$KA$-wavelets on semisimple Lie groups 
and quasi-orthogonality of matrix coefficients 

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§1 Introduction.

First we brief the history of continuous wavelet transforms. Originally the (continuous) wavelet transform, introduced by Morlet around 1980, was the following one. We denote by $H^2(\mathbb{R})$ the closed subspace of $L^2(\mathbb{R})$ consisting of all $L^2$ functions $f$ on $\mathbb{R}$ with $\text{supp}(\hat{f}) \subset [0, \infty)$, and we fix $\psi \in H^2(\mathbb{R})$ satisfying the so-called admissible condition

$$c_\psi = \int_0^\infty \frac{\left|\hat{\psi}(\lambda)\right|^2}{\lambda} d\lambda < \infty.$$ 

Then the wavelet transform $W_\psi$ associated to $\psi$ is defined on $H^2(\mathbb{R})$ as

$$W_\psi f(u, v) = \int_{-\infty}^{\infty} f(x)e^{-u/2}\tilde{\psi}(e^{-u}x + v)dx \quad (u, v \in \mathbb{R}).$$

**Theorem 1.1.** $W_\psi$ is an isometric isomorphism from $H^2(\mathbb{R})$ onto $L^2(\mathbb{R}^2)$: For any $f \in H^2(\mathbb{R})$

$$\|f\|^2 = \frac{1}{c_\psi} \|W_\psi f\|^2.$$ 

Furthermore, for any $f \in H^2(\mathbb{R})$ and $x \in \mathbb{R}$ at which $f$ is continuous,

$$f(x) = \frac{1}{c_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(u, v)e^{-u/2}\tilde{\psi}(e^{-u}x + v)dudv.$$

In [GMP] Grossmann-Morlet-Paul pointed out the group-theoretical interpretation of the wavelet transform $W_\psi$. Let $G$ be the affine group $\mathbb{R}^2$ with multiplication law:

$$(u, v)(u', v') = (u + u', e^{-u}v + v'),$$
and let \((T, \mathcal{H}^2(\mathbb{R}))\) be an irreducible unitary representation of \(G\) defined by
\[
(T(u,v)f)(x) = e^{-u/2}f(e^{-u}x + v) \quad (f \in \mathcal{H}^2(\mathbb{R})).
\]
In this scheme \(W_\psi\) can be rewritten as
\[
W_\psi f(u,v) = \langle f, T(u,v)\psi \rangle,
\]
where \(\langle \cdot, \cdot \rangle\) is the inner product of \(\mathcal{H}^2(\mathbb{R})\). Furthermore, since \(dudv\) is a left invariant Haar measure on \(G\), Theorem 1.1 yields the square-integrability and the orthogonality of the matrix coefficients \(\langle f, T(u,v)\psi \rangle\) of \(T\) on \(G\). In this sense the theory of the continuous wavelet transform \(W_\psi\) on \(\mathcal{H}^2(\mathbb{R})\) is nothing but the one of the square-integrable representation \((T, \mathcal{H}^2(\mathbb{R}))\) of \(G\).

General theory of square-integrable representations of locally compact groups has been investigated by various mathematicians; Weyl [W] for compact groups, Godement [G] for unimodular locally compact groups, and Duflo-Moore [DM] for general locally compact groups. Explicit theory based on the construction of the square-integrable representations was obtained by Harish-Chandra [HC] for semisimple Lie groups and by Moore-Wolf [MW] for nilpotent groups.

How to extend the theory of square-integrable representations of locally compact groups \(G\)? One of the ways is to replace the square-integrability on \(G\) by the one on a quotient space \(G/H\) for a closed subgroup \(H\) of \(G\). More generally, find a representation \((T, \mathcal{H})\) of \(G\), a measurable subset \((S, ds)\) of \(G\), and \(\psi \in \mathcal{H}\) for which, for any \(f \in \mathcal{H}\)
\[
\tag{*}
\|f\|^2 = \frac{1}{c_{S,\psi}} \int_S |\langle f, T(s)\psi \rangle|^2 ds.
\]
Then, it is easy to see that the transform defined by \(\langle f, T(s)\psi \rangle\) is an isometric isomorphism from \(\mathcal{H}\) onto \(L^2(S, ds)\), and each \(f \in \mathcal{H}\) has an \(L^2\) decomposition in the weak sense:
\[
f = \frac{1}{c_{S,\psi}} \int_S \langle f, T(s)\psi \rangle T(s)\psi ds.
\]
For the last decade researches has been done in this scheme and many wavelet transforms has been constructed on locally compact groups, for example, on \(\mathbb{R}^*_+ \times SO(n)\) by Murenzi [M], on \(\mathbb{R}^*_+ \times SO(1, n)\) by A-J. Unterberger [U], on \(\mathbb{R}^*_+ \times SO(1, n) \times \mathbb{R}^{n+1}\) by Bhonke [B], on \(S \times V\), \(V\) is a vector space and \(S\) is
a subgroup of $GL(V)$, by De Bièvre [DB], on $SO(2,1) \times \mathbb{R}^3$ by Ali, Antoine, Gazeau [AAG], on $\mathbb{R}^*_+ \times SO(n) \times H_n$ by Kalisa-Toréssani [KT], Toréssani [T1,2], on $GL(n, \mathbb{R})$ by Bernier-Taylor [BT], on $SO(2,1)$ by Wu-Zhong [WZ], and on Iwasawa $AN$ groups by Kawazoe [K3] and Liu [L].

In this paper we shall consider the case that $G$ is a semisimple Lie group and $S = KA$, where $K$ and $A$ are respectively the maximal compact and abelian subgroups of $G$. More precisely, let $G$ be a semisimple Lie group with finite center and $G = KAK$ the Cartan decomposition of $G$. $dg$ denotes a Haar measure on $G$ and $dg = D(a)dk adk$ the corresponding decomposition of $dg$. Then we take $S = KA$ and $ds = D(a)dk da$ in the above scheme, and we try to find a representation $(T, \mathcal{H})$ of $G$ and $\psi \in \mathcal{H}$ satisfying ($\star$). Unfortunately, the condition ($\star$) is very strong, so I feel that we have no answer for $T$ and $\psi$. Therefore, we shall consider a weak condition; there exist constants $0 < C_1, C_2 < \infty$ such that

$$\tag{**} C_1 \| f \|^2 \leq \int_{\mathcal{H}} |\langle f, T(s)\psi \rangle|^2 ds \leq C_2 \| f \|^2$$

and we shall obtain a sufficient condition on $\psi$ for which $\langle f, T(s)\psi \rangle$ satisfies (**). In §4 we shall treat the case of $G = SU(1,1)$ and $(T_{1/2}, \mathcal{H}_{1/2})$ the limit of the holomorphic discrete series of $G$. We note that $T_{1/2}$ is not square-integrable on $G$. Then we shall find a $\psi \in \mathcal{H}_{1/2}$ satisfying ($\star$). Moreover, we shall deduce that, if we ignore a finite dimensional subspace of $\mathcal{H}_{1/2}$, then we can find a $\psi \in \mathcal{H}_{1/2}$ satisfying ($\star$) (see Theorem 4.4). In this process we use the facts that some differences of the matrix coefficients of $T_{1/2}$ are square-integrable on $\mathbb{R}$ with respect to $D(a)da$ and moreover, they satisfy a quasi-orthogonality. These facts are summarized in Lemmas 4.1, 4.2, and 4.3.

After the lecture, the author noticed that J.-P. Antoine and P. Vanderghynst [AV1,2] had the same idea and they obtained an example in the case of $SO(3,1)$.

§2. Notation.

Let $G$ be a semisimple Lie group with finite center and $G = KAN$ the Iwasawa decomposition of $G$. Let $\Sigma$ be the set of roots for $(G, A)$ and $\Sigma^+$ the one of positive roots corresponding to $N$. Let $A^+$ denote the closed positive Weyl chamber in $A$ and $G = KA^+K$ the Cartan decomposition of $G$. Let
$dg$ denote a Haar measure on $G$, and $dk$, $da$, and $dn$ ones for $K$, $A$, and $N$ respectively. We normalize $dk$ as $\int_K dk = 1$. According to the Iwasawa and Cartan decompositions of $G$, there are decompositions of $dg$ such that

$$dg = e^{\rho \log a} dk da dn = D(a) dk da dk',$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ and

$$D(a) = \prod_{\alpha \in \Sigma^+} \left( \sinh \alpha \log a \right)^{m_\alpha},$$

$m_\alpha$ stands for the multiplicity of $\alpha$.

§3. $KA$-wavelets.

Let $(T, \mathcal{H})$ be a unitary representation of $G$ and

$$\mathcal{H} = \bigoplus_{\tau \in \hat{K}} \mathcal{H}_\tau,$$

the $K$-type decomposition of $\mathcal{H}$. In the following argument we assume that

$$[T, \tau] \leq 1,$$

and we denote by $\hat{K}_T$ the set of all $\tau \in \hat{K}$ such that $[T, \tau] = 1$. Then, as a representation of $K$, $(T|_K, \mathcal{H}_\tau)$ is equivalent with $\tau$ for each $\tau \in \hat{K}_T$. We choose a complete orthonormal basis of $\mathcal{H}$ such that

$$\{ e_{\tau}^n; e_{\tau}^n \in \mathcal{H}_\tau, 1 \leq n \leq \dim \tau, \tau \in \hat{K}_T \}$$

and we denote by $I$ the set of the indexes $\{ (\tau, n); 1 \leq n \leq \dim \tau, \tau \in \hat{K}_T \}$. For each $f \in \mathcal{H}$ the Fourier expansion of $f$ is given by

$$f = \sum_{(\tau, n) \in I(f)} f_{n}^\tau e_{n}^\tau,$$

where $f_{n}^\tau = \langle f, e_{n}^\tau \rangle_{\mathcal{H}}$ and $I(f)$ the subset of $I$ consisting of all $(\tau, n)$ such that $f_{n}^\tau \neq 0$. Here we put

$$I_A(f) = \{ (\tau, n); (T(\cdot)f)_{n}^\tau = \langle T(\cdot) f, e_{n}^\tau \rangle \text{ is not identically } 0 \text{ on } A \}.$$
We say that \( \psi \in \mathcal{H} \) is admissible if there exist constants \( 0 < C_1, C_2 < \infty \) such that, if \((\tau, n) \in I_A(\psi),\)

\[
C_1 \leq c_{\psi, \tau, n} = \int_A |\langle T(a)\psi, e_n^* \rangle|^2 D(a) \, da \leq C_2.
\]

We put

\[
\mathcal{H}_\psi = \{ f \in \mathcal{H}; I(f) \subset I_A(\psi) \}.
\]

Then, by using the bounded constants \( c_{\psi, \tau, n} \) we shall define a Fourier multiplier \( M_\psi \) on \( \mathcal{H}_\psi \) as follows. For each \( f = \sum_{(\tau, n) \in I(f)} f_n^* e_n \) in \( \mathcal{H}_\psi \)

\[
M_\psi f = \sum_{(\tau, n) \in I(f)} c_{\psi, \tau, n}^{-1/2} f_n^* e_n.
\]

**Theorem 3.1.** Let \( \psi \) be admissible in \( \mathcal{H} \). Then for any \( f \in \mathcal{H}_\psi \)

\[
(1) \quad C_1 \| f \|^2 \leq \int \int_{K^2} |\langle f, T(ka)\psi \rangle|^2 D(a) \, dk \, da \leq C_2 \| f \|^2,
\]

\[
(2) \quad \| f \|^2 = \int \int_{K^2} |\langle f, M_\psi T(ka)\psi \rangle|^2 D(a) \, dk \, da,
\]

\[
(3) \quad f = \int \int_{K^2} \langle f, M_\psi T(ka)\psi \rangle M_\psi T(ka)\psi \, D(a) \, dk \, da.
\]

**Proof.** We note that

\[
T(k^{-1})f = \sum_{(\tau, n) \in I(f)} f_n^* T(k^{-1}) e_n = \sum_{(\tau, n) \in I(f), (\tau', n') \in I} f_n^* T(k^{-1}) e_n^* e_n'^* e_n'^*.
\]

Then the orthogonality of the matrix coefficients of \( T|_K \) yields that

\[
\int \int_{K^2} |\langle f, T(ka)\psi \rangle|^2 D(a) \, dk \, da
= \int_A \sum_{(\tau, n) \in I(f)} |f_n^*|^2 |\langle T(a)\psi, e_n^* \rangle|^2 D(a) \, da
= \sum_{(\tau, n) \in I(f)} |f_n^*|^2 \left( \int_A |\langle T(a)\psi, e_n^* \rangle|^2 D(a) \, da \right).
\]
Since
\[ \|f\|^2 = \sum_{(\tau, n) \in I(f)} |f_{\tau}^{n}|^2 \quad \text{and} \quad I(f) \subset I_A(\psi), \]
(1) easily follows from the definition of the admissible vector \( \psi \). We replace \( f \) by \( M_v f \) in the above calculation. Then \( |f_{\tau}^{n}|^2 \) in the last equation turns to \( |f_{\tau}^{n}|^2 c_{\psi, \tau, n}^{-1} \) and then, \( c_{\psi, \tau, n}^{-1} \) cancels the integral over \( A \). Thereby (2) follows. As for (3) we put \( \mathcal{H}(f) = \text{Span}\{e_{\tau}^{\gamma}; (\tau, n) \in I(f)\} \) and define an operator \( Q \) on \( \mathcal{H}(f) \) by
\[ h \mapsto \int_{KA} \langle f, M_v T(ka) \psi \rangle \langle h, M_v T(ka) \psi \rangle D(a) \, dk \, da. \]
Then (2) and the Schwarz inequality yield that \( Q \) is bounded and \( \|Q\| \leq \|f\|^2 \), and thereby, there exists \( f_0 \in \mathcal{H}(f) \) such that \( Q(h) = \langle h, f_0 \rangle \) and \( \|f_0\| = \|Q\| \). Since \( Q(f) = \langle f, f_0 \rangle = \|f\|^2 \) by (2), it easily follows that \( f = f_0 \) (cf. [K]). Clearly, \( Q(h) = \langle h, f \rangle \) means (3).

**Remark 3.2.** When \((T, \mathcal{H})\) is an irreducible square-integrable representation of \( G \), it is well-known that each \( \psi \in \mathcal{H} \) is admissible and satisfies
\[ c_{\psi, \tau, n} = d_T^{-1} \|\psi\|^2, \]
where \( c_T \) is the formal degree of \( T \) (cf.[V]). Furthermore, applying the orthogonality of the matrix coefficients on \( G \), we can replace the integrals over \( KA \) in Theorem 3.1 by the ones over \( G \).

§4. Example in \( SU(1, 1) \).

Let \( G \) be \( SU(1, 1) \). Then
\[ K = \{k_\theta = \left( \begin{array}{cc} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{array} \right); 0 \leq \theta < 4\pi \}, \]
\[ A = \{a_t = \left( \begin{array}{cc} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{array} \right); t \in \mathbb{R} \}, \]
and \( A^+ = \{a_t; t > 0 \} \). In what follows we put
\[ x = \tanh t. \]
Let \((T_h, \mathcal{H}_h)\) \((h \in \mathbb{Z}/2, h \geq 1)\) be the holomorphic discrete series of \(G\) realized on the weighted Bergman space \(\mathcal{H}_h\) on the unit disk \(D = G/K\):

\[
\mathcal{H}_h = \{ f : D \to \mathbb{C}; f \text{ is holomorphic on } D \text{ and } \| f \|^2_h = (2h - 1)^{-1} \int_D |f(z)|^2 (1 - |z|^2)^{2(h-1)} \, dz < \infty \},
\]

and \((T_{1/2}, \mathcal{H}_{1/2})\) the limit of holomorphic discrete series of \(G\) realized on the Hardy space \(\mathcal{H}_{1/2}\) on \(D\):

\[
\mathcal{H}_{1/2} = \{ f : D \to \mathbb{C}; f \text{ is holomorphic on } D \text{ and } \| f \|^2_{1/2} = \lim_{h \to 1/2} \| f \|^2_h < \infty \}.
\]

For \(h \in \mathbb{Z}/2, h \geq 1/2\) we denote by \(\langle \cdot, \cdot \rangle_h\) the inner product of \(\mathcal{H}_h\) and we put

\[
e_h^n(z) = \left( (2h + n), (2h), (n + 1) \right)^{1/2} z^n \quad (n \in \mathbb{N}).
\]

Then \(\{e_h^n; n \in \mathbb{N}\}\) is an orthonormal basis of \(\mathcal{H}_h\). For simplicity we denote

\[
\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{1/2} \quad \text{and} \quad e_n(z) = e_{1/2}^n(z) = z^n.
\]

According to this basis the matrix coefficients of \(T_h\) are given as follows (see [Sa]):

\[
\langle T_h(g)e_h^n, e_h^m \rangle_h = e^{i(n \theta + m \varphi)} \langle T_h(a_t)e_h^n, e_h^m \rangle_h \quad (g = kga_kk\varphi)
\]

where for \(n \geq m\),

\[
M(h; n, m; x) = C^h_{n,m} (1 - x^2)^{h} (-x)^{-m-n} F(-m, n + 2h, n - m + 1; x^2),
\]

\[
C^h_{n,m} = \left( (n + 1), (n + 2h), (m + 1), (m + 2h) \right)^{1/2} \frac{1}{(n - m + 1)}
\]

and \(F(a, b, c; x)\) is the hypergeometric function, and for \(m > n\) we change \(n\) and \(m\) by \(m\) and \(n\) respectively. Since

\[
D(a_t)dt = \sinh(2t)dt = \frac{2x}{(1 - x^2)^2} dx,
\]

\[7\]
$M(h; n, m; x)$ $(n, m \in \mathbb{N})$ are square-integrable on $G$ if and only if $h > 1/2$. Here we note that for $n \geq m$,

$$
\lim_{x \to 1} (1 - x^2)^{-h} M(h; n, m; x)
= C_{n,m}^h (-1)^n, (1 - m + n), (m + 2h) \over (2h), (n + 1)
= (-1)^n \frac{1}{(2h)} \left( (n + 2h), (m + 2h) \right)^{1/2}
= (-1)^n D_{n,m}^h
$$

and for $m > n$, $\lim_{x \to 1} (1 - x^2)^{-h} M(h; n, m; x) = (-1)^m D_{m,n}^h = (-1)^m D_{n,m}^h$.

Then we shall define the normalized matrix coefficients $NM(h; n, m, x)$ as

$$
NM(h; n, m, x) = (D_{n,m}^h)^{-1} M(h; n, m; x)
$$

and the differences of the normalized matrix coefficients $DM(h; n, m; x)$ as

$$
DM(h; n, m; x) = NM(h; n, m; x) - NM(h; n + 2, m; x).
$$

The key lemmas are the following.

**Lemma 4.1.** Let notations be as above. Then

$$
DM(h; n, m; x) = \frac{(1 - x^2)^{1/2}}{x}
$$

\[x \left( \frac{m}{2h} NM(h + 1/2; n, m - 1; x) - \frac{m + 2h}{2h} NM(h + 1/2; n + 1, m; x) \right).\]

**Proof.** We realize $T_h$ on the circle and let $z = e^{i\theta}$ ($0 \leq \theta < 2\pi$) (see [Sa]).

We first note that

$$
(D_{n,m}^h)^{-1} e_n^h = \left( \frac{(2h), (m + 1)}{(m + 2h)} \right)^{1/2} z^n,
$$

$$
(D_{n+2,m}^h)^{-1} e_{n+2}^h = \left( \frac{(2h), (m + 1)}{(m + 2h)} \right)^{1/2} z^{n+2},
$$

8
and moreover,

\[
T_h(a_i)(z^n - z^{n+2}) = \frac{1}{(-z \sinh t/2 + \cosh t/2)^2} \bigg( \frac{z \cosh t/2 - \sinh t/2}{-z \sinh t/2 + \cosh t/2} \bigg)^n \\
\times \left( 1 - \left( \frac{z \cosh t/2 - \sinh t/2}{-z \sinh t/2 + \cosh t/2} \right)^2 \right)
\]

\[
= \frac{1}{(-z \sinh t/2 + \cosh t/2)^2} \bigg( \frac{z \cosh t/2 - \sinh t/2}{-z \sinh t/2 + \cosh t/2} \bigg)^n \\
\times \frac{1 - z^2}{(-z \sinh t/2 + \cosh t/2)^2}
\]

\[
= \frac{1}{\sinh t/2} \frac{1}{(-z \sinh t/2 + \cosh t/2)^{2h+1}} \bigg( \frac{z \cosh t/2 - \sinh t/2}{-z \sinh t/2 + \cosh t/2} \bigg)^n \\
\times \left( - \left( \frac{z \cosh t/2 - \sinh t/2}{-z \sinh t/2 + \cosh t/2} \right) + z \right).
\]

On the other hand, we easily see that

\[
\langle \left( \frac{(2h), (m+1)}{(m+2h)} \right)^{1/2} z^{n+1}, e^h_{m+1} \rangle_h \\
= \langle (D_{n+1,m}^{h+1/2})^{-1} e_{n+1}^{h+1/2}, e^h_{m+1} \rangle_h \\
= \frac{m + 2h}{2h} \langle (D_{n+1,m}^{h+1/2})^{-1} e_{n+1}^{h+1/2}, e^h_{m} \rangle_h^{h+1/2}
\]

and

\[
\langle \left( \frac{(2h), (m+1)}{(m+2h)} \right)^{1/2} z^n, e^h_{m-1} \rangle_h \\
= \langle (D_{n,m-1}^{h+1/2})^{-1} e_{n}^{h+1/2}, e^h_{m-1} \rangle_h \\
= \frac{m}{2h} \langle (D_{n,m-1}^{h+1/2})^{-1} e_{n}^{h+1/2}, e^h_{m-1} \rangle_h^{h+1/2}.
\]

Then the desired result follows.

**Lemma 4.2.** Let notations be as above. Then for each \( n, m \in \mathbb{N} \),

\[
0 < \int_0^1 D M(h; n, m; x)^2 \frac{2x}{(1 - x^2)^2} dx < \infty,
\]

9
and especially, for \( m > n \)

\[
\int_0^1 DM(h; n, m; x)^2 \frac{2x}{(1 - x^2)^2} dx
\]

\[
= \frac{(2h)^2}{2} (n + h + 1) \frac{(m + 1)}{(m + 2h)} \frac{(n + 1)}{(n + 2h + 2)}.
\]

**Proof.** The case of \( m > n \): We note that

\[
\frac{(1 - x^2)^{1/2} m}{2h} \frac{NM(h + 1/2, n, m; x)}{N(M)}
\]

\[
= Ax^{m-n-2}(1 - x^2)^{h+1} G_n(m - n + 2h, m - n; x^2)
\]

and

\[
\frac{(1 - x^2)^{1/2} m + 2h}{2h} \frac{NM(h + 1/2, n + 1, m; x)}{N(M)}
\]

\[
= \frac{m + 2h}{n + 2h + 1} Ax^{m-n-2}(1 - x^2)^{h+1} G_{n+1}(m - n + 2h, m - n; x^2),
\]

where

\[
A = \frac{(2h)}{(n + 2h + 1)}, \frac{(m + 1)}{(m - n)}
\]

and \( G_n(x) = G_n(\alpha, \gamma, x) (\alpha = m-n+2h, \gamma = m-n) \) is the Jacobi polynomial. Hence,

\[
I = \int_0^1 DM(h; n, m; x)^2 \frac{2x}{(1 - x^2)^2} dx
\]

\[
= A^2 \int_0^1 x^{2(m-n-2)}(1 - x^2)^{2h} \left( G_n(x^2) - \frac{m + 2h}{n + 2h + 1} G_{n+1}(x^2) \right)^2 2x dx
\]

\[
= A^2 \int_0^1 x^{\gamma-1}(1 - x)^{\alpha-\gamma} \left( G_n(x) - \frac{m + 2h}{n + 2h + 1} G_{n+1}(x) \right)^2 \frac{dx}{x}.
\]

We here consider the case of \( m > n + 1 \). Then, \( \gamma - 2 = m - n - 2 \geq 0 \). We note that \( G_n^2 = (G_n - 1)G_n + G_n \) and \((G_n - 1)/x \) is the polynomial of
degree \( n - 1 \). So the orthogonality relations for the Jacobi polynomials and the definition of \( G_n(x) \):

\[
G_n(x) = \frac{\gamma}{\gamma + n} x^{1-\gamma} (1 - x)^{\gamma-\alpha} \left( \frac{d}{dx} \right)^n (x^{\gamma+n-1}(1 - x)^{\alpha+n-\gamma})
\]

yield that

\[
\begin{align*}
\int_0^1 x^{\gamma-1} (1 - x)^{\alpha-\gamma} G_n(x)^2 \frac{dx}{x} \\
= \int_0^1 x^{\gamma-1} (1 - x)^{\alpha-\gamma} G_n(x) \frac{dx}{x} \\
= \frac{\gamma}{\gamma + n} x^{1-\gamma} (1 - x)^{\gamma-\alpha} \left( \frac{d}{dx} \right)^n (x^{\gamma+n-1}(1 - x)^{\alpha+n-\gamma})
\end{align*}
\]

and similarly,

\[
\begin{align*}
\int_0^1 x^{\gamma+1} (1 - x)^{\alpha-\gamma} G_{n+1}(x)^2 \frac{dx}{x} \\
= \int_0^1 x^{\gamma+1} (1 - x)^{\alpha-\gamma} G_{n+1}(x) \frac{dx}{x} \\
= \frac{\gamma+1}{\gamma + n+1} x^{1-\gamma} (1 - x)^{\gamma-\alpha} \left( \frac{d}{dx} \right)^n (x^{\gamma+n-1}(1 - x)^{\alpha+n-\gamma})
\end{align*}
\]

Therefore,

\[
I = A^2 B \left( 1 - 2 \frac{n+1}{m} + \frac{n+1}{m} \frac{m+2h+1}{n+2h+1} \right)
\]

and hence, the desired result follows.

In the case of \( m = n+1 \) we note that \( (G_n(x) - G_{n+1}(x))/x \) is a polynomial of degree \( n \) and thus, the integral \( I \) is well-defined. Then the analytic continuation on \( \gamma \), letting \( \gamma \to 1 \) in the previous case, yields the desired formula for \( m = n + 1 \).
The case of $m \leq n$: Since $M(h+1/2; n, m-1; x)$ and $M(h+1/2; n+1, m; x)$ have the term $x^{n-m+1}$ and $n - m + 1 \geq 1$, it easily follows from Lemma 4.1 that the desired integral is positive and finite.

This completes the proof of the lemma.

**Lemma 4.3.** Let notations be as above and suppose that

$$n, m \in 2\mathbb{N} \text{ or } n, m \in 2\mathbb{N} + 1.$$

Then, for $p > n, m$

$$\int_{0}^{1} DM(h; n, p, x) DM(h; m, p, x) \frac{2x}{(1-x^2)^2} dx$$

$$= \delta_{nm}, \frac{(2h)^2}{2(n+h+1)} \frac{(p+1)}{(p+2h)(n+2h+2)}.$$

**Proof.** When $n = m$, it follows from Lemma 4.2. We may suppose that $n > m$ and hence, $n - m \geq 2$ and even. Then, applying the same argument used in the proof of Lemma 4.2, we see that the desired integral equals to

$$\int_{0}^{1} x^{p-n-1+(n-m)/2} \frac{2x}{(1-x^2)^2} dx$$

$$\times \left( G_n(x) - \frac{p+2h}{n+2h+1} G_{n+1}(x) \right) \left( G_m(x) - \frac{p+2h}{m+2h+1} G_{m+1}(x) \right) dx.$$

Since $(n-m)/2$ is integer, $0 \leq (n-m)/2 - 1 \leq n - 1$, and

$$\left( G_m(x) - \frac{p+2h}{m+2h+1} G_{m+1}(x) \right)$$

is a polynomial of degree $m+1 < n$, the orthogonality relations for the Jacobi polynomials yield that the integral equals to 0.

We here note that, if $h = 1/2$, then $D_{n,m}^h = 1$ and hence,

$$DM(1/2; n, m; x) = M(1/2; n, m; x) - M(1/2; n+2, m; x)$$

$$= \langle T_{1/2}(a_t)(e_n - e_{n+2}), e_m \rangle.$$
Therefore, Lemma 4.2 implies that
\[ 0 < \int_A |\langle T_{1/2}(a_t)(e_n - e_{n+2}), e_m \rangle|^2 D(a_t) dt < \infty \]
and for \( m > n \) this integral equals to
\[ \frac{(2n + 3)}{(n + 1)(n + 2)}. \]
Furthermore, these differences \( \langle T_{1/2}(a_t)(e_n - e_{n+2}), e_m \rangle \) satisfy the quasi-orthogonality relations stated in Lemma 4.3 with \( h = 1/2 \). Thereby, as an application of Theorem 3.1, we see the following.

**Theorem 4.4.** Let \( G = SU(1, 1) \) and \( (T_{1/2}, \mathcal{H}_{1/2}) \) the limit of the discrete series of \( G \).

1. Let \( \psi \) be a finite linear combination of \( e_{n+2} - e_n \). Then there exist constants \( 0 < C_1, C_2 < \infty \) such that for any \( f \) in \( \mathcal{H}_{1/2} \)
\[ C_1 \| f \|^2 \leq \int \int_{KA} |\langle f, T_{1/2}(ka_t)\psi \rangle|^2 \sinh 2t \, dk \, dt \leq C_2 \| f \|^2. \]

2. Let
\[ \psi = \sum c_n \left( \frac{(2n + 3)}{(n + 1)(n + 2)} \right)^{-1/2} (e_{n+2} - e_n), \]
where the sum is taken over \( 0 \leq n \leq N, n \in 2\mathbb{N} \) or \( 0 \leq n \leq N, n \in 2\mathbb{N} + 1 \), and let \( \| \psi \|_0^2 = \sum |c_n|^2 \). Then for any \( f \) in the \( L^2 \)-span of \( \{ e_p, p \geq N + 1 \} \),
\[ f(x) = \frac{1}{\| \psi \|_0} \int \int_{KA} \langle f, T_{1/2}(ka_t)\psi \rangle T_{1/2}(ka_t)\psi \sinh 2t \, dk \, dt. \]

**References**


