore, $f*\ \phi^{(\epsilon)}$ converges to a function h in $L^1(G//K)$. Clearly, h must be f, because $(\phi^{(\epsilon)})^\wedge(\lambda) \to \psi_0^{\sim}(0) = 1$ as $\epsilon \to 0$. So, letting $\epsilon \to 0$, we obtain the desired inequality for f. This completes the proof of Theorem 4.1. □

We now define the Hardy space $H_s^1(G//K)$ $(s \ge 0)$ on G. Let $\mathcal{Q}_s$ $(s \ge 0)$ denote the Fourier multiplier of $\mathbf{R}$ defined by

$$(\mathcal{Q}_s F)^{\sim}(\lambda) = \left(\frac{1+|\lambda|}{|\lambda|}\right)^s F^{\sim}(\lambda) \quad (\lambda \in \mathbf{R}).$$

Definition 4.3. *For $s \geq 0$*

$$H^1_s(G//K) = \{f \in L^1_{loc}(G//K); \mathcal{Q}_s\mathcal{C}^1_+ F^1_f \text{ is well-defined and is in } H^1(\mathbf{R})\}$$

and $\|f\|_{H^1_s(G)} = \|\mathcal{Q}_s\mathcal{C}^1_+ F^1_f\|_{H^1(\mathbf{R})}$.

Theorem 4.4. *If $s' \geq s \geq 0$, then $H^1_{s'}(G//K) \subset H^1_s(G//K) \subset L^1(G//K)$ and for all $f \in H^1_{s'}(G//K)$*

$$\|f\|_{L^1(G)} \leq c\|f\|_{H^1_s(G)} \leq c'\|f\|_{H^1_{s'}(G)},$$

where c and c' are independent of f.

Proof. Theorem 4.1 asserts that $\|f\|_{L^1(G)} \leq c\|f\|_{H^1_0(G)}$. Let $\mathcal{R}_s$ denote the Fourier multiplier on $\mathbf{R}$ defined by $(\mathcal{R}_s F)^\sim(\lambda) = F^\sim(\lambda)(|\lambda|/\ 1+|\lambda|)^s$ $(\lambda \in \mathbf{R})$. We notice that $(|\lambda|/\ 1+|\lambda|)^s$ $(s \geq 0)$ satisfies the Hörmander condition and thus, $\mathcal{R}_s$ is a bounded linear operator on $H^1(\mathbf{R})$ (see [**12**, Theorem 4.2]). Therefore, for all $f \in H^1_{s'}(G//K)$, $\|f\|_{H^1_s(G)} = \|\mathcal{R}_{s'-s}\ \mathcal{Q}_{s'}\mathcal{C}^1_+ F^1_f\|_{H^1(\mathbf{R})} \leq c\|\mathcal{Q}_{s'}\ \mathcal{C}^1_+ F^1_f\|_{H^1(\mathbf{R})} = \|f\|_{H^1_{s'}(G)}$. □

Theorem 4.5. *For $s \geq 0$, $H^1_s(G//K) \cap \mathcal{L}(G//K)$ is dence in $H^1_s(G//K)$.*

Proof. We retain the notation used in the proof of Theorem 4.1 and suppose that $f \in H^1_s(G//K)$. Since $\mathcal{Q}_s\mathcal{C}^1_+ F^1_f \in H^1(\mathbf{R})$ and $\mathcal{Q}_s\ \mathcal{C}^1_+ F^1_{f*\phi^{(\epsilon)}} = \mathcal{Q}_s\mathcal{C}^1_+(F^1_f *\ \psi^{(\epsilon)}_\epsilon) = (\mathcal{Q}_s\mathcal{C}^1_+ F^1_f)*\ \psi^{(\epsilon)}_\epsilon$ (cf. [**4**, Theorem 5]), it follows from Lemma 4.2 that $f * \phi^{(\epsilon)} \in H^1_s(G//K) \cap \mathcal{L}(G//K)$ and $\|f - f*\phi^{(\epsilon)}\|_{H^1_s(G)} \to 0$ $(\epsilon \to 0)$. □

Remark 4.6. $H^1_0(G//K)$ contains all $f \in C^\infty_c(G//K)$ with $\int_G f(g)dg = 0$. Indeed, we suppose that $\mathrm{supp}(f) \subset B(r)$, and we observe that $\hat{f}(\lambda)$ is holomorphic on $\mathcal{F}_c$ of exponential type r and $C(-\lambda)^{-1}$ is holomorphic and temperd on the upper half plane (cf. [**14**, 9.2.3] and [**4**, Lemma 8]). We here recall the technique used in the proof of the Paley-Wiener theorem (cf. [**14**, 9.2.3]). Hence, shifting the integral line $\mathbf{R}$ of the integral $\int_{\mathbf{R}} C(-\lambda - i\rho)^{-1}\hat{f}(\lambda + i\rho)\ e^{i\lambda x}d\lambda$ defining $\mathcal{C}^1_+ F^1_f$ to $\mathbf{R} + i\eta$ $(\eta \to +\infty)$, we can deduce that $\mathrm{supp}(\mathcal{C}^1_+ F^1_f) \subset (-\infty, r]$. Thereby, $\|\mathcal{C}^1_+ F^1_f\|_{L^2(\mathbf{R})} = \|e^{\rho x}(\mathcal{C}^0_+ F^0_f)\|_{L^2(\mathbf{R})} \leq e^{\rho r}$ $\|\mathcal{C}^0_+\ F^0_f\|_{L^2(\mathbf{R})} = e^{\rho r}\|f\|_{L^2(G)}$ by the Plancherel formulas on $\mathbf{R}$ and G, and similarly, $\|x(\mathcal{C}^1_+ F^1_f)\|_{L^2(\mathbf{R})} \leq c(1+re^{\rho r})\ \|f\|_{L^2(G)}$. On the other hand, since $\phi_{i\rho} \equiv 1$ and $C(-i\rho) = 1$ (cf. [**4**, §3 and Lemma 8]), we have $\int_{\mathbf{R}}(\mathcal{C}^1_+ F^1_f)(x)dx = C(-i\rho)^{-1}\hat{f}(i\rho) = \int_G f(g)dg = 0$. These follow immediately that $\mathcal{C}^1_+ F^1_f$ is a (1,2,0,1/2)-molecule on $\mathbf{R}$ and hence, in $H^1(\mathbf{R})$ (see [**12**, Theorem 2.9]).

5. Criteria for boundedness. We fix $0 < r_1 < r_2 \leq \infty$ and a function $\phi(r,g)$ on $(r_1,r_2) \times G$ satisfying $\phi(r,\cdot) \in L^1(G//K)$ for each $r \in (r_1,r_2)$. We define a radial maximal operator T^G_{ϕ,r_1,r_2} on G as follows: for $f \in L^1_{loc}(G//K)$

$$(T^G_{\phi,r_1,r_2}f)(g) = \sup_{r_1<r<r_2} |(f*\phi(r,\cdot))(g)| \quad (g \in G).$$

In this section we obtain some conditions on ϕ under which T^G_{ϕ,r_1,r_2} is a bounded sub-linear operator of $H^1_s(G//K)$ $(s \geq 0)$ to $L^1(G//K)$.

Let B_{r_1,r_2} be the set of all functions $\beta(r,\lambda)$ on $(r_1,r_2)\times\mathbf{R}$ for which there exists a continuous function $\Theta=\Theta_\beta$ on $\mathbf{R}$ such that

$$\begin{aligned}(16)\qquad &(a_1)\quad \Theta(\lambda)\in L^1(\mathbf{R}),\\ &(a_2)\quad \lim_{|\lambda|\to\infty}\Theta(\lambda)=0,\\ &(b)\quad |(\frac{d}{d\lambda})^M\beta(r,\lambda)|\le r^M\Theta(r\lambda)\quad (M=0,1,2),\end{aligned}$$

and $B^+_{r_1,r_2}$ the subset of B_{r_1,r_2} defined by replacing $(16)(a_1),(a_2)$ with

$$\begin{aligned}(17)\qquad &(a_1)\quad \lambda\Theta(\lambda)\in L^1(\mathbf{R}),\\ &(a_2)\quad \lim_{|\lambda|\to\infty}\lambda\Theta(\lambda)=0.\end{aligned}$$

Lemma 5.1. *If $\hat\phi(r,\lambda+i\rho)\in B_{r_1,r_2}$, then the maximal operator defined by $\sup_{r_1<r<r_2}|(F^1_{\phi(r,\cdot)}*F)(x)|$ of $F\in L^1_{loc}(\mathbf{R})$ satisfies the $L^p(\mathbf{R})$-norm inequalities for $p>1$, and a weak type $L^1(\mathbf{R})$ estimate.*

Proof. Since $(F^1_{\phi(r,\cdot)})^\sim(\lambda)=\hat\phi(r,\lambda+i\rho)$ (see (9)), it follows from (16) of $\hat\phi(r,\lambda+i\rho)$ that $|x^M F^1_{\phi(r,\cdot)}(x)|\le|\int_{\mathbf{R}}(d/d\lambda)^M\,\hat\phi(r,\lambda+i\rho)\cdot e^{i\lambda x}\,d\lambda|\le\|\Theta\|_{L^1(\mathbf{R})}r^{M-1}$ $(M=0,1,2)$ and thus, $|F^1_{\phi(r,\cdot)}(x)|\le cr^{-1}(1+r^{-1}|x|)^{-2}$ $(x\in\mathbf{R})$. This inequality easily yields that $\sup_{r_1<r<r_2}|(F^1_{\phi(r,\cdot)}*F)(x)|\le c(M_{HL}F)(x)$ $(x\in\mathbf{R})$, where M_{HL} is the Hardy-Littlewood maximal operator on $\mathbf{R}$. Hence the desired results follow from those for M_{HL} (cf. [**5**, Theorem 2.4]). □

Lemma 5.2. *If $\hat\phi(r,\lambda+i\rho)\in B^+_{r_1,r_2}$, then for each (1,2,0)-atom a on $\mathbf{R}$ with supp(a) $\subset[x_0-\ell,x_0+\ell]$,*

$$\sup_{r_1<r<r_2}|(F^1_{\phi(r,\cdot)}*a)(x)|\le c\ell|x-x_0|^{-2}\quad \textit{if } |x-x_0|\ge 2\ell,$$

where c is independent of a.

Proof. The moment condition $\int_{\mathbf{R}}a(x)dx=0$ and the mean value theorem yield that for $x\in\mathbf{R}_+$

$$\begin{aligned}(F^1_{\phi(r,\cdot)}*a)(x)&=\int_{\mathbf{R}}F^1_{\phi(r,\cdot)}(x-y)a(y)dy\\ &=\int_{\mathbf{R}}\left(F^1_{\phi(r,\cdot)}(x-y)-F^1_{\phi(r,\cdot)}(x-x_0)\right)a(y)dy\\ &=\int_{-\ell}^{\ell}\left(\frac{d}{dx}F^1_{\phi(r,\cdot)}(x-y_0)\right)(x_0-y)a(y)dy,\end{aligned}$$

where y_0 is on the line segment from y to x_0. Since $(dF^1_{\phi(r,\cdot)}/dx)^\sim(\lambda) = i\lambda\hat{\phi}(r, \lambda + i\rho)$ (see (9)), it follows from (16)(b) and (17) of $\hat{\phi}(r, \lambda + i\rho)$ that

$$\begin{aligned}|(x-y_0)^2 \frac{d}{dx} F^1_{\phi(r,\cdot)}(x-y_0)| &= |\int_{\mathbf{R}} (\frac{d}{d\lambda})^2 \left(\lambda\hat{\phi}(r, \lambda + i\rho)\right) e^{i\lambda(x-y_0)} d\lambda| \\ &\leq 2r \int_{\mathbf{R}} |\Theta(r\lambda)| d\lambda + r^2 \int_{\mathbf{R}} |\lambda\Theta(r\lambda)| d\lambda \\ &= 2\|\Theta\|_{L^1(\mathbf{R})} + \|\lambda\Theta\|_{L^1(\mathbf{R})}.\end{aligned}$$

Let $|x-x_0| \geq 2\ell$. Since $|x_0-y| < \ell$ and y_0 is located between y and x_0, we have $|x-y_0| > \ell$ and thus, $|x-x_0||x-y_0|^{-1} \leq 1 + |x_0-y_0||x-y_0|^{-1} \leq 2$. Therefore, if $|x-x_0| \geq 2\ell$, $\sup_{r_1<r<r_2} |(F^1_{\phi(r,\cdot)} * a)(x)| \leq c|x-x_0|^{-2} \int_{\mathbf{R}} |x_0-y||a(y)|dy \leq c\ell|x-x_0|^{-2}$. □

Proposition 5.3. *If $\hat{\phi}(r, \lambda + i\rho) \in B^+_{r_1,r_2}$, then T^G_{ϕ,r_1,r_2} is a bounded sublinear operator of $H^1_0(G//K)$ to $L^1(G//K)$.*

Proof. From Theorem 4.5 it suffices to obtain the boundedness for $f \in H^1_0(G//K) \cap \mathcal{L}(G//K)$. Since $(f* \phi(r,\cdot))^\wedge(\lambda) = \hat{f}(\lambda)\hat{\phi}(r, \lambda) = \hat{f}(\lambda)(F^0_{\phi(r,\cdot)})^\sim(\lambda)$ (cf. [**4**, Theorem 5]), the same manipulation as in (10) yields that for $x \in \mathbf{R}_+$

$$(f * \phi(r,\cdot))(x) = e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} (F^1_{\phi(r,\cdot)} * (\Gamma^1_m \mathcal{C}^1_+ F^1_f))(x)$$

and thereby,

$$(18) \qquad (T^G_{\phi,r_1,r_2} f)(x) \leq e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} \sup_{r_1<r<r_2} |(F^1_{\phi(r,\cdot)} * (\Gamma^1_m \mathcal{C}^1_+ F^1_f))(x)|.$$

Since $\hat{\phi}(r, \lambda + i\rho) \in B^+_{r_1,r_2}$, Lemmas 5.1 and 5.2 imply that the maximal operator $\sup_{r_1<r<r_2} |(F^1_{\phi(r,\cdot)} * F)(x)|$ of $F \in L^1_{loc}(\mathbf{R})$ satisfies the properties (a) and (b) in Lemma 3.1. Therefore, the result follows from Lemma 3.1 and Definition 4.3. □

Let $B^\delta_{r_1,r_2}$ ($\delta \in \mathbf{R}$) be the set of all functions $\beta(r, \lambda)$ on $(r_1, r_2) \times \mathbf{R}$ for which there exists a continuous function $\Theta = \Theta_\beta$ on $\mathbf{R}$ such that

$$(19) \qquad \begin{aligned}&(a_1) \quad \lambda^{\delta+1}\Theta(\lambda) \in L^1(\mathbf{R}), \\ &(a_2) \quad \lim_{|\lambda|\to\infty} \lambda^{\delta+1}\Theta(\lambda) = 0, \\ &(b) \quad |(\frac{d}{d\lambda})^M \beta(r, \lambda)| \leq r^{M+\delta}\Theta(r\lambda) \quad (M = 0, 1, 2).\end{aligned}$$

Proposition 5.4. *If* $\hat{\phi}(r, \lambda + i\rho) \in B^{\delta}_{r_1,r_2}$ $(\delta \geq 2)$*, then* T^{G}_{ϕ,r_1,r_2} *is a bounded sublinear operator of* $H^1_\delta(G//K)$ *to* $L^1(G//K)$*.*

Proof. As we have deduced (18), we see that for $f \in H^1_\delta(G//K) \cap \mathcal{L}(G//K)$

$$(T^G_{\phi,r_1,r_2}f)(x) \leq e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} |(T\Gamma^1_m \mathcal{Q}_\delta \mathcal{C}^1_+ F^1_f)(x)| \quad (x \in \mathbf{R}_+),$$

where T is a maximal operator on $L^1_{loc}(\mathbf{R})$ defined by

$$(TF)(x) = \sup_{r_1<r<r_2} \int_{\mathbf{R}} F^{\sim}(\lambda) \left(\frac{|\lambda|}{1+|\lambda|}\right)^{\delta} \hat{\phi}(r, \lambda + i\rho) e^{i\lambda x} d\lambda \quad (x \in \mathbf{R}).$$

Then, as in the proof of Proposition 5.3 it suffices to show that $(|\lambda|/1+|\lambda|)^{\delta}\ \hat{\phi}(r, \lambda+i\rho) \in B^{+}_{r_1,r_2}$. Indeed, it follows from (19)(b) of $\hat{\phi}(r, \lambda + i\rho)$ that for $M = 0, 1, 2$

$$\begin{aligned}
&|(\frac{d}{d\lambda})^M \left(\left(\frac{|\lambda|}{1+|\lambda|}\right)^{\delta} \hat{\phi}(r, \lambda + i\rho)\right)| \\
&\leq c \sum_{m=0}^{M} |r^{\delta} (\frac{d}{d\lambda})^{M-m} \left(\frac{|\lambda|}{1+|\lambda|}\right)^{\delta} | \cdot |r^{-\delta} (\frac{d}{d\lambda})^{m} \hat{\phi}(r, \lambda + i\rho)| \\
&\leq c \sum_{m=0}^{M} \delta(\delta-1)\dots(\delta - M + m + 1)|r\lambda|^{\delta-M+m} r^{M-m} \cdot r^m \Theta(r\lambda) \\
&\leq c r^M \Psi(r\lambda)\Theta(r\lambda),
\end{aligned}$$

where $\Psi(\lambda) = (\delta(\delta-1)|\lambda|^{\delta-2} + \delta|\lambda|^{\delta-1} + |\lambda|^{\delta})$. Therefore, if we define Θ in (16) and (17) by $\Psi\Theta$, the above calculation and $(19)(a_1), (a_2)$ imply that $(|\lambda|/1+|\lambda|)^{\delta}\ \hat{\phi}(r, \lambda + i\rho) \in B^{+}_{r_1,r_2}$. □

Let $B^{\delta,+}_{r_1,r_2}$ $(\delta \in \mathbf{R})$ be the set of all functions $\beta(r, \lambda)$ on $(r_1, r_2) \times \mathbf{R}$ for which there exists a continuous function $\Theta = \Theta_\beta$ on $\mathbf{R}$ such that

(20)

$$\begin{aligned}
&(a_1) \quad \lambda^{\delta+1}\Theta(\lambda) \in L^1(\mathbf{R}), \\
&(a_2) \quad \lim_{|\lambda|\to\infty} \lambda^{\delta+1}\Theta(\lambda) = 0, \\
&(b_1) \quad |\beta(r, \lambda)| \leq \Theta(r\lambda), \\
&(b_2) \quad |(\frac{d}{d\lambda})\beta(r, \lambda)| \leq r^{\delta}\Theta(r\lambda), \\
&(b_3) \quad |(\frac{d}{d\lambda})^2\beta(r, \lambda)| \leq r^{2+\delta}\Theta(r\lambda).
\end{aligned}$$

Proposition 5.5. *Let $r_1 \geq 1$ and $\delta \geq 2$. If $\hat{\phi}(r, \lambda + i\rho) \in B^{\delta,+}_{r_1,r_2}$, then T^G_{ϕ,r_1,r_2} is a bounded sublinear operator of $H^1_{\delta-1+\epsilon}(G//K)$ to $L^1(G//K)$ for $\epsilon > 0$.*

Proof. We first notice that under the assumption that $r_1 \geq 1$ and $\delta \geq 2$, $\beta(r, \lambda)$ in $B^{\delta,+}_{r_1,r_2}$ also satisfies

$$(b_4) \quad |\beta(r, \lambda)| \leq r^{\delta-1}\Theta(r\lambda),$$

$$(b_5) \quad |(\frac{d}{d\lambda})^M \beta(r, \lambda)| \leq r^{M+\delta}\Theta(r\lambda) \quad (M = 0, 1, 2).$$

In what follows we modify the proofs of Lemmas 5.1 and 5.2. As in the proof of Theorem 4.4, $\mathcal{R}_s$ ($s \in \mathbf{R}$) denotes the Fourier multiplier on $\mathbf{R}$ defined by $(\mathcal{R}_s F)^{\sim}(\lambda) = F^{\sim}(\lambda)(|\lambda|/1 + |\lambda|)^s$ ($\lambda \in \mathbf{R}$). Since $(\mathcal{R}_{\delta-1}\ F^1_{\phi(r,\cdot)})^{\sim}(\lambda) = \hat{\phi}(r, \lambda + i\rho)\ (|\lambda|/1 + |\lambda|)^{\delta-1}$, we see from (a), (b_2), and (b_4) of $\hat{\phi}(r, \lambda + i\rho)$ that

$$(21) \qquad |(\mathcal{R}_{\delta-1} F^1_{\phi(r,\cdot)})(x)| \leq cx^{-1}\left((\delta - 1)\|\lambda^{\delta-2}\Theta\|_{L^1(\mathbf{R})} + \|\lambda^{\delta-1}\Theta\|_{L^1(\mathbf{R})}\right)$$

and similarly from (a) and (b_5) that

$$|(\mathcal{R}_\delta F^1_{\phi(r,\cdot)})(x)| \leq crx^{-2}\left(\delta(\delta - 1)\|\lambda^{\delta-2}\Theta\|_{L^1(\mathbf{R})} + \delta\|\lambda^{\delta-1}\Theta\|_{L^1(\mathbf{R})} + \|\lambda^{\delta}\Theta\|_{L^1(\mathbf{R})}\right).$$

We here fix r and x, and we observe that $(\mathcal{R}_z F^1_{\phi(r,\cdot)})(x)$ makes sense for $z \in \mathbf{C}$ with $\delta - 1 \leq \Re(z) \leq \delta$ and, as a function of z, it is holomorphic on $\delta - 1 < \Re(z) < \delta$. Moreover, $|(\mathcal{R}_z F^1_{\phi(r,\cdot)})(x)| \leq r^{-1}\|\Theta\|_{L^1(\mathbf{R})}$ by (a) and (b_1), and when $\Re(z) = \delta - 1$ (resp. δ), $(\mathcal{R}_z\ F^1_{\phi(r,\cdot)})(x)$ satisfies the same inequality for $z = \delta - 1$ (resp. δ) obtained above. Therefore, Three Lines Lemma (cf. [**9**, p. 69]) yields that $|(\mathcal{R}_{\delta-1+\epsilon}\ F^1_{\phi(r,\cdot)})(x)| \leq cr^{\epsilon}|x|^{-(1+\epsilon)}$ $(0 \leq \epsilon \leq 1)$. Since $|(\mathcal{R}_{\delta-1+\epsilon}\ F^1_{\phi(r,\cdot)})(x)| \leq r^{-1}\ \|\Theta\|_{L^1(\mathbf{R})}$, we have $|(R_{\delta-1+\epsilon}$ $F^1_{\phi(r,\cdot)})(x)| \leq cr^{-1}(1 + r^{-1}|x|)^{-(1+\epsilon)}$ $(0 \leq \epsilon \leq 1)$. In particular, if $0 < \epsilon \leq 1$, the maximal function defined by $\sup_{r_1<r<r_2} |((\mathcal{R}_{\delta-1+\epsilon}\ F^1_{\phi(r,\cdot)}) * F)(x)|$ of $F \in L^1_{loc}(\mathbf{R})$ is pointwisely dominated by $c(M_{HL}F)(x)$ and hence, is bounded on $L^2(\mathbf{R})$.

We next observe that (a) and (b_5) imply that $\hat{\phi}(r, \lambda + i\rho) \in B^{\delta}_{r_1,r_2}$ and thus, Lemma 5.2 and the proof of Proposition 5.4 yield that for each (1,2,0)-atom a on $\mathbf{R}$ with supp(a) $\subset [x_0 - \ell, x_0 + \ell]$,

$$\begin{aligned} |((\mathcal{R}_\delta F^1_{\phi(r,\cdot)}) * a)(x)| \leq c\ell|x - x_0|^{-2}(\delta(\delta - 1)(2\|\lambda^{\delta-2}\Theta\|_{L^1(\mathbf{R})} + \|\lambda^{\delta-1}\Theta\|_{L^1(\mathbf{R})}) \\ + \delta(2\|\lambda^{\delta-1}\Theta\|_{L^1(\mathbf{R})} + \|\lambda^{\delta}\Theta\|_{L^1(\mathbf{R})}) + 2\|\lambda^{\delta}\Theta\|_{L^1(\mathbf{R})} + \|\lambda^{\delta+1}\Theta\|_{L^1(\mathbf{R})}) \end{aligned}$$

if $|x - x_0| \geq 2\ell$. On the other hand, we notice that if $|x_0 - y| \leq \ell$ and $|x - x_0| \geq 2\ell$, then $|x - x_0||x - y|^{-1} \leq 1 + |x_0 - y||x - y|^{-1} \leq 2$ and thereby, from (21) that

$$\begin{aligned} |((\mathcal{R}_{\delta-1} F^1_{\phi(r,\cdot)}) * a)(x)| &= |\int_{\mathbf{R}} (\mathcal{R}_{\delta-1} F^1_{\phi(r,\cdot)})(x - y)a(y)dy| \\ &\leq c|x - x_0|^{-1}\left((\delta - 1)\|\lambda^{\delta-2}\Theta\|_{L^1(\mathbf{R})} + \|\lambda^{\delta-1}\Theta\|_{L^1(\mathbf{R})}\right) \end{aligned}$$

if $|x-x_0| \geq 2\ell$. Moreover, $|((\mathcal{R}_z F^1_{\phi(r,\cdot)}) * a)(x)| \leq \|\mathcal{R}_z\ F^1_{\phi(r,\cdot)}\|_\infty\ \|a\|_{L^1(\mathbf{R})} \leq r^{-1}$ $\|\Theta\|_{L^1(\mathbf{R})}$ for $\delta - 1 \leq \Re(z) \leq \delta$. Then, applying Three Lines Lemma again, we see that for $0 \leq \epsilon \leq 1$, $|((\mathcal{R}_{\delta-1+\epsilon}\ F^1_{\phi(r,\cdot)}) * a)(x)| \leq c\ell^\epsilon\ |x-x_0|^{-(1+\epsilon)}$ if $|x-x_0| \geq 2\ell$.

We have therefore proved that if $0 < \epsilon \leq 1$, the maximal operator defined by $\sup_{r_1<r<r_2} |((\mathcal{R}_{\delta-1+\epsilon}\ F^1_{\phi(r,\cdot)}) * F)(x)|$ of $F \in L^1_{loc}(\mathbf{R})$ satisfies the properties (a) and (b) in Lemma 3.1. Since $(T^G_{\phi,r_1,r_2} f)(x)$ $(x \in \mathbf{R}_+)$ is dominated by

$$e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} \sup_{r_1<r<r_2} |((\mathcal{R}_{\delta-1+\epsilon} F^1_{\phi(r,\cdot)}) * (\mathit{\Gamma}^1_m \mathcal{Q}_{\delta-1+\epsilon} \mathcal{C}^1_+ F^1_f))(x)|,$$

the desired result follows from Lemma 3.1, Definition 4.3 and Theorem 4.4. □

6. Maximal operators. We apply the criteria obtained in the previous section to some radial maximal operators. Actually, as $\phi(r,g)$ in §5, we shall take $\phi^\natural_r(g)$, $\phi^\sharp_r(g)$, $\phi^\flat_r(g)$, $h_r(g)$, and $p_r(g)$ respectively (see **A**∼**E** below). For simplicity, we denote henceforth the set of bounded sublinear operators of $H^1_s(G//K)$ to $L^1(G//K)$ by (H^1_s, L^1) and that on $L^p(G//K)$ by (L^p, L^p). We also say that an operator T of $L^1(G//K)$ is of weak type $(1,1)$ provided there exists a constant c such that

$$|\{g \in G; |(Tf)(g)| > \alpha\}| \leq \frac{c}{\alpha} \|f\|_{L^1(G)}$$

for all $f \in L^1(G//K)$ and for every $\alpha > 0$.

A. Let $A^\natural_\delta(G)$ $(\delta \in \mathbf{R})$ be the set of all functions $\phi \in L^1(G//K)$ satisfying $|(d/d\lambda)^M \hat{\phi}(\lambda)| \leq (1+ |\lambda|)^{-\delta}$ $(\lambda \in \mathcal{F}(\rho))$ for $M = 0, 1, 2$.

Definition 6.1. *For* $f \in L^1_{loc}(G//K)$

$$(M^{G,\natural}_{\delta,0} f)(g) = \sup_{\substack{\phi \in A^\natural_\delta(G) \\ 0<r<\infty}} |(f * \phi^\natural_r)(g)| \quad (g \in G),$$

where $\phi^\natural_r(g) = \int_{\mathbf{R}} \hat{\phi}(r\lambda) \phi_\lambda(g) |C(\lambda)|^{-2} d\lambda$.

Theorem 6.2. *If* $\delta > 2$, *then* $M^{G,\natural}_{\delta,0} \in (H^1_0, L^1)$.

Proof. We easily see that if $\delta > 2$, $\hat{\phi}(r\lambda)$ belongs to $B^+_{0,\infty}$ and hence, by Proposition 5.3 we have the desired result. □

B. Let $A_N(G)$ $(N \in \mathbf{N})$ be the set of all functions $\phi \in C^N(G//K)$ whose restriction on A satisfies $\mathrm{supp}(\phi) \subset [-1, 1]$, $\|(d/dx)^n \phi\|_\infty \leq 1$ $(0 \leq n \leq N)$, and $\phi(x) = \mathcal{O}(x^N)$.

Definition 6.3. *For* $\epsilon \geq 0$ *and* $f \in L^1_{loc}(G//K)$

$$(M^{G,\sharp}_{N,\epsilon} f)(g) = \sup_{\substack{\phi \in A_N(G) \\ 0<r<\infty}} (1+r)^{-\epsilon} |(f * \phi^\sharp_r)(g)| \quad (g \in G),$$

where $\phi^\sharp_r(g) = \frac{1}{|B(r)|} \phi(\frac{\sigma(g)}{r})$.

Theorem 6.4. *(1) $M_{N,\epsilon}^{G,\sharp}$ is in (L^p, L^p) for $p > 1$ and of weak type (1,1),*
(2) If $N \geq 6$, then $M_{N,0}^{G,\sharp} \in (H_\delta^1, L^1)$ for $\delta > 2$,
(3) If $N \geq 4$ and $\epsilon > 1$, then $M_{N,\epsilon}^{G,\sharp} \in (H_0^1, L^1)$.

Proof. (1) is obvious from Theorem 1.2 because $M_{N,\epsilon}^{G,\sharp} f$ is pointwisely dominated by $M_{HL}^G f$. To prove (2) and (3) we first obtain the following,

Lemma 6.5. *Let $\phi \in A_{2N}(G)$ $(N \in \mathbf{N})$. Then, for $M = 0, 1, 2$*

$$|(\frac{d}{d\lambda})^M (\phi_r^\sharp)^\wedge(\lambda + i\rho)| \leq c r^M (1+r)^s (1 + |r\lambda|)^{-2s} \quad (0 \leq s \leq N).$$

Proof. We recall that $|(d/d\lambda)^m \phi_{\lambda+i\rho}(x)| \leq x^m \phi_{i\rho}(x) \leq x^m$ $(\lambda \in \mathcal{F}, x \in \mathbf{R}_+)$, ϕ_λ is an eigenfunction of the Laplace-Beltrami operator Ω on G with eigenvalue $-p(\lambda) = -\lambda^2 - \rho^2$, and the radial component of Ω on $\mathbf{R}_+$ is of the form $D^{-1} \cdot (d/dx(D \cdot d/dx))$. For these facts we refer to [**4**], Lemma 14, Proposition 3, and §2 respectively. We now apply them to the integral defining the Fourier transform $(\phi_r^\sharp)^\wedge(\lambda + i\rho)$ and thereby, we can deduce that for each $m, n \in \mathbf{N}$ and $0 \leq n \leq N$

$$\begin{aligned} &|(\frac{d}{d\lambda})^m \left(p(\lambda+i\rho)^n \cdot (\phi_r^\sharp)^\wedge(\lambda+i\rho)\right)| \\ &\leq \int_0^\infty |\Omega^n \phi_r^\sharp(x) \cdot (\frac{d}{d\lambda})^m \phi_{\lambda+i\rho}(x)| D(x) dx \\ &\leq c|B(r)|^{-1} r^m \int_0^r |\left(D(x)^{-1} \frac{d}{dx}(D(x)\frac{d}{dx})\right)^n \phi(\frac{x}{r})| D(x) dx. \end{aligned} \tag{22}$$

For simplicity, we put $\mathbf{D} = D^{-1}(dD/dx)$ and $\Phi(x) = \phi(x/r)$ $(r \in \mathbf{R}_+)$, and we observe that $(d/dx)^p \Phi(x) \sim r^{-2N} x^{2N-p} \leq r^{-2n} x^{2n-p}$ if $0 < x \leq r$, $\mathbf{D}(x) \sim (1+x)/x$, and $(d/dx)^q \mathbf{D}(x) \leq c x^{-q-1} e^{-2x}$ if $q > 0$. Hence we have

$$\begin{aligned} &\|(\frac{d}{dx})^{2n} \Phi\|_\infty \leq c r^{-2n}, \\ &\|(\frac{d}{dx})^p \Phi \cdot (\frac{d}{dx})^q \mathbf{D} \cdot \mathbf{D}^s\|_\infty \leq c r^{-2n} \quad \text{if } p + q + s < 2n \text{ and } q > 0, \\ &\|(\frac{d}{dx})^p \Phi \cdot \mathbf{D}^s\|_\infty \leq c r^{-2n} (1+r)^s \quad \text{if } p + s = 2n. \end{aligned}$$

Then, by using these estimates to handle the derivatives of $\phi(x/r)$ in (22) we can deduce that $|(d/d\lambda)^m (p(\lambda+i\rho)^n \cdot (\phi_r^\sharp)^\wedge(\lambda+i\rho))| \leq c r^{m-2n}(1+r)^n$. Especially, letting $m = 0$, we obtain the inequality for $M = 0$ and $s = n$. Since $|p(\lambda+i\rho)^n \cdot (d/d\lambda)^m (\phi_r^\sharp)^\wedge(\lambda+i\rho)|$ is dominated by $|(d/d\lambda)^m (p(\lambda+i\rho)^n \cdot (\phi_r^\sharp)^\wedge(\lambda+i\rho))| + c\sum_{i=1}^m |(d/d\lambda)^i (p(\lambda+i\rho)^n) \cdot (d/d\lambda)^{m-i} (\phi_r^\sharp)^\wedge(\lambda+i\rho)|$, the rest inequalities follow by induction and interpolation. □

As for (2) let $\phi \in A_N(G)$ $(N \geq 6)$. By Lemma 6.5, $(\phi_r^\sharp)^\wedge(\lambda+i\rho) \in B_{0,1}^+$ if $0 < r \leq 1$ and $B_{1,\infty}^\delta$ $(2 < \delta \leq 3)$ if $1 < r < \infty$. Therefore, from Propositions 5.3, 5.4, and Theorem 4.4 it follows that $M_{N,0}^{G,\sharp} \in (H_\delta^1, L^1)$ for $\delta > 2$. As for (3) let $\phi \in A_N(G)$ $(N \geq 4)$. Since $(1+r)^{-\epsilon} (\phi_r^\sharp)^\wedge(\lambda+i\rho) \in B_{0,\infty}^+$ $(\epsilon > 1)$ by Lemma 6.5, it follows from Proposition 5.3 that $M_{N,\epsilon}^{G,\sharp} \in (H_0^1, L^1)$ for $\epsilon > 1$. □

C. Let $A_N(G)$ be the same as in **B**. We here introduce a dilation which preserves the L^1-norm on G (see [**7**]).

Definition 6.6. *For $\epsilon \geq 0$ and $f \in L^1_{loc}(G//K)$*

$$(M^{G,\flat}_{N,\epsilon} f)(g) = \sup_{\substack{\phi \in A_N(G) \\ 0<r<\infty}} (1+r)^{-\epsilon} |(f * \phi^\flat_r)(g)| \quad (g \in G),$$

where $\phi^\flat_r(g) = \frac{1}{r}\frac{1}{D(\sigma(g))} D(\frac{\sigma(g)}{r})\phi(\frac{\sigma(g)}{r})$.

Theorem 6.7. *(1) $M^{G,\flat}_{N,\epsilon}$ is in (L^p, L^p) for $p > 1$ and is of weak type (1,1),*
(2) If $N \geq 6$, then $M^{G,\flat}_{N,0} \in (H^1_\delta, L^1)$ for $\delta > 1$,
(3) If $N \geq 4$ and $\epsilon > 2$, then $M^{G,\flat}_{N,\epsilon} \in (H^1_0, L^1)$.

Proof. It is easy to verify that $\|\phi^\flat_r\|_{L^1(G)} = \|\phi^\flat_1\|_{L^1(G)}$ and moreover, $\phi^\flat_r(x) \leq c\phi^\sharp_r(x)$ if $0 < r \leq 1$ and $\phi^\flat_r(x) \leq ce^{-2\rho x}$ $(x \in \mathbf{R}_+)$ if $1 < r < \infty$. Therefore, as in [**7**, Theorem 3.4], it follows that $M^{G,\flat}_{N,\epsilon}$ is in (L^p, L^p) for $p > 1$ and of weak type $(1,1)$. We now suppose that $\phi \in A_{2N}(G)$ and we observe that for each $0 \leq n \leq 2N$ and $0 \leq x \leq r$, $(d/dx)^n \phi(x/r) \sim cr^{-2N}x^{2N-n}$ and

$$(\frac{d}{dx})^n D(\frac{x}{r}) \leq cr^{-n}\left(\frac{r+x}{x}\right)^n D(\frac{x}{r}) \leq cx^{-n}D(\frac{x}{r}),$$
$$(\frac{d}{dx})^n D(x)^{-1} \leq c\left(\frac{1+x}{x}\right)^n D(x)^{-1}.$$

Then the same argument as in the proof of Lemma 6.5 yields that for $M = 0, 1, 2$,

$$|(\frac{d}{d\lambda})^M (\phi^\flat_r)^\wedge(\lambda + i\rho)| \leq cr^M(1+r)^s(1+|r\lambda|)^{-2s} \quad (0 \leq s \leq N),$$

and therefore, the desired result follows. □

D. The heat maximal operator $M^G_{H,\epsilon}$ $(\epsilon \geq 0)$ on G is given as follows.

Definition 6.8. *For $\epsilon \geq 0$ and $f \in L^1_{loc}(G//K)$*

$$(M^G_{H,\epsilon} f)(g) = \sup_{0<r<\infty} (1+r)^{-\epsilon}|(f * h_r)(g)| \quad (g \in G),$$

where $h_r(g) = \int_{\mathbf{R}} e^{-(\lambda^2+\rho^2)r}\phi_\lambda(g)|C(\lambda)|^{-2}d\lambda$.

Theorem 6.9. *(1) $M^G_{H,\epsilon}$ is in (L^p, L^p) for $p > 1$ and of weak type (1,1),*
(2) $M^G_{H,0} \in (H^1_\delta, L^1)$ for $\delta > 1$,
(3) $M^G_{H,1} \in (H^1_0, L^1)$.

Proof. (1) is well-known (cf. [**9**, p.73] and [**1**, Corollary 3.2]). Since $\hat{h}_r(\lambda + i\rho) = e^{-\lambda^2 r}e^{-2i\rho\lambda r}$, we see that $\hat{h}_{r^2}(\lambda + i\rho) \in B^+_{0,1}$ if $0 < r \leq 1$ and $B^{2,+}_{1,\infty}$ if $1 < r < \infty$, and moreover, $(1+r^2)^{-1}\hat{h}_{r^2}(\lambda + i\rho) \in B^+_{0,\infty}$. Therefore, (2) follows from Propositions 5.3 and 5.5, and by changing r^2 to r, (3) follows from Proposition 5.3. □

E. The Poisson maximal operator $M^G_{P,\epsilon}$ $(\epsilon \geq 0)$ on G is given as follows.

Definition 6.10. *For* $\epsilon \geq 0$ *and* $f \in L^1_{loc}(G//K)$

$$(M^G_{P,\epsilon} f)(g) = \sup_{0<r<\infty} (1+r)^{-\epsilon} |(f * p_r)(g)| \quad (g \in G),$$

where $p_r(g) = \int_{\mathbf{R}} e^{-(\lambda^2+\rho^2)^{1/2} r} \phi_\lambda(g) |C(\lambda)|^{-2} d\lambda.$

Theorem 6.11. *(1)* $M^G_{P,\epsilon}$ *is in* (L^p, L^p) *for* $p > 1$ *and of weak type (1,1),*
(2) $M^G_{P,0} \in (H^1_{1/2}, L^1)$,
(3) $M^G_{P,2} \in (H^1_0, L^1)$.

Proof. As shown by Stein [**9**, p.48], it is well-known that $M^G_{P,\epsilon}$ is in (L^p, L^p) for $p > 1$ and of weak type (1,1). Let us recall the calculation in [**9**, p.49] and modify it slightly: the subordination formula gives an integral representation of $f * p_r$ such as

$$(f * p_r)(g) = \frac{1}{r^2} \int_0^\infty (f * h_y)(g) \phi(\frac{y}{r^2}) dy \quad (g \in G),$$

where $\phi(y) = 1/(2\pi^{1/2}) e^{-1/4y} y^{-3/2}$, and the integral by part yields the estimate of $(1+r)^{-2}(f * p_r)(g)$ such as

$$\begin{aligned}
&|\frac{1}{(1+r)^2 r^2}| \int_0^\infty \left(\int_0^y \frac{1}{1+s} (f * h_s)(g) ds \right) \frac{d}{dy} \left((1+y)\phi(\frac{y}{r^2}) \right) dy| \\
&\leq \sup_{0<y<\infty} \frac{1}{y} \int_0^y \frac{1}{1+s} |(f * h_s)(g)| ds \cdot \frac{1}{(1+r)^2 r^2} \int_0^\infty y |\frac{d}{dy} \left((1+y)\phi(\frac{y}{r^2}) \right)| dy \\
&\leq C (M^G_{H,1} f)(g),
\end{aligned}$$

where $C = \|y\phi\|_{L^1(\mathbf{R})} + \|(1+y)y \cdot d\phi/dy\|_{L^1(\mathbf{R})}$. Therefore, it follows from Theorem 6.9 that $M^G_{P,2} \in (H^1_0, L^1)$. Let $M^{G,0}_{P,0}$ (resp. $M^{G,1}_{P,0}$) denote the maximal operator defined by replacing $\sup_{0<r<\infty}$ in the definition of $M^G_{P,0}$ by $\sup_{0<r\leq 1}$ (resp. $\sup_{1<r<\infty}$). Obviously, $M^{G,0}_{P,0}$ is dominated by $cM^G_{P,2}$ and hence, $M^{G,0}_{P,0} \in (H^1_0, L^1)$ as remarked above. Especially, $M^{G,0}_{P,0} \in (H^1_{1/2}, L^1)$ by Theorem 4.4. On the other hand, we observe from the proof of [**1**, Corollary 6.3] that $\|M^{G,1}_{P,0} f\|_{L^1(G)} \leq c \|T^G_{1/\sqrt{p}} f\|_{L^1(G)}$, where $p(\lambda) = \lambda^2 + \rho^2$ and $T^G_{1/\sqrt{p}}$ is the Fourier multiplier on G defined by Definition 7.1 below. Then, since $p(\lambda + i\rho)^{-1/2}$ $(|\lambda|/1+|\lambda|)^{1/2}$ satisfies the Hörmander condition, Theorem 7.2 below yields that $T^G_{1/\sqrt{p}}$ $\in (H^1_{1/2}, L^1)$ and hence, $M^{G,1}_{P,0} \in (H^1_{1/2}, L^1)$. □

7. Other operators.

A. We shall define the Fourier multiplier on G corresponding to an even bounded function $m(\lambda)$ on $\mathcal{F}$.

Definition 7.1. *For* $f \in L^1_{loc}(G//K)$

$$(T_m^G f)(g) = \int_{\mathcal{F}} \hat{f}(\lambda) m(\lambda) \phi_\lambda(g) |C(\lambda)|^{-2} d\lambda \quad (g \in G).$$

We easily see that $T_m^G \in (L^2, L^2)$ by the Plancherel formula on G. We now suppose that $m(\lambda)$ has a holomorphic extension on $\mathcal{F}(\rho)$ and there exists $s \geq 0$ such that

$$\sup_{0 \leq \xi \leq \rho} \|m_{\xi,s}\| < \infty, \quad \text{where} \quad m_{\xi,s}(\lambda) = m(\lambda + i\xi) \left(\frac{|\lambda + i(\xi - \rho)|}{1 + |\lambda + i(\xi - \rho)|} \right)^s.$$

Then by the same manipulation as in (10) we see that for $f \in \mathcal{L}(G//K)$, $T_m^G f$ has the expansion:

$$\begin{aligned}(T_m^G f)(x) &= e^{-2\rho x} \sum_{n=0}^{\infty} e^{-2nx} T_{m_{\rho,0}} \Gamma_n^1 \mathcal{C}_+^1 F_f^1(x) \quad (x \in \mathbf{R}_+) \\ &= (T_{\Phi,D} T_{m_{\rho,s}})(\mathcal{Q}_s \mathcal{C}_+^1 F_f^1)(x) D(x)^{-1},\end{aligned}$$

where T_w is the Fourier multiplier on $\mathbf{R}$ defined by $(T_w\ F)^\sim(\lambda) = w(\lambda) F^\sim(\lambda)$ $(\lambda \in \mathbf{R})$ and $T_{\Phi,D}$ is the pseudo-differential operator given in Remark 3.2. Since $T_{\Phi,D}$ is a bounded operator of $H^1(\mathbf{R})$ to $L^1(\mathbf{R})$, we can obtain from Definition 4.3 and Theorem 4.5 that

Theorem 7.2. *Let* $m(\lambda)$ *be an even holomorphic function on* $\mathcal{F}(\rho)$. *If there exists* $s \geq 0$ *such that* $\sup_{0 \leq \xi \leq \rho} \|m_{\xi,s}\| < \infty$ *and* $T_{m_{\rho,s}}$ *is bounded on* $H^1(\mathbf{R})$, *then* $T_m^G \in (H_s^1, L^1)$.

B. We last treat the Riesz transform on G, and we henceforth use the standard notation as in [**9**]. Especially, we denote the Laplacian on G by Δ and the covariant differentiation on G by ∇. We put $|\nabla|^2(h) = \Delta(h^2) - 2\Delta h \cdot h$ for $h \in C^\infty(G)$. Then the Riesz transform $\mathcal{R}^G$ on G is given as follows:

Definition 7.3. *For* $f \in L^1_{loc}(G//K)$

$$(\mathcal{R}^G f)(g) = (|\nabla|(-\Delta)^{-1/2})(f)(g) \quad (g \in G).$$

$\mathcal{R}^G f$ is a K-biinvariant function on G. Indeed, $(-\Delta)^{-1/2} = T^G_{1/\sqrt{p}}$ where $p(\lambda) = \lambda^2 + \rho^2$ and we notice that for $h \in C^\infty(G)$, $|\nabla h|^2(g) = \sum_{m=1}^n |X_i h|^2(g)$ $(g \in G)$, here $\{X_i; 1 \leq i \leq n\}$ is denoted as an orthonormal basis of the Lie algebra $\mathfrak{g}$ of G and each X_i is regarded as a left (or right) invariant differential operator on G, and furthermore, if h is K-biinvariant on G, $|\nabla h|^2(g)$ is simply expressed as $c|(d/dx)h(a_x)|^2$ provided $\sigma(g) = x$ (cf. [**4**, §2]).

Theorem 7.4. *(1) $\mathcal{R}^G$ is in (L^p, L^p) for $p > 1$ and of weak type (1,1),*
(2) $\mathcal{R}^G \in (H^1_{1/2}, L^1)$.

Proof. (1) is well-known (see [**1**, Corollary 5.2]). As for (2) we may assume that $f \in H^1_{1/2}(G//K) \cap \mathcal{L}(G//K)$ and then, we have

$$((-\Delta)^{-1/2} f)(x) = e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} (T_{1/\sqrt{p_{\rho,0}}} \varGamma^1_m \mathcal{C}^1_+ F^1_f)(x) \quad (x \in \mathbf{R}_+).$$

As remarked after Definition 7.3, in order to obtain the estimate of $\mathcal{R}^G f$ it suffices to calculate the action of d/dx on the right side of the above equation. Actually, d/dx acts on $e^{-2\rho x}$, e^{-2mx}, and $(T_{1/\sqrt{p_{\rho,0}}} \varGamma^1_m \mathcal{C}^1_+ F^1_f)(x)$, so we denote the results by $(I_1 f)(x)$, $(I_2 f)(x)$, and $(I_3 f)(x)$ respectively. Since $P(\lambda + i\rho)^{-1/2}(|\lambda|/1 + |\lambda|)^{1/2}$ satisfies the Hörmander condition, it follows from Theorem 7.2 that $\|I_1 f \cdot D\|_{L^1(\mathbf{R}_+)} \leq c\|f\|_{H^1_{1/2}(G)}$. Similarly, since $(d/dx)\ (T_{1/\sqrt{p_{\rho,0}}} \varGamma^1_m \mathcal{C}^1_+ F^1_f)(x) = (T_{1/\sqrt{p_{\rho,0}} \cdot (i\lambda)} \varGamma^1_m \mathcal{C}^1_+ F^1_f)(x)$ and $p(\lambda + i\rho)^{-1/2}(i\lambda)$ satisfies the Hörmander condition, we have $\|I_3 f \cdot D\|_{L^1(\mathbf{R}_+)} \leq c\|f\|_{H^1_0(G)}$. In obtaining the estimate for $I_2 f$, we rewrite it as follows:

$$(I_2 f)(x) = e^{-2\rho x} \sum_{m=1}^{\infty} (-2m) e^{-2mx} (\gamma^1_m \mathcal{Q}_{1/2} \mathcal{C}^1_+ F^1_f)(x) \quad (x \in \mathbf{R}_+),$$

where γ^1_m is the Fourier multiplier on $\mathbf{R}$ defined by

$$(\gamma^1_m F)^{\sim}(\lambda) = \Gamma_m(\lambda + i\rho) p(\lambda + i\rho)^{-1/2} \left(\frac{|\lambda|}{1 + |\lambda|}\right)^{1/2} F^{\sim}(\lambda) \quad (\lambda \in \mathbf{R}).$$

We here observe from (25), (26) and Lemma 8.1 below that for $M = 0, 1$,

$$|(\frac{d}{d\lambda})^M \Gamma_m(\lambda + i\rho)| \leq c m^{2\alpha+1} (1 + |\lambda|)^{-M} \frac{\rho|2\rho - i\lambda|}{m|m + \rho - i\lambda|}$$

and hence,

$$|(\frac{d}{d\lambda})^M \left(\Gamma_m(\lambda + i\rho) p(\lambda + i\rho)^{-1/2} \cdot \left(\frac{|\lambda|}{1 + |\lambda|}\right)^{1/2}\right)| \leq c m^{2\alpha-1} |\lambda|^{-M}.$$

Therefore, $\Gamma_m(\lambda + i\rho) p(\lambda + i\rho)^{-1/2} \cdot (|\lambda|/1 + |\lambda|)^{1/2}$ satisfies the Hörmander condition with the constant $cm^{4\alpha-2}$. We now repeat the proof of Lemma 3.1 when T is the identity operator, in which case we replace $\varGamma^1_m$ and F by γ^1_m and $\mathcal{Q}_{1/2} \mathcal{C}^1_+ F^1_f$ respectively. Fortunately, the extra order of m that appears in the differentiation of e^{-2mx} is canceled, because the operator norm $cm^{2\alpha}$ of $\varGamma^1_m$ is changed to $cm^{2\alpha-1}$ of γ^1_m. So the exactly same proof of Lemma 3.1 is valid in this case and it follows that $\|I_2 f \cdot \Delta\|_{L^1(\mathbf{R}_+)} \leq c\|\mathcal{Q}_{1/2} \mathcal{C}^1_+ F^1_f\|_{H^1(\mathbf{R})} = c\|f\|_{H^1_{1/2}(G)}$. This completes the proof of (2). □

Remark 7.5. Theorem 6.11(2) also follows from Theorem 7.4(2) and [**1**, Corollary 6.3].

Remark 7.6. Let $\mathcal{R} = T_{i\lambda/|\lambda|}$ be the Riesz transform on $\mathbf{R}$ and $\mathcal{Q}$ the Fourier multiplier on $\mathbf{R}$ defined by

$$(\mathcal{Q}F)^{\sim}(\lambda) = \left(\frac{\lambda^2+4\rho^2}{\lambda^2}\right)^{1/4} \cdot e^{i2^{-1}\tan^{-1}(2\rho/\lambda)} \cdot \frac{\lambda}{|\lambda|} \cdot F^{\sim}(\lambda) \quad (\lambda \in \mathbf{R}).$$

We observe that $e^{-i2^{-1}\tan^{-1}(2\rho/\lambda)} \cdot \lambda/|\lambda| \cdot (\lambda^2+4\rho^2)^{-1/4}\ (1+|\lambda|)^{1/2}$ satisfies the Hörmander condition and hence, if $\mathcal{Q}F \in H^1(\mathbf{R})$, then $Q_{1/2}F \in H^1(\mathbf{R})$. Especially, Definition 4.3 and Theorem 7.4(2) yield that if $f \in H^1_{1/2}(G//K)$, then

$$\|\mathcal{R}^G f\|_{L^1(G)} + \|\mathcal{Q}\mathcal{C}^1_+ F^1_f\|_{L^1(\mathbf{R})} \le c\|\mathcal{Q}\mathcal{C}^1_+ F^1_f\|_{H^1(\mathbf{R})}. \tag{23}$$

We now suppose that $f \in L^1_{loc}(G//K)$ satisfies $\|\mathcal{R}^G f\|_{L^1(G)} + \|\mathcal{Q}F^1_f\|_{L^1(\mathbf{R})} < \infty$. If we take X_1 in $\mathfrak{a}$ in the remark after Definition 7.3, we see that $|\nabla h|(a_x g) \ge |(d/dx)h(a_x g)|$ $(g \in G)$ for $h \in C^\infty(G)$, and thereby,

$$\begin{aligned}
\|\mathcal{R}^G f\|_{L^1(G)} &= \int_A \int_N |(\mathcal{R}^G f)(a_x n)| e^{2\rho x}\, dn\, dx \\
&\ge \int_A \int_N |\frac{d}{dx}((-\Delta)^{-1/2} f)(a_x n)| e^{2\rho x}\, dn\, dx \\
&\ge \int_A |\frac{d}{dx} \int_N ((-\Delta)^{-1/2} f)(a_x n)\, dn| e^{2\rho x}\, dx \\
&= \|\frac{d}{dx} F^{-1}_{(-\Delta)^{-1/2} f} \cdot e^{2\rho x}\|_{L^1(\mathbf{R})} = \|\mathcal{R}\mathcal{Q}F^1_f\|_{L^1(\mathbf{R})},
\end{aligned}$$

because $(\lambda + 2i\rho)(\lambda^2 + 2i\rho\lambda)^{-1/2} = e^{i2^{-1}\tan^{-1}(2\rho/\lambda)} \cdot (\lambda^2 + 4\rho^2/\lambda^2)^{1/4}$. Therefore, the assumption on f yields that $\mathcal{Q}F^1_f \in H^1(\mathbf{R})$ and

$$\|\mathcal{Q}F^1_f\|_{H^1(\mathbf{R})} \le c(\|\mathcal{R}^G f\|_{L^1(G)} + \|\mathcal{Q}F^1_f\|_{L^1(\mathbf{R})}). \tag{24}$$

If we let $\rho \to 0$ formally, then F^1_f goes to an even function f on $\mathbf{R}$, $\mathcal{R}^G$ to $\mathcal{R}$ on $\mathbf{R}$, and $\mathcal{C}^1_+$, $\mathcal{Q}$ to the identity operators on $\mathbf{R}$, so from (23) and (24) we can recover the classical result: $\|f\|_{H^1(\mathbf{R})} \sim \|\mathcal{R}f\|_{L^1(\mathbf{R})} + \|f\|_{L^1(\mathbf{R})}$ (cf. [**10**], p. 123).

We last restrict our attention to functions $f \in \mathcal{L}(G//K)$ such that $\mathcal{Q}\mathcal{C}^1_+ F^1_f \in L^1(\mathbf{R})$ (see §2 and Remark 7.6), and we define

$$\mathcal{L}^1(G//K) = \{f \in \mathcal{L}(G//K); \|f\|_{\mathcal{L}^1(G)} = \|f\|_{\mathcal{L}(G)} + \|\mathcal{Q}\mathcal{C}^1_+ F^1_f\|_{L^1(\mathbf{R})} < \infty\}$$

and

$$\mathcal{H}^1(G//K) = \{f \in \mathcal{L}(G//K); \|f\|_{\mathcal{H}^1(G)} = \|f\|_{\mathcal{L}(G)} + \|\mathcal{Q}\mathcal{C}^1_+ F^1_f\|_{H^1(\mathbf{R})} < \infty\}.$$

Obviously, $\mathcal{H}^1(G//K) \subset \mathcal{L}^1(G//K)$ and $\|\mathcal{R}^G f\|_{L^1(G)} + \|f\|_{\mathcal{L}^1(G)} \leq c\|f\|_{\mathcal{H}^1(G)}$ (see (23)). We now suppose that $f \in \mathcal{L}^1(G//K)$ and we recall the proof of Theorem 7.4(2). Especially, $|(\mathcal{R}^G f)(x)| = |\sum_{k=1}^3 (I_k f)(x)|$ $(x \in \mathbf{R}_+)$ where

$$\begin{aligned}(I_1 f)(x) + (I_3 f)(x) &= e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} (T_{\frac{i(\lambda+2i\rho)}{(\lambda^2+2i\rho\lambda)^{1/2}}} \Gamma_m^1 \mathcal{C}_+^1 F_f^1)(x) \quad (x \in \mathbf{R}_+) \\ &= e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} (\Gamma_m^1 \mathcal{R}\mathcal{Q}\mathcal{C}_+^1 F_f^1)(x), \\ I_2 f(x) &= e^{-2\rho x} \sum_{m=1}^{\infty} (-2m) e^{-2mx} (\gamma_m^1 \mathcal{Q}_{1/2} \mathcal{C}_+^1 F_f^1)(x).\end{aligned}$$

Therefore, we can deduce that for $x \in \mathbf{R}_+$

$$\begin{aligned}|(\mathcal{R}\mathcal{Q}\mathcal{C}_+^1 F_f^1)(x)| \leq e^{2\rho x} |(\mathcal{R}^G f)(x)| &+ \sum_{m=1}^{\infty} e^{-2mx} |(\Gamma_m^1 \mathcal{R}\mathcal{Q}\mathcal{C}_+^1 F_f^1)(x)| \\ &+ \sum_{m=1}^{\infty} 2m e^{-2mx} |(\gamma_m^1 \mathcal{Q}_{1/2} \mathcal{C}_+^1 F_f^1)(x)|.\end{aligned}$$

Since $\|(\mathcal{Q}\ \mathcal{C}_+^1 F_f^1)^{\sim}\|_\infty \leq \|\mathcal{Q}\ \mathcal{C}_+^1 F_f^1\|_{L^1(\mathbf{R})} \leq \|f\|_{\mathcal{L}^1(G)}$ and $\|(\mathcal{C}_+^1 F_f^1)^{\sim}\|_{L^1(\mathbf{R})} \leq \|f\|_{\mathcal{L}(G)} \leq \|f\|_{\mathcal{L}^1(G)}$, it follows that $|(\Gamma_m^1\ \mathcal{R}\ \mathcal{Q}\ \mathcal{C}_+^1 F_f^1)(x)| \leq c m^{2\alpha}\ \|f\|_{\mathcal{L}^1(G)}$ and similarly, $|(\gamma_m^1\ \mathcal{Q}_{1/2}\ \mathcal{C}_+^1 F_f^1)(x)| = |(\gamma_m^1\ \mathcal{Q}_{1/2}\ \mathcal{Q}^{-1})\ (\mathcal{Q}\ \mathcal{C}_+^1 F_f^1)(x)| \leq c m^{2\alpha-1}\ \|f\|_{\mathcal{L}^1(G)}$ (see §2 and the proof of Theorem 7.4(2)). Hence we have,

$$\begin{aligned}\int_1^\infty |(\mathcal{R}\mathcal{Q}\mathcal{C}_+^1 F_f^1)(x)| dx &\leq c \int_1^\infty |(\mathcal{R}^G f)(x)| D(x) dx + c\|f\|_{\mathcal{L}^1(G)} \sum_{m=1}^{\infty} m^{2\alpha} \int_1^\infty e^{-2mx} dx \\ &\leq c(\|\mathcal{R}^G f\|_{L^1(G)} + \|f\|_{\mathcal{L}^1(G)}).\end{aligned}$$

On the other hand, since

$$\begin{aligned}|(\mathcal{R}\mathcal{Q}\mathcal{C}_+^1 F_f^1)(x)| &= |\int_{\mathbf{R}} \frac{i(\lambda+2i\rho)}{(\lambda^2+2i\rho\lambda)^{1/2}} \hat{f}(\lambda+i\rho) C(-\lambda-i\rho)^{-1} e^{i\lambda x} d\lambda| \quad (x \in \mathbf{R}) \\ &= e^{\rho x} |\int_{\mathbf{R}} \frac{i(\lambda+i\rho)}{(\lambda^2+\rho^2)^{1/2}} \hat{f}(\lambda) C(-\lambda)^{-1} e^{i\lambda x} d\lambda| \\ &\leq c\|f\|_{\mathcal{L}(G)} e^{\rho x},\end{aligned}$$

it follows that $\int_{-\infty}^1 |(\mathcal{R}\mathcal{Q}\mathcal{C}_+^1 F_f^1)(x)| dx \leq c\|f\|_{\mathcal{L}^1(G)}$. Then, we can deduce that $\|\mathcal{R}\ \mathcal{Q}\ \mathcal{C}_+^1 F_f^1\|_{L^1(\mathbf{R})} \leq c(\|\mathcal{R}^G f\|_{L^1(G)} + \|f\|_{\mathcal{L}^1(G)})$. In particular, $\|f\|_{\mathcal{H}^1(G)} = \|f\|_{\mathcal{L}(G)} + \|\mathcal{Q}\ \mathcal{C}_+^1 F_f^1\|_{H^1(\mathbf{R})} \leq c(\|f\|_{\mathcal{L}(G)} + \|\mathcal{Q}\ \mathcal{C}_+^1 F_f^1\|_{L^1(\mathbf{R})} + \|\mathcal{R}\ \mathcal{Q}\ \mathcal{C}_+^1 F_f^1\|_{L^1(\mathbf{R})}) \leq c(\|\mathcal{R}^G f\|_{L^1(G)} + \|f\|_{\mathcal{L}^1(G)})$. We have therefore proved the following,

Theorem 7.7. *$f \in \mathcal{H}^1(G//K)$ if and only if $f \in \mathcal{L}^1(G//K)$ and $\mathcal{R}^G \in L^1(G//K)$. Especially,*

$$\|f\|_{\mathcal{H}^1(G)} \sim \|\mathcal{R}^G f\|_{L^1(G)} + \|f\|_{\mathcal{L}^1(G)}.$$

8. Appendix. We shall obtain an estimate for the derivatives of $\Gamma_m(\lambda)$ (see (6)), which yields the Hörmander condition of Γ_m. In what follows we denote Γ_m by Γ_{2m} and refer to the notation and the proof of Lemma 7 in Flensted-Jensen [**4**]. Actually, Γ_m ($m \in \mathbf{N}$) is recurrently defined by $\Gamma_0 = 1$, $\Gamma_{2n+1} = 0$ ($n \in \mathbf{N}$) and

$$4m(m - i\lambda)\Gamma_{2m}(\lambda) = \sum_{k=0}^{m-1}(2k - i\lambda + \rho)4\left((\alpha - \beta) + \delta_k^m(2\beta + 1)\right)\Gamma_{2k}(\lambda),$$

where $\delta_k^m = 0$ for $k \equiv m + 1 (\text{mod } 2)$, $\delta_k^m = 1$ for $k \equiv n (\text{mod } 2)$. For each $m \in \mathbf{N}$ we put

$$C_k(\lambda) = 4k(k - i\lambda) \quad \text{and} \quad R_k(\lambda) = 4\theta(2k - i\lambda + \rho),$$

where $\theta = \alpha - \beta$ if $k \equiv m + 1$ and $\theta = \rho$ if $k \equiv m$. Then it follows that

$$\begin{aligned}\Gamma_{2m}(\lambda) &= C_m(\lambda)^{-1} \sum_{k=0}^{m-1} R_k(\lambda)\Gamma_{2k}(\lambda) \\ &= \frac{R_0(\lambda)}{C_m(\lambda)} \prod_{k=1}^{m-1}\left(1 + \frac{R_k(\lambda)}{C_k(\lambda)}\right).\end{aligned}$$

Therefore, if we put $c_k(\lambda) = |C_k(\lambda)|$, $r_k(\lambda) = |R_k(\lambda)|$ and

$$b_0(\lambda) = 1 \quad \text{and} \quad b_m(\lambda) = c_m(\lambda)^{-1} \prod_{k=0}^{m-1} b_k(\lambda)c_k(\lambda) \quad \text{for } m \geq 1,$$

we easily see that

$$|\Gamma_{2m}(\lambda)| \leq b_m(\lambda) = \frac{\rho|\rho - i\lambda|}{m|m - i\lambda|} \prod_{k=1}^{m-1}\left(1 + \frac{r_k(\lambda)}{c_k(\lambda)}\right). \tag{25}$$

Lemma 8.1. *Let $\lambda = \xi + i\eta$ and suppose that $\eta \geq \rho$. Then there exists a positive constant c such that for all $m \in \mathbf{N}$*

$$\prod_{k=1}^{m-1}\left(1 + \frac{r_k(\lambda)}{c_k(\lambda)}\right) \leq cm^{2\alpha+1}.$$

Proof. Since $\eta \geq \rho$, we have $|2k + \rho + \eta - i\xi|/|k + \eta - i\xi| \leq 2$ and thereby,

$$\left(k\frac{r_k(\lambda)}{c_k(\lambda)}\right)^2 \leq (2\theta)^2.$$

Then it follows that

$$\prod_{k=1}^{m-1}\left(1+\frac{r_k(\lambda)}{c_k(\lambda)}\right) \le \exp\left(\sum_{k=1}^{m-1}\frac{r_k(\lambda)}{c_k(\lambda)}\right)$$
$$\le \exp\left(\rho \sum_{\substack{k\equiv m\\ 1\le k\le m-1}} 2k^{-1} + (\alpha-\beta)\sum_{\substack{k\equiv m+1\\ 1\le k\le m-1}} 2k^{-1}\right)$$
$$\le c\exp\left((\rho+\alpha-\beta)\log m\right)$$
$$\le cm^{2\alpha+1} \qquad \square$$

Lemma 8.2. *Let* $\Im(\lambda) \ge 0$ *and fix* $M \in \mathbf{N}$. *Then there exists a positive constant* c *such that for all* $m \in \mathbf{N}$

$$(1) \quad |(\frac{d}{d\lambda})^M \frac{R_0(\lambda)}{C_m(\lambda)}| \le c\frac{r_0(\lambda)}{c_m(\lambda)}\frac{1}{(1+|\lambda|)^M},$$
$$(2) \quad |(\frac{d}{d\lambda})^M \left(1+\frac{R_k(\lambda)}{C_k(\lambda)}\right)| \le c\left(1+\frac{r_k(\lambda)}{c_k(\lambda)}\right)\frac{1}{(k+|\lambda|)^{M+1}} \quad (M\ge 1).$$

Proof. (1)

$$|(\frac{d}{d\lambda})^M \frac{R_0(\lambda)}{C_m(\lambda)}| = c|\frac{m-\rho}{C_m(\lambda)(m-i\lambda)^M}|$$
$$\le c\frac{r_0(\lambda)}{c_m(\lambda)}|\frac{m-\rho}{(\rho-i\lambda)(m-i\lambda)^M}|$$
$$\le c\frac{r_0(\lambda)}{c_m(\lambda)}\frac{1}{(1+|\lambda|)^M}.$$

(2) For $M \ge 1$

$$|(\frac{d}{d\lambda})^M\left(1+\frac{R_k(\lambda)}{C_k(\lambda)}\right)| = c|\frac{k-\rho}{C_k(\lambda)(k-i\lambda)^M}|$$
$$\le c\left(1+\frac{r_k(\lambda)}{c_k(\lambda)}\right)|\frac{k-\rho}{((k^2+\theta(2k+\rho))-(k+\theta)i\lambda)(k-i\lambda)^M}|$$
$$\le c\left(1+\frac{r_k(\lambda)}{c_k(\lambda)}\right)\frac{1}{(k+|\lambda|)^{M+1}}. \qquad \square$$

We now suppose that $\eta = \Im(\lambda) \ge 0$ and observe by Lemma 8.2 that

$$|(\frac{d}{d\lambda})\Gamma_{2m}(\lambda)| = |(\frac{d}{d\lambda})\frac{R_0(\lambda)}{C_m(\lambda)}\prod_{k=1}^{m-1}\left(1+\frac{R_k(\lambda)}{C_k(\lambda)}\right) + \frac{R_0(\lambda)}{C_m(\lambda)}(\frac{d}{d\lambda})\prod_{k=1}^{m-1}\left(1+\frac{R_k(\lambda)}{C_k(\lambda)}\right)|$$
$$\le cb_m(\lambda)\left((1+|\lambda|)^{-1}+\sum_{j=0}^{m-2}(1+j+|\lambda|)^{-2}\right)$$
$$\le cb_m(\lambda)(1+|\lambda|)^{-1}. \tag{26}$$

Hence, by Lemma 8.1 and the above estimate we can deduce the following,

Proposition 8.3. *Let $M = 0, 1$. If $\Im(\lambda) \geq \rho$, then there exists a positive constant c such that for all $m \in \mathbf{N}$*

$$|(\frac{d}{d\lambda})^M \Gamma_{2m}(\lambda)| \leq cm^{2\alpha}(1 + |\lambda|)^{-M}.$$

Corollary 8.4. *Suppose that $\eta \geq \rho$. Then there exists a positive constant c such that for all $m \in \mathbf{N}$*

$$R^{2M-1} \int_{R<|\xi|\leq 2R} |(\frac{d}{d\lambda})^M \Gamma_{2m}(\xi + i\eta)|^2 d\xi \leq cm^{4\alpha}$$

for $M = 0, 1$ and $R > 0$.

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