L^1 ESTIMATES FOR MAXIMAL FUNCTIONS AND RIESZ
TRANSFORM ON REAL RANK 1 SEMISIMPLE LIE GROUPS

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Abstract. Let G be a real rank one semisimple Lie group and K a maximal compact
subgroup of G. Radial maximal operators for suitable dilations, the heat and Poisson
maximal operators, and the Riesz transform, which act on K-biinvariant functions on G,
satisfy the L^p-norm inequalities for p > 1 and a weak type L^1 estimate. In this paper,
through the Fourier theories on R and G we shall duplicate the Hardy space H^1(R) to a
subspace H^1_s(G) (s ≥ 0) of L^1(G) and show that these operators are bounded from H^1_s(G)
to L^1(G).

1. Introduction. Let G be a real rank one connected semisimple Lie group with finite
center, G = KAN an Iwasawa decomposition of G, and dg = dkda dn a corresponding
decomposition of a Haar measure dg on G. Let L^1_{loc}(G//K) denote the space of locally
integrable, K-biinvariant functions on G and L^p(G//K) (0 < p ≤ ∞) the subspace of
L^1_{loc}(G//K) consisting of functions with finite L^p-norm on G. In the following, if we
say that T is an operator on G, it means that T is an operator acting on these spaces.
The first problem we shall treat in this paper is concerned with the Abel transform on
G. For f ∈ L^1(G//K) the Abel transform F_f of f is defined by

F_f(x) = e^{px} \int_N f(a x n)dn \ (x \in R),

where A is parametrized as \{ a_x; x \in R \} (for the definition of p see (3) below). We let
F_f^1(x) = e^{px} F_f(x). Then the integral formula for the Iwasawa decomposition of G (cf.
[6, p.373]) yields that f ∈ L^1(G) if and only if F_f^1 ∈ L^1(R), and thus,

Proposition 1.1. If f ∈ L^1(G), then F_f^1 ∈ L^1(R).

The reverse of this proposition is not true: even if F_f^1 is well-defined for f ∈ L^1_{loc}(G//K)
and it belongs to L^1(R), f does not always belong to L^1(G//K). The first problem we offer
is the following.

Problem A. For f ∈ L^1_{loc}(G//K) find a condition on F_f^1 under which f belongs to
L^1(G//K).

We next consider a problem concerning with a maximal operator on G. Let σ : G → R_+ denote
the K-biinvariant function on G induced by the distance function on
X = G/K (cf. [13, p.320]). For each r ∈ R_+ we let B(r) = { g ∈ G; σ(g) ≤ r } and χ_B(r)

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the characteristic function of $B(r)$. The Hardy-Littlewood maximal operator $M_{HL}^G$ on $G$ is defined as follows: for $f \in L^1_{loc}(G//K)$

$$(M_{HL}^G f)(g) = \sup_{0 < r < \infty} |B(r)|^{-1} (|f| * \chi_{B(r)})(g) \quad (g \in G).$$

This definition makes sense for $f \in L^p(X) \ (p \geq 1)$. Clerc and Stein [3] and Strömberg [11] have shown the following,

**Theorem 1.2.** $M_{HL}^G$ satisfies the $L^p$-norm inequalities for $p > 1$, and a weak type $L^1$ estimate, that is, for all $\alpha \in \mathbb{R}_+$ and all $f \in L^1(X)$

$$|\{g \in G; (M_{HL}^G f)(g) > \alpha\}| \leq \frac{c}{\alpha} \|f\|_{L^1(G)},$$

where $c$ is independent of $\alpha$ and $f$.

We here define a radial maximal operator on $G$ as an analogue of the one on $\mathbb{R}$. We fix $\phi \in C_c^\infty(G//K)$, the space of $C^\infty$ compactly supported $K$-biinvariant functions on $G$, and suppose that $\phi$ is sufficiently zero at the origin of $G$. The radial maximal operator $M_{\phi, \epsilon}^{G, \sharp} \ (\epsilon \geq 0)$ on $G$ is defined as follows: for $f \in L^1_{loc}(G//K)$

$$(M_{\phi, \epsilon}^{G, \sharp} f)(g) = \sup_{0 < r < \infty} (1 + r)^{-\epsilon} \|(f * \phi_r^\sharp)(g)\| \quad (g \in G),$$

where the dilation $\phi_r^\sharp \ (r \in \mathbb{R}_+)$ of $\phi$ is given by

$$\phi_r^\sharp(g) = |B(r)|^{-1} \phi(a_r \sigma(g)/r) \quad (g \in G).$$

This dilation is different from the one used in [7] and see §6 for other dilations. Since $M_{\phi, \epsilon}^{G, \sharp} f \leq c M_{HL}^G f$ pointwisely, $M_{\phi, \epsilon}^{G, \sharp}$ also satisfies the property stated in Theorem 1.2. As Folland and Stein have shown in their book [5], when a Lie group $G$ is of homogeneous type, a radial maximal operator $M_\phi^G$ on $G$ is bounded from the atomic Hardy space $H^{1}_{\infty, 0}(G)$ to $L^1(G)$. More precisely, integrability of the maximal function $M_\phi^G f = \sup_\phi M_\phi^N f$ where the supremum is taken over $\phi$ in a suitable class, is equivalent to that $f$ belongs to $H^{1}_{\infty, 0}(G)$ (see [5, Chapter 3]). At present, we have no definition of the (atomic) Hardy space on $G$ on which $M_{\phi, \epsilon}^{G, \sharp}$ is bounded. The second problem, which was also treated in [7], is the following,

**Problem B.** Find a subspace of $L^1(G//K)$ on which $M_{\phi, \epsilon}^{G, \sharp}$ is bounded to $L^1(G//K)$.

These two problems look unrelated each other. However, the answers obtained in this paper indicate that they are deeply related. Let $C(\lambda) \ (\lambda \in \mathbb{R})$ be the Harish-Chandra’s $C$-function. Since $C(\lambda)$ has a meromorphic extension on $\mathbb{C}$, the following definition of the Fourier multiplier $C_1^+ \ | \ \mathbb{R}$ makes sense:

$$(C_1^+ F)^\sim(\lambda) = C(-\lambda - i\rho)^{-1} F^\sim(\lambda) \quad (\lambda \in \mathbb{R}),$$

where $F^\sim$ denotes the Euclidean Fourier transform of $F$. Let $H^1(\mathbb{R})$ be the $H^1$-Hardy space on $\mathbb{R}$ (cf. [10, Chap.3]). Then our answer of Problem A can be stated as
**Answer A.** If $C^1_F \in H^1(\mathbb{R})$, then $f \in L^1(G//K)$.

Let $Q_\lambda$ ($s \geq 0$) denote the Fourier multiplier on $\mathbb{R}$ defined by

$$(Q_\lambda f)^\circ(\lambda) = \left(\frac{1 + |\lambda|}{|\lambda|}\right)^s f^\circ(\lambda) \quad (\lambda \in \mathbb{R}).$$

We set

$$H^1_s(G//K) = \{ f \in L^1_{loc}(G//K); Q_\lambda C^1_F \in H^1(\mathbb{R}) \}$$

and $\| f \|_{H^1_s(G//K)} = \| Q_\lambda C^1_F \|_{H^1(\mathbb{R})}$. Answer A means that $H^1_0(G//K)$ is a subspace of $L^1(G//K)$ and moreover, the fact that $(|\lambda|/1 + |\lambda|)^s$ satisfies the Hörmander condition on Fourier multiplier yields that $H^1_s(G//K)$ ($s > 0$) is a subspace of $H^1_0(G//K)$ (see §4). Our answer of Problem B is given as follows.

**Answer B.** For $\delta > 0$, $M^{G, \delta}_{\phi, 0}$ is a bounded operator of $H^1_{2+\delta}(G//K)$ to $L^1(G//K)$ and $M^{G, \delta}_{\phi, 1+\delta}$ is one of $H^1_0(G//K)$ to $L^1(G//K)$.

In order to understand a true character of $C^1_F$ we need the Fourier analysis on $G$, so we refer to the Warner’s book [14]. Let $\phi_\lambda(g)$ ($g \in G$, $\lambda \in \mathbb{R}$) be the zonal spherical function on $G$ where the dual space of the Lie algebra of $A$ is identified with $\mathbb{R}$. Then, for $f \in L^1(G//K)$ the Fourier transform $\hat{f}(\lambda)$ of $f$ is defined by $\hat{f}(\lambda) = \int_G f(g)\phi_\lambda(g)dg$. The inversion formula is of the following form: for $f \in C^\infty_c(G//K)$

$$(1) \quad \hat{f}(g) = \int_\mathbb{R} \hat{f}(\lambda)\phi_\lambda(g)|C(\lambda)|^{-2}d\lambda,$$

in which case, $\hat{f}(\lambda)$ is a holomorphic function of exponential type. Hence, by regarding each $K$-biinvariant function on $G$ as an even function on $\mathbb{R}$, by substituting the expansion of $\phi_\lambda$ (see (5) and (6) below) into (1), and then, by shifting the integral line $\mathbb{R}$ to $\mathbb{R} + i\rho$, we can rewrite (1) as follows ([14, p.356]):

$$(2) \quad f(x) = e^{-\rho x} \int_\mathbb{R} \hat{f}(\lambda)\Phi(\lambda, x)C(-\lambda)^{-1}e^{i\lambda x}d\lambda \quad (x \in \mathbb{R}_+)$$

$$= e^{-2\rho x} \sum_{m=0}^\infty e^{-2m x} \int_\mathbb{R} C(-\lambda - i\rho)^{-1}\Gamma_m(\lambda + i\rho)\hat{f}(\lambda + i\rho)e^{i\lambda x}d\lambda.$$

Since $(F_f)^\circ(\lambda) = \hat{f}(\lambda)$ and $\Gamma_0 \equiv 1$ ([14, Proposition 9.2.2.3 and 9.1.5]), the leading term corresponding to $m = 0$ in the right side of (2) is nothing but $e^{-2\rho x}(C^1_F)^\circ(x)$. Roughly speaking, Answers A and B can be restated as follows: if the leading term of $e^{2\rho x}f(x)(x \in \mathbb{R}_+)$ belongs to $H^1(\mathbb{R})$, then $f$ and $M^{G, \delta}_{\phi, 1+\delta} f$ belong to $L^1(G//K)$.

We state the organization of this paper. In §3 we shall obtain a key lemma for the integrability of $f$ satisfying $C^1_F \in H^1(\mathbb{R})$. Then the proof of Answer A and some basic properties of $H^1_s(G//K)$ are given in §4. In §5 we shall obtain some criteria by which we can judge whether a radial maximal operator is bounded from $H^1_s(G//K)$ to $L^1(G//K)$. Actually, we apply a criterion to $M^{G, \delta}_{\phi, \epsilon}$ and obtain the proof of Answer B in
§6. Moreover, we consider the same problem for the heat and Poisson maximal operators $M_{H,\epsilon}^G$ and $M_{P,\epsilon}^G$ ($\epsilon \geq 0$) on $G$, which are defined as follows: for $f \in L^1_{\text{loc}}(G//K)$

$$(M_{H,\epsilon}^G f)(g) = \sup_{0 < r < \infty} (1 + r)^{-\epsilon} |(f * h_r)(g)| \quad (g \in G),$$

$$(M_{P,\epsilon}^G f)(g) = \sup_{0 < r < \infty} (1 + r)^{-\epsilon} |(f * p_r)(g)| \quad (g \in G),$$

where $h_r(\lambda) = e^{-(\lambda^2 + r^2)r}$ and $p_r(\lambda) = e^{-(\lambda^2 + r^2)^{1/2}r}$. As shown by Stein [9] in great generality these operators also satisfy the property stated in Theorem 1.2. In §6 we shall show that $M_{H,0}^G$ (resp. $M_{P,0}^G$) is a bounded sublinear operator of $H^{1+\delta}(G//K)$ ($\delta > 0$) (resp. $H^{1}_{1/2}(G//K)$) to $L^1(G//K)$ and moreover, $M_{H,1}^G$ and $M_{P,2}^G$ are ones of $H^{1}(G//K)$ to $L^1(G//K)$. In §7 we shall treat the Riesz transform $\mathcal{R}^G$ on $G$ which are defined as follows: for $f \in L^1_{\text{loc}}(G//K)$

$$(\mathcal{R}^G f)(g) = (|\Delta|^{-1/2})(f)(g) \quad (g \in G),$$

where $\Delta$ is the Laplacian on $G$ and $|\Delta|^2(h) = \Delta(h^2) - 2\Delta h \cdot h$ for $h \in C^\infty(G)$. As studied by Anker [1] and Lohoué [8] $\mathcal{R}^G f$ is deeply related with $M_{P,\epsilon}^G f$, especially, the $L^1$-norm of $M_{P,\epsilon}^G f$ is controlled by the one of $\mathcal{R}^G f$. Therefore, we can expect that the (Hardy) space $H^1(G//K)$ might be useful to obtain an $L^1$ estimate for $\mathcal{R}^G$. Indeed, we shall prove that $\mathcal{R}^G$ is bounded from $H^{1}_{1/2}(G//K)$ to $L^1(G//K)$.

2. Notations. Let $G$ be a real rank one connected semisimple Lie group with finite center and $G = KAN$ an Iwasawa decomposition of $G$. Let $\mathfrak{a}$ be the Lie algebra of $A$ and $\mathcal{F} = \mathfrak{a}^*$ the dual space of $\mathfrak{a}$. Let $H$ be the unique element in $\mathfrak{a}$ satisfying $\gamma(H) = 1$ where $\gamma$ is the positive simple root of $(G, A)$ determined by $N$. We parametrize each element in $A$, $\mathfrak{a}$, and $\mathcal{F}$ as $a_s = \exp(xH), xH, \text{and } x\gamma \ (x \in \mathbb{R})$ respectively. In what follows we often identify these spaces with $\mathbb{R}$ and also $\mathcal{F}_c$, the complexification of $\mathcal{F}$, with $\mathbb{C}$ without making mention of the identification. We put $\mathcal{F}(s) = \{ \lambda \in \mathcal{F}_c; |\Im(\lambda)| < s \}$ ($s \in \mathbb{R}_+$) and $A_+ = \{ a_s; x \in \mathbb{R}_+ \}$. Then, according to the Cartan decomposition $G = KCL(A_+)K$ of $G$ and the action of the Weyl group of $(G, A)$ on $A$, every $K$-biinvariant functions $f$ on $G$ are determined by their restriction to $CL(A_+)$ and hence, they are identified with even functions on $\mathbb{R}$. We denote them by the same letter, that is, if $g \in Ka_+(K)$ and $x(g) \in \mathbb{R}_+$,

$$f(g) = f(a_{x(g)}) = f(x(g)) = f(-x(g)).$$

Let $dg$ (resp. $dk$ and $dn$) denote the Haar measure on $G$ (resp. $K$ and $N$), normalized as $\int_K dk = 1$ and the following integral formula holds for all integrable, $K$-biinvariant functions $f$ on $G$:

$$\int_G f(g)dg = \int_0^\infty f(x)D(x)dx,$$

where $D(x) = (\sin x)^{m_1}(\sin 2x)^{m_2} \ (x \in \mathbb{R}_+), m_1$ and $m_2$ are the multiplicities of $\gamma$ and $2\gamma$ respectively. We put

$$a = \frac{m_1 + m_2 - 1}{2} \quad \text{and} \quad \rho = \frac{m_1 + 2m_2}{2}.$$
Then the order of $D(x)$ is given by

$$
D(x) \sim \begin{cases} 
  x^{2\alpha+1} & (0 < x \leq 1) \\
  e^{2px} & (1 < x < \infty),
\end{cases}
$$

where the symbol " $\sim$ " means that the ratio of the left side and the right side is bounded below and above by positive constants. Let $L_{loc}^1(G//K)$ denote the space of locally integrable, $K$-biinvariant functions on $G$. Let $L^p(G//K)$ ($0 < p \leq \infty$) and $C_c^\infty(G//K)$ denote the subspaces of $L_{loc}^1(G//K)$ consisting of, respectively, functions with finite $L^p$-norm on $G$ and $C^\infty$, compactly supported functions on $G$. Henceforth, for each normed space $V$ we denote the norm of $v \in V$ as $\|v\|_V$, excepting that $G//K$ is abbreviated by $G$ and the $L^\infty$-norm is denoted by $\|v\|_\infty$.

We recall the bases of the Fourier analysis on $G$ and refer to [4] and [14]. Let $\phi_\lambda(g)$ ($\lambda \in \mathcal{F}$, $g \in G$) be the zonal spherical function of $G$. The Harish-Chandra expansion of $\phi_\lambda$ is given as

$$
\phi_\lambda(x) = e^{-px} \left( \Phi(\lambda, x)C(\lambda)e^{i\lambda x} + \Phi(-\lambda, x)C(-\lambda)e^{-i\lambda x} \right) \quad (x \in \mathbb{R}_+)
$$

and furthermore, $\Phi(\lambda, x)$ has the so-called Gangolli expansion:

$$
\Phi(\lambda, x) = \sum_{m=0}^{\infty} \Gamma_m(\lambda)e^{-2mx} \quad (\lambda \in \mathcal{F}, x \in \mathbb{R}_+)
$$

([14, 9.1.4 and 9.1.5]). Their explicit forms and some basic properties of $C(\lambda)$, $\Phi(\lambda, x)$, and $\Gamma_m(\lambda)$, which will be used in the following arguments, are summarized in [4, §2 and §3], and a sharp estimate for the derivatives of $\Gamma_m(\lambda)$ will be obtained in the appendix of this paper (see §8). For $f \in L^1(G//K)$ the Fourier transform $\hat{f}(\lambda)$ ($\lambda \in \mathcal{F}$) of $f$ is defined by

$$
\hat{f}(\lambda) = \int_G f(g)\phi_\lambda(g)dg \quad (\lambda \in \mathcal{F}).
$$

From the Riemann-Lebesgue’s lemma on $G$ ([4, Lemma 11]) it follows that $\hat{f}(\lambda)$ is an even holomorphic function on $\mathcal{F}(\rho)$ satisfying $|\hat{f}(\lambda)| \to 0$ as $|\lambda| \to \infty$ in $CL(\mathcal{F}(\rho))$ and hence,

$$
\sup_{\lambda \in CL(\mathcal{F}(\rho))} |\hat{f}(\lambda)| \leq \|f\|_{L^1(G)}.
$$

When $f$ is in $C_c^\infty(G//K)$, the Paley-Wiener theorem on $G$ ([14, 9.2.3]) implies that $\hat{f}(\lambda)$ is an even holomorphic function on $\mathcal{F}_c$ of exponential type, in which case, the Fourier inversion formula:

$$
f(g) = \int_{\mathbb{R}} \hat{f}(\lambda)\phi_\lambda(g)|C(\lambda)|^{-2}d\lambda \quad (g \in G)
$$

and the Plancherel formula:

$$
\int_0^\infty |f(x)|^2 D(x)dx = \int_0^\infty |\hat{f}(\lambda)|^2|C(\lambda)|^{-2}d\lambda
$$
hold. Thereby, the Fourier transform \( f \mapsto \hat{f} \) of \( C_c^\infty(G//K) \) is uniquely extended to an isometry between \( L^2(G//K) \) and \( L^2(\mathbf{R}, |C(\lambda)|^{-2}d\lambda) \) ([14, Theorem 9.2.2.13]).

We now introduce some operators on \( G \) and \( \mathbf{R} \). In what follows most of operators on \( G \) and \( \mathbf{R} \) are denoted by scripts: \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots \), excepting \( M \) and \( T \). Especially, \( \mathcal{AB} \ldots \mathcal{C} \) \( f \) means that \( \mathcal{A}(\mathcal{B}(...(\mathcal{C}(f)))) \) and \( \mathcal{A}^\mathcal{G} \) an operator on \( G \).

For \( s \in \mathbf{R} \) we define the Abel transform \( F_s^f \) of \( f \in L^1_{loc}(G//K) \) as

\[
F_s^f(x) = e^{s\rho x} F_f(x) = e^{(s+1)\rho x} \int_N f(a_x n)dn \quad (x \in \mathbf{R})
\]

and the Fourier multipliers \( C_s^+ \) and \( \Gamma_m^s \) (\( m \in \mathbf{N} \)) on \( \mathbf{R} \) as

\[
(C_s^+ F)(x) = \int_\mathbf{R} C(-\lambda - is\rho)^{-1} F^\sim(\lambda)e^{i\lambda x}d\lambda \quad (x \in \mathbf{R}),
\]

\[
(\Gamma_m^s F)(x) = \int_\mathbf{R} \Gamma_m(\lambda + is\rho) F^\sim(\lambda)e^{i\lambda x}d\lambda \quad (x \in \mathbf{R}),
\]

where \( F^\sim \) denotes the Euclidean Fourier transform of \( F \). Of course, these definitions make sense if the integrals of the right sides exist. Since \( \hat{f}(\lambda) = (F_f)^\sim(\lambda) \) (\( \lambda \in \mathcal{F}_c \)) for \( f \in C_c^\infty(G//K) \) ([14, Proposition 9.2.2.3]), the relation

\[
(F_s^f)^\sim(\lambda) = \hat{f}(\lambda + is\rho)
\]

holds if the both sides exist for \( \lambda \in \mathcal{F}_c \). We here suppose \( f \in C_c^\infty(G//K) \) and we observe that on \( \mathcal{F}(\rho) \), \( \hat{f}(\lambda) \) is holomorphic and rapidly decreasing, \( C(-\lambda)^{-1} \) is holomorphic and tempered, and \( \Gamma_m(\lambda) \) is holomorphic and uniformly dominated by a polynomial of \( m \) ([14, Proposition 9.1.7.2 and p.334]). Hence, by substituting the expansions in (5) and (6) into (7), by changing the order of integration and summation, and by shifting the integral line from \( \mathbf{R} \) to \( \mathbf{R} + i \rho \), we can deduce that

\[
f(x) = e^{-\rho x} \int_\mathbf{R} \hat{f}(\lambda) \Phi(\lambda, x) C(-\lambda)^{-1} e^{i\lambda x}d\lambda \quad (x \in \mathbf{R}_+)
\]

\[
= e^{-\rho x} \sum_{m=0}^\infty e^{-2mx} (\Gamma_m^0 C_+^0 F_f^0)(x)
\]

\[
= e^{-2\rho x} \sum_{m=0}^\infty e^{-2mx} (\Gamma_m^1 C_+^1 + F_f^1)(x).
\]

This manipulation is valid provided that \( \hat{f}(\lambda) \) is holomorphic on \( \mathcal{F}(\rho) \) and satisfies

\[
\sup_{0 \leq \xi \leq \rho} \int_{\mathbf{R} + i \xi} |\hat{f}(\lambda)| C(-\lambda)^{-1} d\lambda < \infty.
\]

In this case, since \( |\Gamma_m(\lambda)| \leq c m^{-2\alpha} \) (\( m \in \mathbf{N} \)) if \( \Re(\lambda) \geq \rho \) (see Proposition 8.3 below) and \( \sum_{m=0}^\infty e^{-2mx} m^{2\alpha} \sim x^{-(2\alpha+1)} \) as \( x \to 0 \) and \( \sim 1 \) as \( x \to \infty \), the right side of (10) is dominated by \( e^{-2\rho x} (1 + D(x))/D(x) \) (see (4)).

We then define

\[
\mathcal{L}(G//K) = \{ f \in L^1_{loc}(G//K); \hat{f} \text{ is holomorphic on } \mathcal{F}(\rho) \text{ and } \}
\]

\[
\|f\|_{\mathcal{L}(G)} = \sup_{0 \leq \xi \leq \rho} \int_{\mathbf{R} + i \xi} |\hat{f}(\lambda)| C(-\lambda)^{-1} d\lambda < \infty .
\]
3. A key lemma.

**Lemma 3.1.** Let $T$ be a sublinear operator of $\mathbf{R}$ satisfying the following properties: there exists $\epsilon > 0$ such that for each $(1,2,0)$-atom $a$ with $\text{supp}(a) \subseteq [x_0 - \ell, x_0 + \ell]$

\[(a) \quad \|Ta\|_{L^2(\mathbf{R})} \leq \|a\|_{L^2(\mathbf{R})},
(b) \quad \|(Ta)(x)\| \leq \ell^\epsilon |x - x_0|^{-(1+\epsilon)} \quad \text{if} \quad |x - x_0| \geq 2\ell.
\]

Then there exists a positive constant $c = c_\epsilon$ such that for any $F \in H^1(\mathbf{R})$

\[
\|e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2m_\epsilon^2 (T\Gamma_m^1 F) \cdot D} \|_{L^1(\mathbf{R}^+)} \leq c \|F\|_{H^1(\mathbf{R})}.
\]

**Proof.** We abbreviate $\| \cdot \|_{L^2(\mathbf{R})}$ as $\| \cdot \|_2$. We first observe that $\|Ta\|_{L^1(\mathbf{R})} \leq \int_{|x-x_0| \leq \ell} |(Ta)(x)| dx + \int_{|x-x_0| > \ell} |(Ta)(x)| dx \leq (2\ell)^{1/2} \|a\|_2 + \ell^\epsilon \int_{|x-x_0| > \ell} |x - x_0|^{-(1+\epsilon)} dx \leq c,$

because $a$ is a $(1,2,0)$-atom. Hence $T$ is a bounded sublinear operator of $H^1(\mathbf{R})$ to $L^1(\mathbf{R})$. Let $F \in H^1(\mathbf{R})$. Since $\Gamma_0 \equiv 1$ (cf. [14, 9.15]), the term corresponding to $m = 0$ in the left side of (11) is dominated as

\[
\|e^{-2\rho x} (TF) \cdot D\|_{L^1(\mathbf{R}^+)} \leq c \|TF\|_{L^1(\mathbf{R})} \leq c \|F\|_{H^1(\mathbf{R})}.
\]

In estimating the rest terms, by substituting a $(1,2,0)$-atomic decomposition of $F$ (cf. [5, Theorem 3.30]) it suffices to show that for any $(1,2,0)$-atom $a$ on $\mathbf{R}$,

\[
\|e^{-2\rho x} \sum_{m=1}^{\infty} e^{-2m_\epsilon^2 (T\Gamma_m^1 a) \cdot D} \|_{L^1(\mathbf{R}^+)} \leq c.
\]

We notice that $\Gamma_m(\lambda + i\rho)$ ($\lambda \in \mathcal{F}$) satisfies the Hörmander condition with the constant $cm^{4\alpha}$ (see Corollary 8.4 below) and therefore, the multiplier theorem obtained by Tableyson and Weiss [12, Theorem 4.2] asserts that $\Gamma_m^1 a$ is a $(1,2,0,1/2)$-molecule with $\|\Gamma_m^1 a\|_{H^1(\mathbf{R})} \leq cm^{2\alpha}$. Moreover, the proof of Theorem 2.9 in [12] yields that $\Gamma_m^1 a$ has a $(1,2,0)$-atomic decomposition such that

\[
\Gamma_m^1 a = \sum_{k=0}^{\infty} d_k^m b_k^m,
\]

where $|d_k^m| = c2^{-k/2}m^{2\alpha}$ and, if we denote by $x_0$ the center of the support of $a$ and we put $\sigma_m = \|\Gamma_m^1 a/m^{2\alpha}\|_2^2$, $b_k^m$ is a $(1,2,0)$-atom on $\mathbf{R}$ supported on the interval $I_k^m = \{x; |x-x_0| \leq 2^{k}\sigma_m\}$. Let $\ell_k^m = |I_k^m|$. Since $a$ and $\Gamma_m^1 a$ are $(1,2,0,1/2)$-molecules on $\mathbf{R}$, it follows from the proof of Theorem 4.2 in [12] that $\sigma_m = m^{4\alpha} \|\Gamma_m^1 a\|_2^{-2} \geq c\|a\|_2^{-2}$ and $\sigma_m = m^{-1}\|a\|_2^{-2}$, that is,

\[
\sigma_m \sim \|a\|_2^{-2} \quad \text{and} \quad \ell_k^m \sim 2^{k}\|a\|_2^{-2}.
\]
We now split the region of integration in (13) into $(0,1]$ and $(1,\infty)$. The integral over 
$(1,\infty)$ is estimated as
\[ \int_{1}^{\infty} \sum_{m=1}^{\infty} e^{-2mx} |(T\Gamma_{m}^{1} a)(x)| e^{-2\rho x} D(x) dx \]
\[ \leq c \sum_{m=1}^{\infty} e^{-2m} \| T\Gamma_{m}^{1} a \|_{L^{1}(\mathbb{R})} \]
\[ \leq c \sum_{m=1}^{\infty} e^{-2m} m^{2\alpha} < \infty. \]

On the other hand, the estimate for $(0,1]$ is obtained as follows:
\[ \int_{0}^{1} \sum_{m=1}^{\infty} e^{-2mx} |(T\Gamma_{m}^{1} a)(x)| e^{-2\rho x} D(x) dx \]
\[ \leq c \int_{0}^{1} \sum_{m=1}^{\infty} e^{-2mx} \sum_{k=1}^{\infty} |d_{k}^{m}| |(Tb_{k}^{m})(x)| D(x) dx \]
\[ \leq c \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \int_{0}^{1} e^{-2mx} 2^{-k/2} m^{2\alpha} |(Tb_{k}^{m})(x)| x^{2\alpha+1} dx \]
\[ = I_{1} + I_{2}, \]

where $I_{1}$ is the integral over the region $D_{1}$: $0 < x \leq 1$, $|x - x_{0}| \geq 2\ell_{k}^{m}$ and $I_{2}$ is the integral over the region $D_{2}$: $0 < x \leq 1$, $|x - x_{0}| < 2\ell_{k}^{m}$. Since $\ell_{k}^{m} \sim 2^{k} \| a \|_{2}^{-2}$ by (14) and $\sum_{m=1}^{\infty} e^{-2mx} m^{2\alpha} \sim x^{-(2\alpha+1)} (0 < x \leq 1)$, the property (b) of $T$ yields that
\[ I_{1} \leq c \sum_{k=0}^{\infty} 2^{-k/2} \sum_{m=1}^{\infty} 2^{k} \| a \|_{2}^{-2} \int_{D_{1}} e^{-2mx} m^{2\alpha} x^{2\alpha+1} |x - x_{0}|^{-1+\epsilon} dx \]
\[ \leq c \sum_{k=0}^{\infty} 2^{-k/2} 2^{k} \| a \|_{2}^{-2} \int_{|x - x_{0}| > c2^{k} \| a \|_{2}^{-2}} |x - x_{0}|^{-1+\epsilon} dx \]
\[ \leq c \sum_{k=0}^{\infty} 2^{-k/2} < \infty. \]

In obtaining the estimate of $I_{2}$ we break the integral over $D_{2}$ as
\[ \int_{D_{2}} = \sum_{j=0}^{\infty} \int_{|x - x_{0}| \leq 2\ell_{j}^{m}, x_{0} \leq 2\ell_{k}^{m}} + \int_{|x - x_{0}| \leq 2\ell_{j}^{m}, x_{0} > 2\ell_{k}^{m}} \]
and we denote the corresponding terms by $I_{21}$ and $I_{22}$ respectively. As for $I_{21}$, if the integral does not vanish, we see that $2^{-j-1} \leq x_{0} + 2\ell_{j}^{m} \leq 5\ell_{k}^{m} \leq c2^{k} \| a \|_{2}^{-2}$ by (14) and moreover, $\| Tb_{k}^{m} \|_{2} \leq \| \tau_{k}^{m} \|_{2} \leq c(\ell_{m}^{k})^{-1/2} \sim 2^{-k/2} \| a \|_{2}$ by the property (a) of $T$.  

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Therefore, taking the maximal values of $e^{-2mx}$ and $x^{2\alpha+1}$ in the integrand and applying the Schwarz inequality to the integral, we can estimate $I_{21}$ as

$$
\leq c \sum_{k=0}^{\infty} 2^{-k/2} \left( \sum_{m=1}^{\infty} \left( \sum_{j \leq 2^k \|\alpha\|^2} e^{-m^2 - j^2 m^{2\alpha} 2^{-j(2\alpha+1)}} \right) 2^{-j/2} 2^{-k/2} \|\alpha\|_2^2 \right) \leq c \sum_{k=0}^{\infty} 2^{-k/2} < \infty.
$$

As for $I_{22}$, if the integral does not vanish, we see that $x \leq x_0 + 2\ell_m^m \leq 5x_0 / 3$ and $x \geq x_0 - 2\ell_m^m > x_0 / 3$. Therefore, we can deduce that

$$
I_{22} \leq c \sum_{k=0}^{\infty} 2^{-k/2} \left( \sum_{m=1}^{\infty} e^{-2mx_0 / 3} m^{2\alpha} \right) \left( 5x_0 / 3 \right)^{2\alpha+1} \|T\ell^m_m\|_{L^1(\mathbb{R})} \leq c \sum_{k=0}^{\infty} 2^{-k/2} < \infty.
$$

This completes the proof of the lemma. □

**Remark 3.2.** Let $T$ be as in Lemma 3.1. Let $T_{\Phi,D}$ denote a pseudo-differential operator with the symbol $\sigma_{\Phi,D}(x, \lambda)$ defined by $e^{-2\rho x} \Phi(\lambda + i\rho, x) D(x)$ if $x \in \mathbb{R}_+$ and 0 otherwise. Then from (6) the conclusion of Lemma 3.1 can be restated as $\|T_{\Phi,D} TF\|_{L^1(\mathbb{R})} \leq c \|F\|_{H^1(\mathbb{R})}$ for all $F \in H^1(\mathbb{R})$. Especially, taking the identity operator as $T$, we see that $T_{\Phi,D}$ is a bounded linear operator of $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$. On the other hand, it follows from Proposition 8.3 below that

$$
|\left( \frac{d}{d\lambda} \right)^M \sigma_{\Phi,D}(x, \lambda)| \leq c (1 + |\lambda|)^{-M}
$$

for $M = 0, 1$. If the same estimate were also true for the derivative of $x$, or a modulus continuity of $x$, we would obtain the boundedness of $T_{\Phi,D}$ directly from Théorème 9 in Coifman and Meyer [2].

4. $L^1$-condition. As stated in Theorem 1.1, $F^1_J$ is integrable on $\mathbb{R}$ if $f$ is integrable on $G$. However, the reverse is not true. In this section we obtain a condition of $F^1_J$ under which $f$ is integrable on $G$.
Theorem 4.1. Let us suppose that $C^1_+ F_j^1$ is well-defined for $f \in L^1_{\text{loc}}(G//K)$ and it belongs to $H^1(R)$. Then $f$ belongs to $L^1(G//K)$. In particular,
\[
\|f\|_{L^1(G)} \leq c \|C^1_+ F_j^1\|_{H^1(R)},
\]
where $c$ is independent of $f$.

Proof. We first prove the inequality for $f \in \mathcal{L}(G//K)$ with $\|C^1_+ F_j^1\|_{H^1(R)} < \infty$, in which case we see from (10) that
\[
f(x) = e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2m\rho x} (T^m_\rho C^1_+ F_j^1)(x) \quad (x \in R_+).
\]
By taking the identity operator as $T$ in Lemma 3.1 we can deduce that
\[
(15) \quad \|f\|_{L^1(G)} = \|fD\|_{L^1(R_+)} \leq c \|C^1_+ F_j^1\|_{H^1(R)}.
\]

For a general $f \in L^1_{\text{loc}}(G//K)$ with $\|C^1_+ F_j^1\|_{H^1(R)} < \infty$ we approximate $f$ by functions in $\mathcal{L}(G//K)$. Let $\psi_0$ be an even $C^\infty$ function on $R$ with $\int \psi_0(x) dx = 0$ and put $\psi^\epsilon(x) = \psi_0(x)e^{-\epsilon x}$ for $\epsilon \in R_+$. Since $\psi_0^\epsilon(\epsilon \lambda)$ is an even holomorphic function of exponential type, the Paley-Wiener theorem on $G$ yields that there exists $\phi^\epsilon \in C^\infty(G//K)$ such that $(\phi^\epsilon)^\wedge(\lambda) = \psi_0^\epsilon(\epsilon \lambda)$. Let $\phi^\epsilon(\lambda) = \delta^{-1} \psi^\epsilon(\delta^{-1} x)$ for $\delta \in R_+$. Then $F^1_{\phi^\epsilon}(x) = \psi^\epsilon(\lambda)$ because $(F^1_{\phi^\epsilon}(\lambda))^\wedge = (\phi^\epsilon)^\wedge(\lambda + i \rho) = \psi_0^\epsilon(\epsilon \lambda + i \rho) = (\psi^\epsilon(\lambda))^\wedge$. and moreover, $f^* \phi^\epsilon \in \mathcal{L}(G//K)$ and $C^1_+ F^1_{f^* \phi^\epsilon} = (C^1_+ F_j^1)^* \psi^\epsilon$. We have the following.

Lemma 4.2. Let $F$ be in $H^1(R)$. Then $F^* \psi^\epsilon \in H^1(R)$ for $0 < \epsilon < 1$ and
\[
\|F - F^* \psi^\epsilon\|_{H^1(R)} \to 0 \quad (\epsilon \to 0).
\]

Proof. We refer to the notations and the results in Folland and Stein [5, §2]. Since $\|\psi^\epsilon\|_{L^1} \leq c$ for all $0 < \epsilon < 1$, it follows from Lemma 3.31 in [5] that there exists $N \geq 1$ such that $\sup_{0 < \delta < \infty} (M_N f^* \psi^\delta) \leq c(M_1 f^* \psi^0)$. Especially, $(M_N f^* \psi^\epsilon) \leq c(M_1 f^* \psi^0)$. Then, $f^* \psi^\epsilon \in H^1(R)$ by Theorem 3.30 in [5]. The rest of the proof follows from the same argument as in the proof of Theorem 3.33 in [5].

Hence, $C^1_+ F^1_{f^* \phi^\epsilon} = (C^1_+ F_j^1)^* \psi^\epsilon \in H^1(R)$ and $C^1_+ F_j^1$ as $\epsilon \to 0$. On the other hand, since $f^* \phi^\epsilon \in \mathcal{L}(G//K)$, it follows from (15) that $\|f^* \phi^\epsilon\|_{L^1(G)} \leq c \|C^1_+ F^1_{f^* \phi^\epsilon}\|_{H^1(R)}$. Therefore, $f^* \phi^\epsilon$ converges to a function $h$ in $L^1(G//K)$. Clearly, $h$ must be $f$, because $(\phi^\epsilon)^\wedge \to \psi_0^\epsilon(0) = 1$ as $\epsilon \to 0$. So, letting $\epsilon \to 0$, we obtain the desired inequality for $f$. This completes the proof of Theorem 4.1.

We now define the Hardy space $H^1_s(G//K)$ $(s \geq 0)$ on $G$. Let $Q_s$ $(s \geq 0)$ denote the Fourier multiplier of $R$ defined by
\[
(Qs f)^\wedge(\lambda) = \left( \frac{1 + |\lambda|^s}{|\lambda|^s} \right) F^\wedge(\lambda) \quad (\lambda \in R).
\]
Definition 4.3. For \( s \geq 0 \)

\[ H^s(G//K) = \{ f \in L^1_{\text{loc}}(G//K); Q_s C^1_F f \text{ is well-defined and is in } H^1(\mathbb{R}) \} \]

and \( \| f \|_{H^s(G//K)} = \| Q_s C^1_F f \|_{H^1(\mathbb{R})} \).

Theorem 4.4. If \( s' \geq s \geq 0 \), then \( H^s_s(G//K) \subset H^{s'}(G//K) \subset L^1(G//K) \) and for all \( f \in H^s_s(G//K) \)

\[ \| f \|_{L^1(G)} \leq c \| f \|_{H^s_s(G)} \leq c' \| f \|_{H^{s'}(G)} , \]

where \( c \) and \( c' \) are independent of \( f \).

Proof. Theorem 4.1 asserts that \( \| f \|_{L^1(G)} \leq c \| f \|_{H^s_s(G)} \). Let \( \mathcal{R}_s \) denote the Fourier multiplier on \( \mathbb{R} \) defined by \( (\mathcal{R}_s F)^\wedge (\lambda) = F^\wedge(\lambda)(|\lambda| / 1 + |\lambda|)^s \) \( (\lambda \in \mathbb{R}) \). We notice that

\[ (|\lambda| / 1 + |\lambda|)^s \quad (s \geq 0) \]

satisfies the Hörmander condition and thus, \( \mathcal{R}_s \) is a bounded linear operator on \( H^1(\mathbb{R}) \) (see [12, Theorem 4.2]). Therefore, for all \( f \in H^1_s(G//K) \),

\[ \| f \|_{H^s_s(G)} = \| \mathcal{R}_{s'}^{-1} Q_s C^1_F f \|_{H^1(\mathbb{R})} \leq c \| Q_s C^1_F f \|_{H^1(\mathbb{R})} = \| f \|_{H^{s'}(G)} . \]

Theorem 4.5. For \( s \geq 0 \), \( H^s_s(G//K) \cap \mathcal{L}(G//K) \) is dense in \( H^s_s(G//K) \).

Proof. We retain the notation used in the proof of Theorem 4.1 and suppose that \( f \in H^s_s(G//K) \). Since \( Q_s C^1_F f_j \in H^1(\mathbb{R}) \) and \( Q_s C^1_F f_{j+\phi_j} = Q_s C^1_F (F_j^{\epsilon_j} \psi(\epsilon_j)) = (Q_s C^1_F f_j)^\wedge \psi(\epsilon) \) \( (\text{cf. } [4, \text{Theorem } 5]) \), it follows from Lemma 4.2 that \( f \ast \phi(\epsilon) \in H^s_s(G//K) \cap \mathcal{L}(G//K) \) and \( \| f - f \ast \phi(\epsilon) \|_{H^s_s(G)} \rightarrow 0 \) \( (\epsilon \rightarrow 0) \).

Remark 4.6. \( H^0_s(G//K) \) contains all \( f \in C^\infty_c(G//K) \) with \( \int_G f(g)dg = 0 \). Indeed, we suppose that \( \text{supp}(f) \subset B(r) \), and we observe that \( \hat{f}(\lambda) \) is holomorphic on \( \mathcal{F}_c \) of exponential type \( r \) and \( C(-\lambda)^{-1} \) is holomorphic and tempered on the upper half plane (cf. [14, 9.2.3] and [4, Lemma 8]). We here recall the technique used in the proof of the Paley-Wiener theorem (cf. [14, 9.2.3]). Hence, shifting the integral line \( \mathbb{R} \) of the integral \( \int_{\mathbb{R}} C(-\lambda - i\rho)^{-1} f(\lambda + i\rho) e^{i\lambda \eta} d\lambda \)

defining \( C^1_F f_j \) to \( \mathbb{R} + i\eta \) \( (\eta \rightarrow +\infty) \), we can deduce that \( \text{supp}(C^1_F f_j) \subset (-\infty, r) \). Thereby, \( \| C^1_F f_j \|_{L^2(\mathbb{R})} \leq e^{r^\eta} \| f \|_{L^2(G)} \) by the Plancherel formulas on \( \mathbb{R} \) and \( G \), and similarly, \( \| C^1_F f_j \|_{L^2(\mathbb{R})} \leq e^{(1+re^{\rho^\eta}) \| f \|_{L^2(G)}} \). On the other hand, since \( \phi \equiv 1 \) and \( C(-i\rho) = 1 \) \( (\text{cf. } [4, \S 3 \text{ and Lemma 8}) \), we have \( \int_{\mathbb{R}} (C^1_F f_j)(x)dx = C(-i\rho)^{-1} f(i\rho) = \int_G f(g)dg = 0 \). These follow immediately that \( C^1_F f_j \) is a (1,2,0,1/2)-molecule on \( \mathbb{R} \) and hence, in \( H^1(\mathbb{R}) \) (see [12, Theorem 2.9]).

5. Criteria for boundedness. We fix \( 0 < r_1 < r_2 \leq \infty \) and a function \( \phi(r,g) \) on \( (r_1, r_2) \times G \) satisfying \( \phi(r,\cdot) \in L^1(G//K) \) for each \( r \in (r_1, r_2) \). We define a radial maximal operator \( T^G_{\phi,r_1,r_2} \) on \( G \) as follows: for \( f \in L^1_{\text{loc}}(G//K) \)

\[ (T^G_{\phi,r_1,r_2} f)(g) = \sup_{r_1 < r < r_2} \| (f \ast \phi(r,\cdot))(g) \| \quad (g \in G) . \]

In this section we obtain some conditions on \( \phi \) under which \( T^G_{\phi,r_1,r_2} \) is a bounded sub-linear operator of \( H^s_s(G//K) \) \( (s \geq 0) \) to \( L^1(G//K) \).
Let $B_{r_1, r_2}$ be the set of all functions $\beta(r, \lambda)$ on $(r_1, r_2) \times \mathbb{R}$ for which there exists a continuous function $\Theta = \Theta_\beta$ on $\mathbb{R}$ such that

\begin{align}
(a_1) & \quad \Theta(\lambda) \in L^1(\mathbb{R}), \\
(a_2) & \quad \lim_{|\lambda| \to \infty} \Theta(\lambda) = 0,
\end{align}

(b) $|\left(\frac{d}{d\lambda}\right)^M \beta(r, \lambda)| \leq r^M \Theta(r \lambda) \quad (M = 0, 1, 2),$

and $B_{r_1, r_2}^+$ the subset of $B_{r_1, r_2}$ defined by replacing (16)(a1), (a2) with

\begin{align}
(a_1) & \quad \lambda \Theta(\lambda) \in L^1(\mathbb{R}), \\
(a_2) & \quad \lim_{|\lambda| \to \infty} \lambda \Theta(\lambda) = 0.
\end{align}

**Lemma 5.1.** If $\hat{\phi}(r, \lambda + i \rho) \in B_{r_1, r_2}$, then the maximal operator defined by $\sup_{r_1 < r < r_2} |(F^1_{\phi(r, \cdot)} \ast F)(x)|$ of $F \in L^1_{loc}(\mathbb{R})$ satisfies the $L^p(\mathbb{R})$-norm inequalities for $p > 1$, and a weak type $L^1(\mathbb{R})$ estimate.

**Proof.** Since $(F^1_{\phi(r, \cdot)})^\sim(\lambda) = \hat{\phi}(r, \lambda + i \rho)$ (see (9)), it follows from (16) of $\hat{\phi}(r, \lambda + i \rho)$ that $|x^M F^1_{\phi(r, \cdot)}(x)| \leq \int_{\mathbb{R}} |(d/d\lambda)^M \hat{\phi}(r, \lambda + i \rho) \cdot e^{i \lambda x} d\lambda| \leq \|\Theta\|_{L^1(\mathbb{R})} r^{M-1} (M = 0, 1, 2)$ and thus, $|F^1_{\phi(r, \cdot)}(x)| \leq cr^{-1}(1 + r^{-1}|x|)^{-2} \ (x \in \mathbb{R})$. This inequality easily yields that $\sup_{r_1 < r < r_2} |(F^1_{\phi(r, \cdot)} \ast F)(x)| \leq c(M_{HL} F)(x) \ (x \in \mathbb{R})$, where $M_{HL}$ is the Hardy-Littlewood maximal operator on $\mathbb{R}$. Hence the desired results follow from those for $M_{HL}$ (cf. [5, Theorem 2.4]). \(\square\)

**Lemma 5.2.** If $\hat{\phi}(r, \lambda + i \rho) \in B_{r_1, r_2}^+$, then for each $(1, 2, 0)$-atom $a$ on $\mathbb{R}$ with $\text{supp}(a) \subset [x_0 - \ell, x_0 + \ell],$

$$
\sup_{r_1, r_2} |(F^1_{\phi(r, \cdot)} \ast a)(x)| \leq c\ell |x - x_0|^{-2} \quad \text{if} \ |x - x_0| \geq 2\ell,
$$

where $c$ is independent of $a$.

**Proof.** The moment condition $\int_{\mathbb{R}} a(x) dx = 0$ and the mean value theorem yield that for $x \in \mathbb{R}_+$

\begin{align}
(F^1_{\phi(r, \cdot)} \ast a)(x) &= \int_{\mathbb{R}} F^1_{\phi(r, \cdot)}(x - y) a(y) dy \\
&= \int_{\mathbb{R}} \left( F^1_{\phi(r, \cdot)}(x - y) - F^1_{\phi(r, \cdot)}(x_0 - y_0) \right) a(y) dy \\
&= \int_{-\ell}^{\ell} \left( \frac{d}{dx} F^1_{\phi(r, \cdot)}(x - y_0) \right) (x_0 - y) a(y) dy,
\end{align}

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where \( y_0 \) is on the line segment from \( y \) to \( x_0 \). Since \((dF^1_{\phi(r,\cdot)}/dx)^*(\lambda) = i\lambda \hat{\phi}(r,\lambda + i\rho)\) (see (9)), it follows from (16)(b) and (17) of \( \hat{\phi}(r,\lambda + i\rho) \) that

\[
|\langle x-y_0 \rangle^2 \frac{d}{dx} F^1_{\phi(r,\cdot)}(x-y_0) | = \left| \int_{\mathbb{R}} \left( \frac{d}{d\lambda} \right)^2 \left( \lambda \hat{\phi}(r,\lambda + i\rho) \right) e^{i\lambda(x-y_0)} d\lambda \right|
\leq 2r \int_{\mathbb{R}} |\Theta(r\lambda)| d\lambda + r^2 \int_{\mathbb{R}} |\lambda \Theta(r\lambda)| d\lambda
= 2\|\Theta\|_{L^1(\mathbb{R})} + \|\lambda \Theta\|_{L^1(\mathbb{R})}.
\]

Let \( |x-x_0| \geq 2\ell \). Since \( |x_0-y| < \ell \) and \( y_0 \) is located between \( y \) and \( x_0 \), we have \( |x-y_0| > \ell \) and thus, \( |x-x_0||x-y_0|^{-1} \leq 1 + |x_0-y_0||x-y_0|^{-1} \leq 2 \). Therefore, if \( |x-x_0| \geq 2\ell \),

\[
\sup_{r_1 < r < r_2} |(F^1_{\phi(r,\cdot)} \ast a)(x)| \leq c|x-x_0|^{-2} \int_{\mathbb{R}} |x_0-y||a(y)|dy \leq c\ell|x-x_0|^{-2}. \quad \Box
\]

**Proposition 5.3.** If \( \hat{\phi}(r,\lambda + i\rho) \in B^+_{r_1,r_2} \), then \( T^G_{\phi,r_1,r_2} \) is a bounded sublinear operator of \( H^1_{\phi}(G//K) \) to \( L^1(G//K) \).

**Proof.** From Theorem 4.5 it suffices to obtain the boundedness for \( f \in H^1_{\phi}(G//K) \cap L(G//K) \). Since \((f \ast \phi(r,\cdot))^{\wedge}(\lambda) = \hat{f}(\lambda)\hat{\phi}(r,\lambda) = \hat{f}(\lambda)(F^0_{\phi(r,\cdot)})^{\wedge}(\lambda) \) (cf. [4, Theorem 5]), the same manipulation as in (10) yields that for \( x \in \mathbb{R}_+ \)

\[
(f \ast \phi(r,\cdot))(x) = e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2m x} (F^1_{\phi(r,\cdot)} \ast (T^1_{m_1}C^1_1 F^1))(x)
\]

and thereby,

\[
(T^G_{\phi,r_1,r_2} f)(x) \leq e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2m x} \sup_{r_1 < r < r_2} |(F^1_{\phi(r,\cdot)} \ast (T^1_{m_1}C^1_1 F^1))(x)|.
\]

Since \( \hat{\phi}(r,\lambda + i\rho) \in B^+_{r_1,r_2} \), Lemmas 5.1 and 5.2 imply that the maximal operator \( \sup_{r_1 < r < r_2} |(F^1_{\phi(r,\cdot)} \ast F)(x)| \) of \( F \in L^1_{\sup}(\mathbb{R}) \) satisfies the properties (a) and (b) in Lemma 3.1. Therefore, the result follows from Lemma 3.1 and Definition 4.3. \( \Box \)

Let \( B^\delta_{r_1,r_2} (\delta \in \mathbb{R}) \) be the set of all functions \( \beta(r,\lambda) \) on \( (r_1, r_2) \times \mathbb{R} \) for which there exists a continuous function \( \Theta = \Theta_\beta \) on \( \mathbb{R} \) such that

\[
\begin{align*}
(a_1) & \quad \lambda^{\delta+1}\Theta(\lambda) \in L^1(\mathbb{R}), \\
(a_2) & \quad \lim_{|\lambda| \to \infty} \lambda^{\delta+1}\Theta(\lambda) = 0, \\
(b) & \quad |(\frac{d}{d\lambda})^M \beta(r,\lambda)| \leq r^{M+\delta} \Theta(r\lambda) \quad (M = 0,1,2).
\end{align*}
\]
Proposition 5.4. If \( \hat{\phi}(r, \lambda + i \rho) \in B^{\delta}_{r_1, r_2} \) (\( \delta \geq 2 \)), then \( T^{G}_{\phi, r_1, r_2} \) is a bounded sublinear operator of \( H^1_\delta(G/\!\!/K) \) to \( L^1(G/\!\!/K) \).

Proof. As we have deduced (18), we see that for \( f \in H^1_\delta(G/\!\!/K) \cap \mathcal{L}(G/\!\!/K) \)

\[
(T^{G}_{\phi, r_1, r_2} f)(x) \leq e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2m \lambda} |(T^{1}_{m} Q_{\delta} C_{1}^{1}(f))(x)| \quad (x \in \mathbb{R}_+),
\]

where \( T \) is a maximal operator on \( L^1_{loc}(\mathbb{R}) \) defined by

\[
(TF)(x) = \sup_{r_1 < r < r_2} \int_{\mathbb{R}} F^\sim(\lambda) \left( \frac{|\lambda|}{1 + |\lambda|} \right)^{\delta} \hat{\phi}(r, \lambda + i \rho) e^{i \lambda x} d\lambda \quad (x \in \mathbb{R}).
\]

Then, as in the proof of Proposition 5.3 it suffices to show that \((|\lambda|/1 + |\lambda|)^{\delta} \hat{\phi}(r, \lambda + i \rho) \in B^{\delta}_{r_1, r_2} \). Indeed, it follows from (19)(b) of \( \hat{\phi}(r, \lambda + i \rho) \) that for \( M = 0, 1, 2 \)

\[
|\left( \frac{d}{d\lambda} \right)^M \left( \left( \frac{|\lambda|}{1 + |\lambda|} \right)^{\delta} \hat{\phi}(r, \lambda + i \rho) \right) |
\]

\[
\leq c \sum_{m=0}^{M} |r^\delta \left( \frac{d}{d\lambda} \right)^{M-m} \left( \frac{|\lambda|}{1 + |\lambda|} \right)^{\delta} \hat{\phi}(r, \lambda + i \rho) |
\]

\[
\leq c \sum_{m=0}^{M} \delta(\delta - 1) \ldots (\delta - M + m + 1) |r\lambda|^{\delta-M+m} r^{M-m} \cdot \Theta(r\lambda)
\]

\[
\leq cr^M \Psi(r\lambda) \Theta(r\lambda),
\]

where \( \Psi(\lambda) = (\delta(\delta - 1)|\lambda|^{\delta-2} + \delta|\lambda|^{\delta-1} + |\lambda|^\delta) \). Therefore, if we define \( \Theta \) in (16) and (17) by \( \Psi \Theta \), the above calculation and (19)(a1), (a2) imply that \((|\lambda|/1 + |\lambda|)^{\delta} \hat{\phi}(r, \lambda + i \rho) \in B^{\delta}_{r_1, r_2} \). \( \square \)

Let \( B^{\delta+1}_{r_1, r_2} \) (\( \delta \in \mathbb{R} \)) be the set of all functions \( \beta(r,\lambda) \) on \( (r_1, r_2) \times \mathbb{R} \) for which there exists a continuous function \( \Theta = \Theta_\beta \) on \( \mathbb{R} \) such that

\[
(20) \quad \begin{align*}
(a_1) & \quad \lambda^{\delta+1} \Theta(\lambda) \in L^1(\mathbb{R}), \\
(a_2) & \quad \lim_{|\lambda| \to \infty} \lambda^{\delta+1} \Theta(\lambda) = 0, \\
(b_1) & \quad |\beta(r,\lambda)| \leq \Theta(\lambda), \\
(b_2) & \quad \left| \left( \frac{d}{d\lambda} \right) \beta(r,\lambda) \right| \leq r^\delta \Theta(r\lambda), \\
(b_3) & \quad \left| \left( \frac{d}{d\lambda} \right)^2 \beta(r,\lambda) \right| \leq r^{2+\delta} \Theta(r\lambda).
\end{align*}
\]
**Proposition 5.5.** Let $r_1 \geq 1$ and $\delta \geq 2$. If $\hat{\phi}(r, \lambda + i\rho) \in B^\delta_{r_1, r_2}$, then $T^G_{\hat{\phi}, r_1, r_2}$ is a bounded sublinear operator of $H^1_{r_1} \to L^1(G)$ for $\epsilon > 0$.

**Proof.** We first notice that under the assumption that $r_1 \geq 1$ and $\delta \geq 2$, $\beta(r, \lambda)$ in $B^\delta_{r_1, r_2}$ also satisfies

$$(b_1) \quad \left| \beta(r, \lambda) \right| \leq r^{\delta-1} \Theta(r\lambda),$$

$$(b_2) \quad \left| \frac{d}{d\lambda} \beta(r, \lambda) \right| \leq r^{M+\delta} \Theta(r\lambda) \quad (M = 0, 1, 2).$$

In what follows we modify the proofs of Lemmas 5.1 and 5.2. As in the proof of Theorem 4.4, $R_s$ ($s \in \mathbb{R}$) denotes the Fourier multiplier on $\mathbb{R}$ defined by $(R_sF)^\sim(\lambda) = F^\sim(\lambda)(|\lambda|/1 + |\lambda|)^{\delta} \ (\lambda \in \mathbb{R})$. Since $(R^-_{\delta-1} F^1_{\hat{\phi}(r, \lambda)})(\lambda) = \hat{\phi}(r, \lambda + i\rho) \ (|\lambda|/1 + |\lambda|)^{\delta-1}$, we see from $(a)$, $(b_2)$, and $(b_2)$ of $\hat{\phi}(r, \lambda + i\rho)$ that

$$|(R^-_{\delta-1} F^1_{\hat{\phi}(r, \lambda)})(x)| \leq c r^{\delta-1} \left( (\delta-1) \parallel \lambda^{\delta-2} \Theta \parallel_{L^1(\mathbb{R})} + \parallel \lambda^{\delta-1} \Theta \parallel_{L^1(\mathbb{R})} \right)$$

and similarly from $(a)$ and $(b_2)$ that

$$|(R^-_{\delta} F^1_{\hat{\phi}(r, \lambda)})(x)| \leq c r^{\delta-1} \left( (\delta-1) \parallel \lambda^{\delta-2} \Theta \parallel_{L^1(\mathbb{R})} + \parallel \lambda^{\delta-1} \Theta \parallel_{L^1(\mathbb{R})} \right).$$

We here fix $r$ and $x$, and we observe that $(R_z F^1_{\hat{\phi}(r, \lambda)})(x)$ makes sense for $z \in \mathbb{C}$ with $\delta - 1 \leq \Re(z) \leq \delta$ and, as a function of $z$, it is holomorphic on $\delta - 1 < \Re(z) < \delta$. Moreover, $|(R_z F^1_{\hat{\phi}(r, \lambda)})(x)| \leq r^{\delta-1} \parallel \Theta \parallel_{L^1(\mathbb{R})}$ by $(a)$ and $(b_1)$, and when $\Re(z) = \delta - 1$ (resp. $\delta$), $(R_z F^1_{\hat{\phi}(r, \lambda)})(x)$ satisfies the same inequality for $z = \delta - 1$ (resp. $\delta$) obtained above. Therefore, Three Lines Lemma (cf. [9, p. 69]) yields that $|(R^-_{\delta-1+\epsilon} F^1_{\hat{\phi}(r, \lambda)})(x)| \leq c r^{\delta-1} \left( (1 + \epsilon) \parallel \lambda^{\delta-2} \Theta \parallel_{L^1(\mathbb{R})} + \parallel \lambda^{\delta-1} \Theta \parallel_{L^1(\mathbb{R})} \right).$

We next observe that $(a)$ and $(b_2)$ imply that $\hat{\phi}(r, \lambda + i\rho) \in B^\delta_{r_1, r_2}$ and thus, Lemma 5.2 and the proof of Proposition 5.4 yield that for each $(1,2,0)$-atom $a$ on $\mathbb{R}$ with supp(a) $\subset [x_0 - \ell, x_0 + \ell],$

$$|((R^-_{\delta} F^1_{\hat{\phi}(r, \lambda)})(x) * a)(x)| \leq c \ell |x - x_0|^{-2} \left( \delta (\delta - 1) (2 \parallel \lambda^{\delta-2} \Theta \parallel_{L^1(\mathbb{R})} + \parallel \lambda^{\delta-1} \Theta \parallel_{L^1(\mathbb{R})}) \right)$$

if $|x - x_0| \geq 2 \ell$. On the other hand, we notice that if $|x_0 - y| \leq \ell$ and $|x - x_0| \geq 2 \ell$, then $|x - x_0| |y - x - y|^{-1} \leq 1 + |x_0 - y| |x - y|^{-1} \leq 2$ and thereby, from (21) that

$$|((R^-_{\delta-1} F^1_{\hat{\phi}(r, \lambda)})(x) * a)(x)| = \left| \int_{\mathbb{R}} (R^-_{\delta-1} F^1_{\hat{\phi}(r, \lambda)})(x - y) a(y) dy \right|$$

lower bound of $\Theta$.
if $|x - x_0| \geq 2\ell$. Moreover, \(|(R_z F^\delta_{\phi(r, \cdot)} \ast a)(x)| \leq \|R_z F^\delta_{\phi(r, \cdot)} \|_\infty \|a\|_{L^1(\mathbb{R})} \leq r^{-1} \Theta_{L^1(\mathbb{R})} \) for $\delta - 1 \leq \Re(z) \leq \delta$. Then, applying Three Lines Lemma again, we see that for $0 \leq \epsilon \leq 1$, \(|(R_{\delta-1+h} F^\delta_{\phi(r, \cdot)} \ast a)(x)| \leq c \epsilon^\ell \ |x - x_0|^{-(1+\epsilon)} \) if $|x - x_0| \geq 2\ell$.

We have therefore proved that if $0 < \epsilon \leq 1$, the maximal operator defined by $\sup_{r_1 < r < r_2} \|(R_{\delta-1+\epsilon} F^\delta_{\phi(r, \cdot)} \ast F)(x)\|_{L^1_\text{loc}(\mathbb{R})}$ satisfies the properties (a) and (b) in Lemma 3.1. Since $(T^{G}_{\phi(r_1, r_2)} f)(x) \ (x \in \mathbb{R_+})$ is dominated by

$$e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2mx} \sup_{r_1 < r < r_2} \|(R_{\delta-1+\epsilon} F^\delta_{\phi(r, \cdot)} \ast (\Gamma^1_{m} \mathcal{Q}_{\delta-1+\epsilon}(C^1_{\epsilon} F^1_{\phi(r, \cdot)}))(x)\|,$$

the desired result follows from Lemma 3.1, Definition 4.3 and Theorem 4.4. \(\square\)

6. Maximal operators. We apply the criteria obtained in the previous section to some radial maximal operators. Actually, as $\phi(r, g)$ in §5, we shall take $\phi^\delta_{\lambda}(g), \phi^\lambda_{\delta}(g), \phi^\lambda_{\delta}(g), h_\lambda(g),$ and $p_\lambda(g)$ respectively (see A~E below). For simplicity, we denote henceforth the set of bounded sublinear operators of $H^1(G/K)$ to $L^1(G/K)$ by $(H^1, L^1)$ and that on $L^p(G/K)$ by $(L^p, L^p)$. We also say that an operator $T$ of $L^1(G/K)$ is of weak type $(1,1)$ provided there exists a constant $c$ such that

$$|\{g \in G; \ |(Tf)(g)| > \alpha \}| \leq \frac{c}{\alpha} \|f\|_{L^1(G)}$$

for all $f \in L^1(G/K)$ and for every $\alpha > 0$.

A. Let $A^\delta_\lambda(G)$ ($\delta \in \mathbb{R}$) be the set of all functions $\phi \in L^1(G/K)$ satisfying $|(d/d\lambda)^M \hat{\phi}(\lambda)| \leq (1 + |\lambda|^{-\delta} \ (\lambda \in \mathcal{F}(\rho))$ for $M = 0, 1, 2.$

**Definition 6.1.** For $f \in L^1_\text{loc}(G//K)$

$$(M_{\delta, \phi}^G f)(g) = \sup_{\phi \in A^\delta_\lambda(G)} \|(f \ast \phi^\delta_{\lambda})(g)\| \ (g \in G),$$

where $\phi^\delta_{\lambda}(g) = \int_{\mathbb{R}} \hat{\phi}(r\lambda)\phi\lambda(g)|C(\lambda)|^{-2}d\lambda$.

**Theorem 6.2.** If $\delta > 2$, then $M_{\delta, \phi}^G \in (H^1, L^1)$.

**Proof.** We easily see that if $\delta > 2$, $\hat{\phi}(\lambda)$ belongs to $B^+_{0, \infty}$ and hence, by Proposition 5.3 we have the desired result. \(\square\)

B. Let $A_N(G)$ ($N \in \mathbb{N}$) be the set of all functions $\phi \in C^N(G/K)$ whose restriction on $A$ satisfies $\text{supp}(\phi) \subset [-1, 1], \|(d/dx)^n \phi\|_{\infty} \leq 1 \ (0 \leq n \leq N),$ and $\phi(x) = O(x^N)$.

**Definition 6.3.** For $\epsilon \geq 0$ and $f \in L^1_\text{loc}(G//K)$

$$(M_{N, \phi}^G f)(g) = \sup_{\phi \in A_N(G)} (1 + r)^{-\epsilon} \|(f \ast \phi^\delta_{\lambda})(g)\| \ (g \in G),$$

where $\phi^\delta_{\lambda}(g) = \frac{1}{|\mathbb{F}|} \phi(\frac{\phi(g)}{r}).$
Theorem 6.4. (1) $M_{N,e}^{G,\delta}$ is in $(L^p, L^p)$ for $p > 1$ and of weak type $(1,1)$.
(2) If $N \geq 6$, then $M_{N,0}^{G,\delta} \in (H^1, L^1)$ for $\delta > 2$.
(3) If $N \geq 4$ and $e > 1$, then $M_{N,e}^{G,\delta} \in (H^1, L^1)$.

Proof. (1) is obvious from Theorem 1.2 because $M_{N,e}^{G,\delta}f$ is pointwisely dominated by $M_{H,L}^0f$. To prove (2) and (3) we first obtain the following,

Lemma 6.5. Let $\phi \in A_N(G)$ ($N \in \mathbb{N}$). Then, for $M = 0, 1, 2$

$$\left| \left( \frac{d}{dx} \right)^M (\phi^\delta)^{\gamma}(\lambda + i \rho) \right| \leq c r^M (1 + |r\lambda|)^{-2s} \quad (0 \leq s \leq N).$$

Proof. We recall that $|(d/d\lambda)^m \phi_{\lambda+i\rho}(x)| \leq x^m \phi_{i\rho}(x) \leq x^m (\lambda \in \mathcal{F}, x \in \mathbb{R}_+), \phi_{\lambda}$ is an eigenfunction of the Laplace-Beltrami operator $\Omega$ on $G$ with eigenvalue $-\rho^2$, and the radial component of $\Omega$ on $\mathbb{R}_+$ is of the form $D^{-1} \cdot (d/dx(D \cdot d/dx)).$

For these facts we refer to [4], Lemma 14, Proposition 3, and §2 respectively. We now apply them to the integral defining the Fourier transform $(\phi^\delta)^{\gamma}(\lambda + i \rho)$ and thereby, we can deduce that for each $m, n \in \mathbb{N}$ and $0 \leq n \leq N$

$$\left| \left( \frac{d}{dx} \right)^m (\phi_{\lambda+i\rho}(x)) \right| \leq \int_0^\infty \left| \Omega^n \phi_{\lambda+i\rho}(x) \right| \left( \frac{d}{dx} \right)^m \phi_{\lambda+i\rho}(x) \left| D(x) \right| dx$$

For simplicity, we put $D = D^{-1}(d/D dx)$ and $\Phi(x) = \phi(x/r)$ ($r \in \mathbb{R}_+$), and we observe that $(d/dx)^m \Phi(x) \sim r^{-2n} x^{2N-p} \leq r^{-2n} x^{2n-p}$ if $0 < x \leq r$, $D(x) \sim (1 + x)/x$, and $(d/dx)^m D(x) \leq c e^{-r-1} e^{-2x}$ if $q > 0$. Hence we have

$$\left\| \left( \frac{d}{dx} \right)^{2n} \phi \right\|_\infty \leq cr^{2n},$$

$$\left\| \left( \frac{d}{dx} \right)^{p \Phi} \cdot \left( \frac{d}{dx} \right)^{q \Phi} \cdot D^s \right\|_\infty \leq cr^{-2n} \quad \text{if } p + q + s < 2n \text{ and } q > 0,$$

$$\left\| \left( \frac{d}{dx} \right)^{p \Phi} \cdot D^s \right\|_\infty \leq cr^{-2n} (1 + r)^s \quad \text{if } p + s = 2n.$$
Definition 6.6. For $\epsilon \geq 0$ and $f \in L^1_{\text{loc}}(G//K)$

$$(M^{G,b,f}_{N,\epsilon})(g) = \sup_{\phi \in A_N(G)} (1 + r)^{-\epsilon} |(f * \phi_r)(g)| \quad (g \in G),$$

where $\phi_r^b(g) = \frac{1}{r} D(\sigma(g)) D(\sigma(g) r) \phi(\sigma(g) r)$.

Theorem 6.7. (1) $M^{G,b}_{N,\epsilon}$ is in $(L^p, L^p)$ for $p > 1$ and is of weak type $(1,1)$.
(2) If $N \geq 6$, then $M^{G,b}_{N,0} \in (H^1_\delta, L^1)$ for $\delta > 1$.
(3) If $N \geq 4$ and $\epsilon > 2$, then $M^{G,b}_{N,\epsilon} \in (H^1_0, L^1)$.

Proof. It is easy to verify that $\|\phi^b_r\|_{L^1(G)} = \|\phi^b_r\|_{L^1(G)}$ and moreover, $\phi^b_r(x) \leq c \phi^b_r(x)$ if $0 < r \leq 1$ and $\phi^b_r(x) \leq c e^{-2pr} (x \in \mathbb{R}_+)$ if $1 < r < \infty$. Therefore, as in [7, Theorem 3.4], it follows that $M^{G,b}_{N,\epsilon}$ is in $(L^p, L^p)$ for $p > 1$ and of weak type $(1,1)$. We now suppose that $\phi \in A_N(G)$ and we observe that for each $0 \leq n \leq 2N$ and $0 \leq x \leq r$, $(d/dx)^n \phi(x/r) \sim cr^{-2N} x^{2N-n}$ and

$$\left(\frac{d}{dx}\right)^n D(\frac{x}{r}) \leq cr^{-n} \left(\frac{r + x}{x}\right)^n D(\frac{x}{r}) \leq c x^{-n} D(\frac{x}{r}).$$

Then the same argument as in the proof of Lemma 6.5 yields that for $M = 0, 1, 2$,

$$\left|\left(\frac{d}{dx}\right)^{M} (\phi^b_r)^N(\lambda + i \rho)\right| \leq c^{M}(1 + r)^n(1 + |r\lambda|)^{-2s} \quad (0 \leq s \leq N),$$

and therefore, the desired result follows. \square

D. The heat maximal operator $M^{G}_{H,\epsilon} (\epsilon \geq 0)$ on $G$ is given as follows.

Definition 6.8. For $\epsilon \geq 0$ and $f \in L^1_{\text{loc}}(G//K)$

$$(M^{G}_{H,\epsilon} f)(g) = \sup_{0 < r < \infty} (1 + r)^{-\epsilon} |(f * h_r)(g)| \quad (g \in G),$$

where $h_r(g) = \int_{\mathbb{R}} e^{-(\lambda^2 + \rho^2) r} \phi_\lambda(g) |C(\lambda)|^{-2} d\lambda$.

Theorem 6.9. (1) $M^{G}_{H,\epsilon}$ is in $(L^p, L^p)$ for $p > 1$ and of weak type $(1,1)$.
(2) $M^{G}_{H,0} \in (H^1_\delta, L^1)$ for $\delta > 1$.
(3) $M^{G}_{H,1} \in (H^1_0, L^1)$.

Proof. (1) is well-known (cf. [9, p.73] and [1, Corollary 3.2]). Since $\hat{h}_r(\lambda + i \rho) = e^{-\lambda^2 r} e^{-2i \rho \lambda r}$, we see that $\hat{h}_{r/2}(\lambda + i \rho) \in B^{+}_{0,1}$ if $0 < r \leq 1$ and $B^{+}_{1,\infty}$ if $1 < r < \infty$, and moreover, $(1 + r^2)^{-1} \hat{h}_{r/2}(\lambda + i \rho) \in B^{+}_{0,\infty}$. Therefore, (2) follows from Propositions 5.3 and 5.5, and by changing $r^2$ to $r$, (3) follows from Proposition 5.3. \square

E. The Poisson maximal operator $M^{G}_{P,\epsilon} (\epsilon \geq 0)$ on $G$ is given as follows.
Definition 6.10. For $\epsilon \geq 0$ and $f \in L^1_{loc}(G//K)$

$$(M^G_{P, \epsilon} f)(g) = \sup_{0 < r < \infty} (1 + r)^{-\epsilon}|(f * p_r)(g)| \quad (g \in G),$$

where $p_r(g) = \int_{\mathbb{R}} e^{-(\lambda^2 + r^2)^{1/2}} r \phi_{\lambda}(g) |C(\lambda)|^{-2} d\lambda$.

Theorem 6.11. (1) $M^G_{P, \epsilon}$ is in $(L^p, L^p)$ for $p > 1$ and of weak type $(1,1)$,
(2) $M^G_{P, 0} \in (H^{1/2}_{1/2}, L^1)$,
(3) $M^G_{P, 2} \in (H^1_{1/2}, L^1)$.

Proof. As shown by Stein [9, p.48], it is well-known that $M^G_{P, \epsilon}$ is in $(L^p, L^p)$ for $p > 1$ and of weak type $(1,1)$. Let us recall the calculation in [9, p.49] and modify it slightly: the subordination formula gives an integral representation of $f * p_r$ such as

$$(f * p_r)(g) = \frac{1}{r^2} \int_0^{\infty} (f * h_y)(g) \phi(\frac{y}{r^2}) dy \quad (g \in G),$$

where $\phi(y) = 1/(2\pi^{1/2}) e^{-y/2} y^{-3/2}$, and the integral by part yields the estimate of $(1 + r)^{-2}(f * p_r)(g)$ such as

$$|\frac{1}{(1 + r)^2 r^2}| \int_0^{\infty} \left( \int_0^{y} \frac{1}{1 + s} (f * h_s)(g) ds \right) \frac{d}{dy} \left( (1 + y) \phi(\frac{y}{r^2}) \right) dy |$$

$$\leq \sup_{0 < y < \infty} \frac{1}{y} \int_0^{y} \frac{1}{1 + s} |(f * h_s)(g)| ds \cdot \frac{1}{(1 + r)^2 r^2} \int_0^{\infty} y \frac{d}{dy} \left( (1 + y) \phi(\frac{y}{r^2}) \right) dy$$

$$\leq C(M^G_{H^1, f})(g),$$

where $C = \|y \phi\|_{L^1(\mathbb{R})} + \|(1 + y) \phi\cdot d\phi/dy\|_{L^1(\mathbb{R})}$. Therefore, it follows from Theorem 6.9 that $M^G_{P, 2} \in (H^1_{1/2}, L^1)$. Let $M^G_{P, 0}$ (resp. $M^G_{P, 1}$) denote the maximal operator defined by replacing $\sup_{0 < r < \infty}$ in the definition of $M^G_{P, 0}$ by $\sup_{0 \leq r \leq 1}$ (resp. $\sup_{1 < r < \infty}$). Obviously, $M^G_{P, 0}$ is dominated by $cM^G_{P, 2}$ and hence, $M^G_{P, 0} \in (H^1_{1/2}, L^1)$ as remarked above. Especially, $M^G_{P, 0} \in (H^1_{1/2}, L^1)$ by Theorem 4.4. On the other hand, we observe from the proof of [1, Corollary 6.3] that $\|M^G_{P, 0} f\|_{L^1(G)} \leq c\|T^G_{1/\sqrt{p}} f\|_{L^1(G)}$, where $p(\lambda) = \lambda^2 + \rho^2$ and $T^G_{1/\sqrt{p}}$ is the Fourier multiplier on $G$ defined by Definition 7.1 below. Then, since $p(\lambda + i\rho)^{-1/2} (|\lambda|/1 + |\lambda|)^{1/2}$ satisfies the Hörmander condition, Theorem 7.2 below yields that $T^G_{1/\sqrt{p}} \in (H^1_{1/2}, L^1)$ and hence, $M^G_{P, 0} \in (H^1_{1/2}, L^1)$. □

7. Other operators.

A. We shall define the Fourier multiplier on $G$ corresponding to an even bounded function $m(\lambda)$ on $\mathcal{F}$. 

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Definition 7.1. For $f \in L^1_{loc}(G//K)$

$$(T_m^G f)(g) = \int_{\mathcal{F}} \hat{f}(\lambda) m(\lambda) |C(\lambda)|^{-2} d\lambda \quad (g \in G).$$

We easily see that $T_m^G \in (L^2, L^2)$ by the Plancherel formula on $G$. We now suppose that $m(\lambda)$ has a holomorphic extension on $\mathcal{F}(\rho)$ and there exists $s \geq 0$ such that

$$\sup_{0 \leq \xi \leq \rho} \|m_{\xi,s}\| < \infty, \quad \text{where} \quad m_{\xi,s}(\lambda) = m(\lambda + i\xi) \left( \frac{|\lambda + i(\xi - \rho)|}{1 + |\lambda + i(\xi - \rho)|} \right)^s.$$

Then by the same manipulation as in (10) we see that for $f \in \mathcal{L}(G//K)$, $T_m^G f$ has the expansion:

$$(T_m^G f)(x) = e^{-2\rho x} \sum_{n=0}^{\infty} e^{-2n x} T_{m_{n,0}} \phi_n(x) \quad (x \in \mathbb{R}_+),$$

$$= (T_{\Phi,D} T_{m_{n,0}})(Q_{\xi} \phi_n(x)D(x)^{-1},$$

where $T_w$ is the Fourier multiplier on $\mathbb{R}$ defined by $(T_w F)^\sim (\lambda) = w(\lambda) F^\sim (\lambda) \ (\lambda \in \mathbb{R})$ and $T_{\Phi,D}$ is the pseudo-differential operator given in Remark 3.2. Since $T_{\Phi,D}$ is a bounded operator of $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$, we can obtain from Definition 4.3 and Theorem 4.5 that

Theorem 7.2. Let $m(\lambda)$ be an even holomorphic function on $\mathcal{F}(\rho)$. If there exists $s \geq 0$ such that $\sup_{0 \leq \xi \leq \rho} \|m_{\xi,s}\| < \infty$ and $T_{m_{n,0}}$ is bounded on $H^1(\mathbb{R})$, then $T_m^G \in (H^1, L^1)$.

B. We last treat the Riesz transform on $G$, and we henceforth use the standard notation as in [9]. Especially, we denote the Laplacian on $G$ by $\Delta$ and the covariant differentiation on $G$ by $\nabla$. We put $|\nabla|^2(h) = -2\Delta h \cdot h$ for $h \in C^\infty(G)$. Then the Riesz transform $R^G$ on $G$ is given as follows:

Definition 7.3. For $f \in L^1_{loc}(G//K)$

$$(R^G f)(g) = (|\nabla|(-\Delta)^{-1/2})(f)(g) \quad (g \in G).$$

$R^G f$ is a $K$-biinvariant function on $G$. Indeed, $(-\Delta)^{-1/2} = T_{1/\sqrt{\rho}}$ where $p(\lambda) = \lambda^2 + \rho^2$ and we notice that $h \in C^\infty(G)$, $|\nabla h|^2(g) = \sum_{i=1}^{n} |X_i h|^2(g) \ (g \in G)$, here $\{X_i; 1 \leq i \leq n\}$ is denoted as an orthonormal basis of the Lie algebra $\mathfrak{g}$ of $G$ and each $X_i$ is regarded as a left (or right) invariant differential operator on $G$, and furthermore, if $h$ is $K$-biinvariant on $G$, $|\nabla h|^2(g)$ is simply expressed as $c((d/dx)h(a_x))^2$ provided $\sigma(g) = x$ (cf. [4, §2]).
Theorem 7.4. (1) $R^G$ is in $(L^p, L^p)$ for $p > 1$ and of weak type $(1, 1)$, 
(2) $R^G \in (H^1_{1/2}, L^1)$.

Proof. (1) is well-known (see [1, Corollary 5.2]). As for (2) we may assume that $f \in H^1_{1/2}(G//K) \cap \mathcal{L}(G//K)$ and then, we have

$$((-\Delta)^{-1/2} f)(x) = e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2m_x} (T_{1/\sqrt{\rho}} \Gamma_1 C_+^1 F_j^1)(x) \quad (x \in R_+).$$

As remarked after Definition 7.3, in order to obtain the estimate of $R^G f$ it suffices to calculate the action of $d/dx$ on the right side of the above equation. Actually, $d/dx$ acts on $e^{-2\rho x}$, $e^{-2m_x}$, and $(T_{1/\sqrt{\rho}} \Gamma_1 C_+^1 F_j^1)(x)$, so we denote the results by $(I_1 f)(x)$, $(I_2 f)(x)$, and $(I_3 f)(x)$ respectively. Since $P(\lambda + i \rho)^{-1/2}(|\lambda|/1 + |\lambda|)^{1/2}$ satisfies the Hörmander condition, it follows from Theorem 7.2 that $\|I_1 f \cdot D\|_{L^1(R_+)} \leq c \|f\|_{H^1_{1/2}(G)}$.

Similarly, since $(d/dx) (T_{1/\sqrt{\rho}} \Gamma_1 C_+^1 F_j^1)(x) = (T_{1/\sqrt{\rho}} \Gamma_1 C_+^1 F_j^1)(x)$ and $p(\lambda + i \rho)^{-1/2}(i \lambda)$ satisfies the Hörmander condition, we have $\|I_3 f \cdot D\|_{L^1(R_+)} \leq c \|f\|_{H^1_{1/2}(G)}$.

In obtaining the estimate for $I_2 f$, we rewrite it as follows:

$$(I_2 f)(x) = e^{-2\rho x} \sum_{m=1}^{\infty} (-2m) e^{-2m_x} (\gamma_m^1 Q_{1/2} C_+^1 F_j^1)(x) \quad (x \in R_+),$$

where $\gamma_m^1$ is the Fourier multiplier on $R$ defined by

$$(\gamma_m^1 F)^\sim(\lambda) = \Gamma_m(\lambda + i \rho)p(\lambda + i \rho)^{-1/2} \left(\frac{|\lambda|}{1 + |\lambda|} \right)^{1/2} F^\sim(\lambda) \quad (\lambda \in R).$$

We here observe from (25), (26) and Lemma 8.1 below that for $M = 0, 1$,

$$|\left(\frac{d}{d\lambda}\right)^M \Gamma_m(\lambda + i \rho) | \leq c m^{2\alpha+1}(1 + |\lambda|)^{-M} \frac{\rho|2\rho - i\lambda|}{m|\lambda - \rho - i\lambda|}$$

and hence,

$$\left|\left(\frac{d}{d\lambda}\right)^M \left(\Gamma_m(\lambda + i \rho)p(\lambda + i \rho)^{-1/2} \cdot \left(\frac{|\lambda|}{1 + |\lambda|} \right)^{1/2} \right)\right| \leq c m^{2\alpha-1}|\lambda|^{-M}.$$ 

Therefore, $\Gamma_m(\lambda + i \rho)p(\lambda + i \rho)^{-1/2} \cdot (|\lambda|/1 + |\lambda|)^{1/2}$ satisfies the Hörmander condition with the constant $c m^{4\alpha-2}$. We now repeat the proof of Lemma 3.1 when $T$ is the identity operator, in which case we replace $\Gamma_m$ and $F$ by $\gamma_m^1$ and $Q_{1/2} C_+^1 F_j^1$ respectively. Fortunately, the extra order of $m$ that appears in the differentiation of $e^{-2m_x}$ is canceled, because the operator norm $c m^{2\alpha}$ of $\Gamma_m^1$ is changed to $c m^{2\alpha-1}$ of $\gamma_m^1$. So the exactly same proof of Lemma 3.1 is valid in this case and it follows that $\|I_2 f \cdot \Delta\|_{L^1(R_+)} \leq c \|Q_{1/2} C_+^1 F_j^1\|_{H^1(R)} = c \|f\|_{H^1_{1/2}(G)}$. This completes the proof of (2). \hfill \Box

Remark 7.5. Theorem 6.11(2) also follows from Theorem 7.4(2) and [1, Corollary 6.3].
Remark 7.6. Let $\mathcal{R} = T_{i\lambda/|\lambda|}$ be the Riesz transform on $\mathbb{R}$ and $Q$ the Fourier multiplier on $\mathbb{R}$ defined by

$$(QF)^{\sim}(\lambda) = \left(\frac{\lambda^2 + 4\rho^2}{\lambda^2}\right)^{1/4} e^{i2\tan^{-1}(2\rho/\lambda)} \cdot \frac{\lambda}{|\lambda|} \cdot F^{\sim}(\lambda) \quad (\lambda \in \mathbb{R}).$$

We observe that $e^{i2\tan^{-1}(2\rho/\lambda)} \cdot \lambda/|\lambda|$: $(\lambda^2 + 4\rho^2)^{-1/4} (1 + |\lambda|)^{1/2}$ satisfies the Hörmander condition and hence, if $QF \in H^1(\mathbb{R})$, then $Q_{1/2}F \in H^1(\mathbb{R})$. Especially, Definition 4.3 and Theorem 7.4(2) yield that if $f \in H^1_{1/2}(G//K)$, then

$$\|\mathcal{R}^G f\|_{L^1(G)} + \|QC^1_+ F^1_f\|_{L^1(\mathbb{R})} \leq c \|QC^1_+ F^1_f\|_{H^1(\mathbb{R})}.$$  

(23)

We now suppose that $f \in L^1_{loc}(G//K)$ satisfies $\|\mathcal{R}^G f\|_{L^1(G)} + \|QF^1_f\|_{L^1(\mathbb{R})} < \infty$. If we take $X_1$ in $\mathfrak{a}$ in the remark after Definition 7.3, we see that $|\nabla h|(a_xg) \geq |(d/dx)h(a_xg)|$ ($g \in G$) for $h \in C^\infty(G)$, and thereby,

$$\|\mathcal{R}^G f\|_{L^1(G)} = \int_A \int_N |(\mathcal{R}^G f)(a_xn)| e^{2\rho x} dn dx \geq \int_A \int_N \frac{d}{dx}((-\Delta)^{-1/2} f)(a_xn)| e^{2\rho x} dn dx$$

$$\geq \int_A \frac{d}{dx} \int_N ((-\Delta)^{-1/2} f)(a_xn) dn \cdot e^{2\rho x} dx$$

$$= \|\frac{d}{dx} F^1_{1/2} f \cdot e^{2\rho x}\|_{L^1(\mathbb{R})} = \|\mathcal{R}QF^1_f\|_{L^1(\mathbb{R})},$$

because $(\lambda + 2i\rho)(\lambda^2 + 4\rho^2)^{-1/2} = e^{i2\tan^{-1}(2\rho/\lambda)} \cdot (\lambda^2 + 4\rho^2/\lambda^2)^{1/4}$. Therefore, the assumption on $f$ yields $QF^1_f \in H^1(\mathbb{R})$ and

$$\|QF^1_f\|_{H^1(\mathbb{R})} \leq c(\|\mathcal{R}^G f\|_{L^1(G)} + \|QF^1_f\|_{L^1(\mathbb{R})}).$$  

(24)

If we let $\rho \to 0$ formally, then $F^1_f$ goes to an even function $f$ on $\mathbb{R}$, $\mathcal{R}^G$ to $\mathcal{R}$ on $\mathbb{R}$, and $C^1_+$, $Q$ to the identity operators on $\mathbb{R}$, so from (33) and (44) we can recover the classical result: $\|f\|_{H^1(\mathbb{R})} \sim \|\mathcal{R} f\|_{L^1(\mathbb{R})} + \|f\|_{L^1(\mathbb{R})}$ (cf. [10], p. 233).

We last restrict our attention to functions $f \in \mathcal{L}(G//K)$ such that $QC^1_+ F^1_f \in L^1(\mathbb{R})$ (see §2 and Remark 7.6), and we define

$$\mathcal{L}^1(G//K) = \{f \in \mathcal{L}(G//K); \|f\|_{\mathcal{L}^1(G)} = \|f\|_{\mathcal{L}(G)} + \|QC^1_+ F^1_f\|_{L^1(\mathbb{R})} < \infty\}$$

and

$$\mathcal{H}^1(G//K) = \{f \in \mathcal{L}(G//K); \|f\|_{\mathcal{H}^1(G)} = \|f\|_{\mathcal{L}(G)} + \|QC^1_+ F^1_f\|_{H^1(\mathbb{R})} < \infty\}.$$
Obviously, $\mathcal{H}^1(G//K) \subset \mathcal{L}^1(G//K)$ and $\|R^G f\|_{L^1(G)} + \|f\|_{\mathcal{L}^1(G)} \leq c \|f\|_{\mathcal{H}^1(G)}$ (see (23)). We now suppose that $f \in \mathcal{L}^1(G//K)$ and we recall the proof of Theorem 7.4(2). Especially, $|(R^G f)(x)| = |\sum_{k=1}^3 \langle I_k f \rangle (x)\rangle (x \in \mathbb{R}_+)$ where

$$(I_1 f)(x) + (I_3 f)(x) = e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2m x} \left( \frac{T(i\lambda + 2i x)}{(\lambda^2 + 2i \rho \lambda)^{1/2}} \Gamma_1^1 C^1_+ F^1_j \right)(x) \quad (x \in \mathbb{R}_+)$$

$$= e^{-2\rho x} \sum_{m=0}^{\infty} e^{-2m x} \left( \Gamma_1^1 R Q C^1_+ F^1_j \right)(x),$$

$$I_2 f(x) = e^{-2\rho x} \sum_{m=1}^{\infty} (-2m) e^{-2m x} \left( \gamma_m^1 Q_1^{1/2} C^1_+ F^1_j \right)(x).$$

Therefore, we can deduce that for $x \in \mathbb{R}_+$

$$|(R Q C^1_+ F^1_j)(x)| \leq e^{2\rho x} |(R^G f)(x)| + \sum_{m=1}^{\infty} e^{-2m x} \left| \left( \Gamma_1^1 R Q C^1_+ F^1_j \right)(x) \right|$$

$$+ \sum_{m=1}^{\infty} 2mc^{-2m x} \left| \left( \gamma_m^1 Q_1^{1/2} C^1_+ F^1_j \right)(x) \right|.$$ 

Since $\|(Q C^1_+ F^1_j)^\sim\|_{\mathcal{L}^1(\mathbb{R})} \leq \|(Q C^1_+ F^1_j)\|_{L^1(\mathbb{R})} \leq \|f\|_{\mathcal{L}^1(D)}$ and $\|(C^1_+ F^1_j)^\sim\|_{L^1(\mathbb{R})} \leq \|f\|_{\mathcal{L}^1(G)}$, it follows that $|(\Gamma_1^1 R Q C^1_+ F^1_j)(x)| \leq cm^{2\alpha} \|f\|_{\mathcal{L}^1(G)}$ and similarly, $|\left( \gamma_m^1 Q_1^{1/2} C^1_+ F^1_j \right)(x)| \leq cm^{2\alpha - 1} \|f\|_{\mathcal{L}^1(G)}$. Hence we have,

$$\int_1^{\infty} |(R Q C^1_+ F^1_j)(x)| dx \leq c \int_1^{\infty} |(R^G f)(x)| D(x) dx + \|f\|_{\mathcal{L}^1(D)} \sum_{m=1}^{\infty} m^{2\alpha} \int_1^{\infty} e^{-2m x} dx$$

$$\leq c \left( \|R^G f\|_{L^1(G)} + \|f\|_{\mathcal{L}^1(G)} \right).$$

On the other hand, since

$$|(R Q C^1_+ F^1_j)(x)| = \left| \int_{\mathbb{R}} \frac{i(\lambda + 2i \rho)}{(\lambda^2 + 2i \rho \lambda)^{1/2}} f(\lambda + i \rho) C(-\lambda - i \rho)^{-1} e^{i \lambda x} d\lambda \right| \quad (x \in \mathbb{R})$$

$$= e^{\rho x} \left| \int_{\mathbb{R}} \frac{i(\lambda + i \rho)}{(\lambda^2 + \rho^2)^{1/2}} f(\lambda) C(-\lambda)^{-1} e^{i \lambda x} d\lambda \right|$$

$$\leq c \|f\|_{\mathcal{L}^1(D)} e^{\rho x},$$

it follows that $\int_1^{\infty} |(R Q C^1_+ F^1_j)(x)| dx \leq c \|f\|_{\mathcal{L}^1(D)}$. Then, we can deduce that $\|R Q C^1_+ F^1_j\|_{L^1(\mathbb{R})} \leq c \|R^G f\|_{L^1(G)} + \|f\|_{\mathcal{L}^1(G)}$. In particular, $\|f\|_{\mathcal{H}^1(G)} = \|f\|_{\mathcal{L}^1(G)} + \|Q C^1_+ F^1_j\|_{H^1(\mathbb{R})} \leq c \|f\|_{\mathcal{L}^1(G)} + \|Q C^1_+ F^1_j\|_{L^1(\mathbb{R})} \leq c \|R^G f\|_{L^1(G)} + \|f\|_{\mathcal{L}^1(G)}$. We have therefore proved the following.
Theorem 7.7. $f \in \mathcal{H}^1(G//K)$ if and only if $f \in \mathcal{L}^1(G//K)$ and $\mathcal{R}^G \in L^1(G//K)$. Especially,

$$\|f\|_{\mathcal{H}^1(G)} \sim \|\mathcal{R}^G f\|_{L^1(G)} + \|f\|_{\mathcal{C}^1(G)}.$$

8. Appendix. We shall obtain an estimate for the derivatives of $\Gamma_m(\lambda)$ (see (6)), which yields the Hörmander condition of $\Gamma_m$. In what follows we denote $\Gamma_m$ by $\Gamma_{2m}$ and refer to the notation and the proof of Lemma 7 in Flensted-Jensen [4]. Actually, $\Gamma_m$ $(m \in \mathbb{N})$ is recurrently defined by $\Gamma_0 = 1$, $\Gamma_{2n+1} = 0$ $(n \in \mathbb{N})$ and

$$4m(m - i\lambda)\Gamma_{2m}(\lambda) = \sum_{k=0}^{m-1} (2k - i\lambda + \rho)4((\alpha - \beta) + \delta_k^m(2\beta + 1)) \Gamma_{2k}(\lambda),$$

where $\delta_k^m = 0$ for $k \equiv m + 1$(mod 2), $\delta_k^m = 1$ for $k \equiv n$(mod 2). For each $m \in \mathbb{N}$ we put

$$C_k(\lambda) = 4k(k - i\lambda) \quad \text{and} \quad R_k(\lambda) = 4\theta(2k - i\lambda + \rho),$$

where $\theta = \alpha - \beta$ if $k \equiv m + 1$ and $\theta = \rho$ if $k \equiv m$. Then it follows that

$$\Gamma_{2m}(\lambda) = C_m(\lambda)^{-1} \sum_{k=0}^{m-1} R_k(\lambda) \Gamma_{2k}(\lambda)$$

$$= \frac{R_0(\lambda)}{C_m(\lambda)} \prod_{k=1}^{m-1} \left(1 + \frac{R_k(\lambda)}{C_k(\lambda)}\right).$$

Therefore, if we put $c_k(\lambda) = |C_k(\lambda)|$, $r_k(\lambda) = |R_k(\lambda)|$ and

$$b_0(\lambda) = 1 \quad \text{and} \quad b_m(\lambda) = c_m(\lambda)^{-1} \prod_{k=0}^{m-1} b_k(\lambda)c_k(\lambda) \quad \text{for } m \geq 1,$$

we easily see that

$$|\Gamma_{2m}(\lambda)| \leq b_m(\lambda) = \frac{\rho |\rho - i\lambda|}{m |m - i\lambda|} \prod_{k=1}^{m-1} \left(1 + \frac{r_k(\lambda)}{c_k(\lambda)}\right).$$

(25)

Lemma 8.1. Let $\lambda = \xi + i\eta$ and suppose that $\eta \geq \rho$. Then there exists a positive constant $c$ such that for all $m \in \mathbb{N}$

$$\prod_{k=1}^{m-1} \left(1 + \frac{r_k(\lambda)}{c_k(\lambda)}\right) \leq cm^{2\alpha + 1}.$$

Proof. Since $\eta \geq \rho$, we have $|2k + \rho + \eta - i\xi|/|k + \eta - i\xi| \leq 2$ and thereby,

$$\left(\frac{kr_k(\lambda)}{c_k(\lambda)}\right)^2 \leq (2\theta)^2.$$
Then it follows that
\[
\prod_{k=1}^{m-1} \left(1 + \frac{r_k(\lambda)}{c_k(\lambda)}\right) \leq \exp \left(\sum_{k=1}^{m-1} \frac{r_k(\lambda)}{c_k(\lambda)}\right)
\leq \exp \left(\rho \sum_{1 \leq k \leq m-1} 2k^{-1} + (\alpha - \beta) \sum_{k=m+1}^{m-1} 2k^{-1}\right)
\leq c \exp \left((\rho + \alpha - \beta) \log m\right)
\leq cm^2 \alpha + 1
\]

**Lemma 8.2.** Let \( \Im(\lambda) \geq 0 \) and fix \( M \in \mathbb{N} \). Then there exists a positive constant \( c \) such that for all \( m \in \mathbb{N} \)

1. \( \left| \frac{d}{d\lambda} \right|^M \frac{R_0(\lambda)}{C_m(\lambda)} \leq c \frac{r_0(\lambda)}{c_m(\lambda)} \frac{1}{(1 + |\lambda|)^M} \),

2. \( \left| \frac{d}{d\lambda} \right|^M \left(1 + \frac{R_k(\lambda)}{C_k(\lambda)}\right) \leq c \left(1 + \frac{r_k(\lambda)}{c_k(\lambda)}\right) \frac{1}{(k + |\lambda|)^M} \quad (M \geq 1). \)

**Proof.**

(1)
\[
\left| \frac{d}{d\lambda} \right|^M \frac{R_0(\lambda)}{C_m(\lambda)} = c \left| \frac{m - \rho}{C_m(\lambda)(m - i\lambda)^M} \right|
\leq c \frac{r_0(\lambda)}{c_m(\lambda)} \left| \frac{m - \rho}{(\rho - i\lambda)(m - i\lambda)^M} \right|
\leq c \frac{r_0(\lambda)}{c_m(\lambda)} \frac{1}{(1 + |\lambda|)^M}.
\]

(2) For \( M \geq 1 \)
\[
\left| \frac{d}{d\lambda} \right|^M \left(1 + \frac{R_k(\lambda)}{C_k(\lambda)}\right) = c \left(1 + \frac{r_k(\lambda)}{c_k(\lambda)}\right) \frac{k - \rho}{(k + \theta(2k + \rho) - (k + \theta)i\lambda)(k - i\lambda)^M}
\leq c \left(1 + \frac{r_k(\lambda)}{c_k(\lambda)}\right) \frac{1}{(k + |\lambda|)^M+1}.
\]

We now suppose that \( \eta = \Im(\lambda) \geq 0 \) and observe by Lemma 8.2 that
\[
\left| \frac{d}{d\lambda} \Gamma_2^m(\lambda) \right| = \left| \frac{d}{d\lambda} \right|^M \frac{R_0(\lambda)}{C_m(\lambda)} \prod_{k=1}^{m-1} \left(1 + \frac{R_k(\lambda)}{C_k(\lambda)}\right) + \frac{R_0(\lambda)}{C_m(\lambda)} \left(1 + \frac{R_k(\lambda)}{C_k(\lambda)}\right) \prod_{k=1}^{m-1} \left(1 + \frac{R_k(\lambda)}{C_k(\lambda)}\right)
\leq c b_m(\lambda) \left(1 + |\lambda|^{-1} + \sum_{j=0}^{m-2} (1 + j + |\lambda|)^{-2}\right)
\leq c b_m(\lambda)(1 + |\lambda|)^{-1}.
\]

Hence, by Lemma 8.1 and the above estimate we can deduce the following,
Proposition 8.3. Let $M = 0, 1$. If $\Im(\lambda) \geq \rho$, then there exists a positive constant $c$ such that for all $m \in \mathbb{N}$

$$|(\frac{d}{d\lambda})^M \Gamma_{2m}(\lambda)| \leq cm^{2\alpha}(1 + |\lambda|)^{-M}.$$ 

Corollary 8.4. Suppose that $\eta \geq \rho$. Then there exists a positive constant $c$ such that for all $m \in \mathbb{N}$

$$R^{2M-1} \int_{R < |\xi| \leq 2R} |(\frac{d}{d\lambda})^M \Gamma_{2m}(\xi + i\eta)|^2 d\xi \leq cm^{4\alpha}$$

for $M = 0, 1$ and $R > 0$.

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