Uncertainty principles for the Jacobi transform

Takeshi KAWAZOE *

Abstract

We obtain some uncertainty inequalities for the Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$, where we suppose $\alpha, \beta \in \mathbb{R}$ and $\rho = \alpha + \beta + 1 \geq 0$. As in the Euclidean case, analogues of the local and global uncertainty principles hold for $\hat{f}_{\alpha,\beta}$. In this paper, we shall obtain a new type of an uncertainty inequality and its equality condition: When $\beta \leq 0$ or $\beta \leq \alpha$, the L^2-norm of $\hat{f}_{\alpha,\beta}(\lambda)\lambda$ is estimated below by the L^2-norm of $\rho f(x)(\cosh x)^{-1}$. Otherwise, a similar inequality holds. Especially, when $\beta > \alpha + 1$, the discrete part of f appears in the Parseval formula and it influences the inequality. We also apply these uncertainty principles to the spherical Fourier transform on $SU(1,1)$. Then the corresponding uncertainty principle depends, not uniformly on the K-types of f.

1. Introduction. The uncertainty principle on $\mathbb{R}$ says that if a function $f(x)$ is concentrated around $x = 0$, then its Fourier transform $\hat{f}(\lambda)$ cannot be concentrated around $\lambda = 0$ unless f is identically zero. As surveyed in [7] and [9], there are various generalizations of this principle on locally compact groups G; the Heisenberg group, motion groups, and semisimple LIe groups, and so on. In this paper we shall obtain a generalization of this principle for the Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$ (see (7)).

On semisimple Lie groups G the local and global uncertainty principles for the spherical Fourier transform of K-finite functions are obtained in [7]. When the real rank of G equals to one, these inequalities correspond to the ones for the Jacobi transforms with specialized α and β. Hence, the results

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in [7] are easily generalized for the Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$. However, it is not clear how the constants appeared in the inequalities depend on α, β, and moreover, how the discrete part of f (see (10)) contributes the uncertainty principles. Hence in §2 and §3, arguing exactly as in the Euclidean case, we shall give the proofs of local and global uncertainty inequalities for the Jacobi transform (see Theorems 3.1, 3.2, 4.2 and 4.3).

On the Euclidean space $\mathbb{R}$, to figure a concentration of $f(x)$ around $x = 0$, we consider a multiplication of x; $f(x)x$, and similarly, for the Fourier transform side, we do a multiplication of λ; $\hat{f}(\lambda)\lambda$. On the other hand, for the global uncertainty inequality for $\hat{f}_{\alpha,\beta}(\lambda)$ (see Theorem 4.1) these x and λ are respectively replaced by

$$V(x) = \int_0^x \Delta(t)dt \quad \text{and} \quad W(\lambda) = \int_{D(\lambda)} d\nu,$$

where $\Delta(t)$ is the weight function on $\mathbb{R}_+$ (see (2)), $D(\lambda) = \{z \in \mathbb{C}; |z| \leq |\lambda|\}$, and $d\nu$ the Plancherel measure for the Jacobi transform (see (13)). In Theorem 4.2 we modify $V(x)$ and $W(\lambda)$ respectively as

$$V_\delta(x) = \min(V(x), \delta^{-1}) \quad \text{and} \quad w_\alpha(\lambda) = (\lambda^2 + \rho^2)^{\alpha+1}$$

for $\delta > 0$. Furthermore, in §5 we shall give a refinement of Theorem 4.2 by replacing $V_\delta(x)$ as

$$v(x) = \frac{V(x)}{\Delta(x)}.$$

We shall obtain a global uncertainty inequality, which figures concentrations of f and $\hat{f}_{\alpha,\beta}$ by the multiplications of $v(x)$ and $w_{-1/2}(\lambda)$ respectively. Especially, we can obtain the equality condition (see Theorem 5.1). We note that functions satisfy the equality condition are neither Gaussian nor heat kernels for the Jacobi transform (see (21b)). In §6, using these inequalities, we shall consider some uncertainty principles for f and $\hat{f}_{\alpha,\beta}$.

In §7 we shall apply these global uncertainty inequalities for the Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$ to the spherical Fourier transform $\tilde{f}(\lambda)$ on $G = SU(1,1)$. Then we can deduce a uncertainty principle for general functions, not K-finite, on G. As in the Euclidean case, to deduce a non-concentration of $\tilde{f}(\lambda)$ around $\lambda = 0$, a concentration of $f(g)$ around $g = e$ is sufficient (see Theorem 7.1). In particular, we see that this sufficient condition depends on the K-types of f and is not uniform on the K-types (see Remark 7.2).

2. Notation. Let $\alpha, \beta \in \mathbb{C}$, $\Re\alpha > -1$ and $\rho = \alpha+\beta+1$. For $\lambda \in \mathbb{C}$, let $\phi_\lambda(x)$ denote the Jacobi function of the first kind, that is, the unique solution of

$$(L + \lambda^2 + \rho^2)f = 0 \tag{1}$$

satisfying $f(0) = 1$ and $f'(0) = 0$, where $L = \Delta(x)^{-1}\frac{d}{dx}\Big(\Delta(x)\frac{d}{dx}\Big)$ and

$$\Delta(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}. \tag{2}$$

For $\lambda \neq -i, -2i, -3i, \dots$, let $\Phi_\lambda(x)$ denote the Jacobi function of the second kind which satisfies

$$2\pi^{1/2}\Gamma(\alpha+1)^{-1}\phi_\lambda(x) = C(\lambda)\Phi_\lambda(x) + C(-\lambda)\Phi_{-\lambda}(x), \tag{3}$$

where $C(\lambda)$ is Harish-Chandra's C-function (cf. [3, §2]). For convenience, we suppose that $\alpha, \beta \in \mathbb{R}$ and $\rho \geq 0$ in the following. Then the following estimates are well-known (cf. [3, 4]): For $x \geq 0$ and $\lambda \in \mathbb{C}$ with $|\Im\lambda| \leq \rho$

$$|\phi_\lambda(x)| \leq 1, \tag{4}$$

and for each $\delta > 0$ there exist a positive constant K_δ such that for all $x \geq \delta$ and $\lambda \in \mathbb{C}$ with $\Im\lambda \geq 0$

$$|\Phi_\lambda(x)| \leq K_\delta e^{-(\Im\lambda+\rho)x}, \tag{5}$$

where K_δ is independent of α, β, and for each $r > 0$ there exist positive constants $K^1_{r,\alpha}$, $K^2_{r,\alpha}$ such that if $\lambda \in \mathbb{C}$ with $\Im\lambda \geq 0$ is at distance larger than r from the poles of $C(-\lambda)^{-1}$ then

$$K^1_{r,\alpha}2^{-\rho}(\rho + |\lambda|)^{\alpha+1/2} \leq |C(-\lambda)|^{-1} \leq K^2_{r,\alpha}2^{-\rho}(\rho + |\lambda|)^{\alpha+1/2}, \tag{6}$$

where $K^i_{r,\alpha}$, $i = 1, 2$, are independent of β.

Let $L^p(\Delta)$, $1 \leq p < \infty$, denote the space of all p-th integrable functions on $\mathbb{R}_+$ with respect to $\Delta(x)dx$ and $C^\infty_{c,e}(\mathbb{R})$ the space of all even C^∞ functions on $\mathbb{R}$ with compact support. For $f \in C^\infty_{c,e}(\mathbb{R})$, the Jacobi transform $\hat{f}(\lambda)$ is defined as

$$\hat{f}(\lambda) = \frac{\sqrt{2}}{\Gamma(\alpha+1)}\int_0^\infty f(x)\phi_\lambda(x)\Delta(x)dx. \tag{7}$$

Clearly (1) and (4) imply that for $\lambda \in \mathbb{C}$,

$$(Lf)^\wedge(\lambda) = -(\lambda^2 + \rho^2)\hat{f}(\lambda) \tag{8}$$

and for $|\Im\lambda| \leq \rho$,

$$|\hat{f}(\lambda)| \leq \frac{\sqrt{2}}{\Gamma(\alpha+1)}\|f\|_{L^1(\Delta)}. \tag{9}$$

This transform $f \to \hat{f}$ satisfies analogous properties of the classical cosine Fourier transform; the inversion formula, the Paley-Wiener theorem, and the Plancherel formula were obtained in [3, 4]: We set

$$D_{\alpha,\beta} = \{i(\beta - \alpha - 1 - 2m); m = 0, 1, 2, \cdots, \beta - \alpha - 1 - 2m > 0\}.$$

Then the inversion formula is given as follows: For $f \in C^\infty_{c,e}(\mathbb{R})$,

$$\begin{aligned} f(x) &= \frac{\sqrt{2}}{\Gamma(\alpha+1)}\Big(\int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda + \sum_{\mu\in D_{\alpha,\beta}} a_\mu\phi_\mu(x)d(\mu)\Big) \\ &= f_P(x) + {}^\circ f(x), \end{aligned} \tag{10}$$

where $a_\mu = \hat{f}(\mu)$ and $d(\mu) = -2\pi i C(\mu)^{-1}\mathrm{Res}_{\lambda=\mu}C(-\lambda)^{-1}$. We call f_P and ${}^\circ f$ the principal part and the discrete part of f respectively. We note that since $\rho \geq 0$, $|\beta| \leq \alpha + 1$ if $\beta \leq 0$ and hence $D_{\alpha,\beta} = \emptyset$ if $\beta \leq 0$. Moreover, there exists a positive constant K_μ such that

$$|\phi_\mu(x)| \leq K_\mu e^{-(\rho+|\mu|)x},\ x \geq 0 \tag{11}$$

and thereby

$$d(\mu)^{-1} = \frac{2}{\Gamma(\alpha+1)^2}\int_0^\infty |\phi_\mu(x)|^2\Delta(x)dx > 0. \tag{12}$$

We denote by $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\mu\})$ a function on $\mathbb{R}_+ \cup D_{\alpha,\beta}$ defined by

$$\boldsymbol{F}(\nu) = \begin{cases} F(\lambda) & \text{if } \nu = \lambda \in \mathbb{R}_+ \\ a_\mu & \text{if } \nu = \mu \in D_{\alpha,\beta}. \end{cases}$$

We put $\overline{\boldsymbol{F}}(\nu) = (\overline{F(\lambda)}, \{\overline{a_\mu}\})$ and define a product of $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\mu\})$ and $\boldsymbol{G}(\nu) = (G(\lambda), \{b_\mu\})$ as

$$(\boldsymbol{F}\boldsymbol{G})(\nu) = (F(\lambda)G(\lambda), \{a_\mu b_\mu\}).$$

Moreover, for a function $h(\lambda)$ on $\mathbb{C}$, we define a multiplication of h as $h(\nu)\boldsymbol{F}(\nu) = (h(\lambda)F(\lambda), \{h(\mu)a_\mu\})$. Let $d\nu$ denote the measure on $\mathbb{R}_+ \cup D_{\alpha,\beta}$ defined by

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} \boldsymbol{F}(\nu) d\nu = \int_0^\infty F(\lambda)|C(\lambda)|^{-2} d\lambda + \sum_{\mu \in D_{\alpha,\beta}} a_\mu d(\mu). \tag{13}$$

For $f \in C_{c,e}^\infty(\mathbb{R})$, we put

$$\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{\hat{f}(\mu)\}).$$

Then the Parseval formula for the Jacobi transform on $C_{c,e}^\infty(\mathbb{R})$ can be stated as follows (see [4, Theorem 2.4] and cf. [2]): For $f, g \in C_{c,e}^\infty(\mathbb{R})$

$$\int_0^\infty f(x)\overline{g(x)}\Delta(x)dx = \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} \hat{\boldsymbol{f}}(\nu)\overline{\hat{\boldsymbol{g}}(\nu)} d\nu. \tag{14}$$

The map $f \to \hat{\boldsymbol{f}}$, $f \in C_{c,e}^\infty(\mathbb{R})$, can be extended to an isometry between $L^2(\Delta)$ and $L^2(\nu) = L^2(\mathbb{R}_+ \cup D_{\alpha,\beta}, d\nu)$. Actually, each function f in $L^2(\Delta)$ is of the form $f = f_P + {}^\circ f$ (see (10)) and their L^2-norms are given as

$$\int_0^\infty |f_P(x)|^2 \Delta(x) = \int_0^\infty |\hat{f}_P(\lambda)|^2 |C(\lambda)|^{-2} d\lambda, \tag{15a}$$

$$\int_0^\infty |{}^\circ f(x)|^2 \Delta(x) dx = \sum_{\mu \in D_{\alpha,\beta}} |a_\mu|^2 d(\mu). \tag{15b}$$

Therefore, if we define $\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{a_\mu\})$, (14) implies that

$$\|f\|_{L^2(\Delta)} = \|\hat{\boldsymbol{f}}\|_{L^2(\nu)}.$$

3. Local uncertainty principles. We define a function $V(x)$ on $\mathbb{R}_+$ by

$$V(x) = \int_0^x \Delta(t) dt \tag{16}$$

and for a measurable subset E of $\mathbb{R}_+ \cup D_{\alpha,\beta}$ we put

$$\sigma(E) = \int_E d\nu.$$

Then as in the Euclidean case, we can deduce the local uncertainty principle (see [5, §3] for semisimple Lie groups and motion groups).

Theorem 3.1. *Let $0 \leq \theta < 1/2$. Then there exists a constant $C_{\theta,\alpha}$ such that for all $f \in L^1(\Delta) \cap L^2(\Delta)$ and $E \subset \mathbb{R}_+ \cup D_{\alpha,\beta}$ with $\sigma(E) < \infty$,*

$$\int_E |\hat{\boldsymbol{f}}(\nu)|^2 d\nu \leq C_{\theta,\alpha}\sigma(E)^{2\theta} \int_0^\infty |f(x)|^2 V(x)^{2\theta}\Delta(x)dx.$$

In order to clear the fact that $C_{\theta,\alpha}$ is independent of β we shall give a sketch of the proof. Let χ_r, $r > 0$, denote the characteristic function of the interval $[0, r]$. We set $g = f\chi_r$ and $h = f - g$. Then

$$\int_E |\hat{\boldsymbol{f}}(\nu)|^2 d\nu \leq 2\Big(\int_E |\hat{\boldsymbol{g}}(\nu)|^2 d\nu + \int_E |\hat{\boldsymbol{h}}(\nu)|^2 d\nu\Big).$$

It follows from (9) and Schwarz' inequality that

$$\begin{aligned}
&\int_E |\hat{\boldsymbol{g}}(\nu)|^2 d\nu \\
\leq\ & \frac{2}{\Gamma(\alpha+1)^2}\|g\|^2_{L^1(\Delta)}\sigma(E) \\
\leq\ & \frac{2}{\Gamma(\alpha+1)^2}\sigma(E)\int_0^r V(x)^{-2\theta}\Delta(x)dx \int_0^r |g(x)|^2 V(x)^{2\theta}\Delta(x)dx \\
=\ & \frac{2}{\Gamma(\alpha+1)^2}\frac{1}{-2\theta+1}\sigma(E)V(r)^{-2\theta+1}\int_0^r |g(x)|^2 V(x)^{2\theta}\Delta(x)dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_E |\hat{\boldsymbol{h}}(\nu)|^2 d\nu &\leq \int_r^\infty |h(x)|^2\Delta(x)dx \\
&\leq V(r)^{-2\theta}\int_r^\infty |h(x)|^2 V(x)^{2\theta}\Delta(x)dx.
\end{aligned}$$

Here we take an r such that $\sigma(E) = V(r)^{-1}$. Then

$$\int_E |\hat{\boldsymbol{f}}(\nu)|^2 d\nu \leq C_{\theta,\alpha}\sigma(E)^{2\theta}\int_0^\infty |f(x)|^2 V(x)^{2\theta}\Delta(x)dx,$$

where $C_{\theta,\alpha} = 2\max\Big(\frac{2}{\Gamma(\alpha+1)^2}\frac{1}{1-2\theta}, 1\Big)$.

We shall modify the above local uncertainty inequality. For each $\delta > 0$ we denote by x_δ the point satisfying $V(x_\delta) = \delta^{-1}$ and we let

$$V_\delta(x) = \begin{cases} V(x) & \text{if } 0 \le x < x_\delta, \\ \delta^{-1} & \text{if } x \ge x_\delta. \end{cases} \tag{17}$$

Theorem 3.2. *Let $\delta > 0$ and $0 \le \theta < 1/2$. Then there exists a constant $C_{\theta,\alpha}$ such that for all $f \in L^1(\Delta) \cap L^2(\Delta)$ and $E \subset \mathbb{R}_+ \cup D_{\alpha,\beta}$ with $\sigma(E) \ge \delta$,*

$$\int_E |\hat{\boldsymbol{f}}(\nu)|^2 d\nu \le C_{\theta,\alpha} \sigma(E)^{2\theta} \int_0^\infty |f(x)|^2 V_\delta(x)^{2\theta} \Delta(x) dx.$$

Proof. Since $\sigma(E) \ge \delta$ and δ is the minimum value of $V_\delta(x)^{-1}$, we can take an r such that $\sigma(E) = V(r)^{-1}$. Therefore, we can repeat the above sketch of the proof replacing V by V_δ. ∎

4. global uncertainty principles. As in the Euclidean case, we can deduce the global uncertainty principles from the local ones. We denote

$$W(r) = \sigma(\{\lambda \in \mathbb{C}; |\lambda| \le r\}).$$

Then the following global uncertainty inequality follows from Theorem 3.1 (see [5, §4] for symmetric spaces).

Theorem 4.1. *Let $0 \le \theta < 1/2$. Then there exists a constant $C_{\theta,\alpha}$ such that for all $f \in L^1(\Delta) \cap L^2(\Delta)$*

$$\|f\|^4_{L^1(\Delta)} \le C_{\theta,\alpha} \int_0^\infty |f(x)|^2 V(x)^{2\theta} \Delta(x) dx \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 W(\nu)^{2\theta} d\nu.$$

We now deduce a global uncertainty inequality from Theorem 3.2. We set $E_r = \{\lambda \in \mathbb{C}; |\lambda|^2 \le r^2 + \rho^2\}$. Since $\sigma(E_r \cap \mathbb{R}_+) = \int_0^{\sqrt{r^2+\rho^2}} |C(\lambda)|^{-2} d\lambda$, substituting the estimate of $C(-\lambda)^{-1}$ (see (6)), we see that there exist positive constants C^i_α, $i = 1, 2$, such that for $\lambda \in \mathbb{R}$

$$C^1_\alpha 2^{-2\rho}(r^2 + \rho^2)^{(\alpha+1)} \le \sigma(E_r \cap \mathbb{R}_+) \le C^2_\alpha 2^{-2\rho}(r^2 + \rho^2)^{(\alpha+1)}. \tag{18}$$

Therefore, if we take $\delta > 0$ as $\delta = C_\alpha^1 2^{-2\rho}\rho^{2(\alpha+1)}$, then $\sigma(E_r \cap \mathbb{R}_+) \geq \delta$. For $\gamma \geq 0$, we define the fractional power of $-L$ as

$$((-L)^\gamma f)(\lambda) = (\lambda^2 + \rho^2)^\gamma \hat{f}(\lambda)$$

(cf. (8)). Then we have the following.

Theorem 4.2. *Let δ, V_δ be as above and let $0 \leq \theta < 1/2$. Then there exists a positive constant $C_{\theta,\alpha}$ such that for all $f = f_P \in L^1(\Delta) \cap L^2(\Delta)$*

$$\|f\|_{L^1(\Delta)}^4 \leq C_{\theta,\alpha} 2^{-4\rho\theta} \int_0^\infty |f(x)|^2 V_\delta(x)^{2\theta} \Delta(x)dx \int_0^\infty |(-L)^{(\alpha+1)\theta} f(x)|^2 \Delta(x)dx.$$

Proof. Let $\gamma = 2(\alpha+1)\theta$ and $f = f_P$. By using the Plancherel formula (16a), we obtain that

$$\begin{aligned} \|f\|_{L^2(\Delta)}^2 &= \int_{\mathbb{R}_+} (\lambda^2+\rho^2)^{-\gamma}(\lambda^2+\rho^2)^\gamma |\hat{f}(\lambda)|^2 |C(\lambda)|^{-2} d\lambda \\ &\leq \rho^{-2\gamma} \int_0^\infty |((-L)^{\gamma/2} f(x)|^2 \Delta(x)dx. \end{aligned} \tag{19}$$

Moreover, if $\hat{f}(\lambda)$ is supported on $E_r^c \cap \mathbb{R}_+$, then $\rho^{-2\gamma}$ can be replaced by $(r^2+\rho^2)^{-\gamma}$, because $\lambda^2+\rho^2 \geq \lambda^2 \geq r^2+\rho^2$ for $\lambda \in E_r^c \cap \mathbb{R}_+$. Then it follows from Theorem 3.2 and (18) that for each $r > 0$

$$\begin{aligned} \|f\|_{L^2(\Delta)}^2 &= \int_{E_r\cap\mathbb{R}_+} |\hat{\boldsymbol{f}}(\nu)|^2 d\nu + \int_{E_r^c\cap\mathbb{R}_+} |\hat{\boldsymbol{f}}(\nu)|^2 d\nu \\ &\leq C_{\theta,\alpha}\sigma(E_r\cap\mathbb{R}_+)^{2\theta} \int_0^\infty |f(x)|^2 V_\delta(x)^{2\theta}\Delta(x)dx \\ &\quad + (r^2+\rho^2)^{-\gamma} \int_0^\infty |((-L)^{\gamma/2} f(x)|^2 \Delta(x)dx \qquad (20) \\ &\leq (r^2+\rho^2)^\gamma 2^{-4\rho\theta} C_{\theta,\alpha} (C_\alpha^2)^{2\theta} \int_0^\infty |f(x)|^2 V_\delta(x)^{2\theta}\Delta(x)dx \\ &\quad + (r^2+\rho^2)^{-\gamma} \int_0^\infty |((-L)^{\gamma/2} f(x)|^2 \Delta(x)dx \\ &= (r^2+\rho^2)^\gamma 2^{-4\rho\theta} C_{\theta,\alpha}(C_\alpha^2)^{2\theta} I_1 + (r^2+\rho^2)^{-\gamma} I_2. \end{aligned}$$

Especially, since $C_{\theta,\alpha} \geq 2$ and $C_\alpha^1 \leq C_\alpha^2$, it follows that

$$\begin{aligned} \|f\|_{L^2(\Delta)}^2 &\leq (r^2+\rho^2)^\gamma 2^{-4\rho\theta} C_{\theta,\alpha}(C_\alpha^2)^{2\theta} I_1 + (r^2+\rho^2)^{-\gamma} C_{\theta,\alpha}(C_\alpha^1/C_\alpha^2)^{-2\theta} I_2 \\ &= (r^2+\rho^2)^\gamma A + (r^2+\rho^2)^{-\gamma} B. \end{aligned}$$

As a function of x on $\mathbb{R}_+$, $x^\gamma A + x^{-\gamma} B$ attains the minimum value $2\sqrt{AB}$ at $x_0 = (B/A)^{1/2\gamma}$. Therefore, it follows from (17) with $\delta = C^1_\alpha 2^{-2\rho}\rho^{2(\alpha+1)}$ and (19) that

$$x_0 = \Big(\frac{C_{\theta,\alpha}(C^1_\alpha/C^2_\alpha)^{-2\theta} I_2}{2^{-4\rho\theta} C_{\theta,\alpha}(C^2_\alpha)^{2\theta} I_1}\Big)^{1/2\gamma} \geq \rho^2.$$

Hence we can take an r such that $x_0 = r^2 + \rho^2$ and therefore,

$$\|f\|^4_{L^2(\Delta)} \leq 2^{-4\rho\theta+2} C^2_{\theta,\alpha}(C^2_\alpha)^{2\theta}(C^1_\alpha/C^2_\alpha)^{-2\theta} I_1 I_2.$$

This completes the proof. ■

For a general $f \in L^1(\Delta) \cap L^2(\Delta)$ we must pay attention to the discrete part $^\circ f$ of f. Let $^\circ f \neq 0$ and thus, $D_{\alpha,\beta} \neq \emptyset$ and $\beta > 0$. In (19) $\mathbb{R}_+$ must be replaced by $\mathbb{R}_+ \cup D_{\alpha,\beta}$ and when $\nu \in D_{\alpha,\beta}$, we see that

$$(\nu^2 + \rho^2)^{-\gamma} \leq (\rho^2 - (\beta - \alpha - 1)^2)^{-\gamma} = (4\beta(\alpha+1))^{-\gamma}.$$

Since $\beta - \alpha - 1 < \rho$, it follows that $E^c_r \cap D_{\alpha,\beta} = \emptyset$. Moreover, in (20) $\sigma(E_r \cap \mathbb{R}_+)$ must be repalced by $\sigma(E_r) = \sigma(E_r \cap \mathbb{R}_+) + \sigma(D_{\alpha,\beta})$. We note that

$$\sigma(D_{\alpha,\beta}) \leq (r^2+\rho^2)^{\alpha+1}\frac{\sigma(D_{\alpha,\beta})}{(r^2+\rho^2)^{\alpha+1}} \leq (r^2+\rho^2)^{\alpha+1}\frac{\sigma(D_{\alpha,\beta})}{\rho^{2(\alpha+1)}}.$$

Hence, applying the same argument, we can deduce the following.

Theorem 4.3. *Let $\delta > 0$ and $0 \leq \theta < 1/2$. Then there exists a positive constant $C_{\theta,\alpha,\beta}$ such that for all $f \in L^1(\Delta) \cap L^2(\Delta)$*

$$\|f\|^4_{L^1(\Delta)} \leq C_{\theta,\alpha,\beta}\int_0^\infty |f(x)|^2 V_\delta(x)^{2\theta}\Delta(x)dx \int_0^\infty |(-L)^{(\alpha+1)\theta} f(x)|^2\Delta(x)dx.$$

5. Main theorem. We retain the notations in the previous sections. We shall obtain a refinement of Theorem 4.3 with $\theta = 1/2(\alpha+1)$. For $x \geq 0$ we put

$$v(x) = \frac{V(x)}{\Delta(x)}$$

and for $\lambda \in \mathbb{C}$

$$w(\lambda) = (\lambda^2 + \rho^2)^{1/2}.$$

Theorem 5.1. *For all $f \in L^1(\Delta) \cap L^2(\Delta)$,*

$$\|fv\|^2_{L^2(\Delta)} \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \geq \frac{1}{4}\|f\|^4_{L^2(\Delta)}, \tag{21a}$$

where the equality holds if and only if f is of the form

$$f(x) = ce^{\gamma \int_0^x v(t)dt} \tag{21b}$$

for some $c, \gamma \in \mathbb{C}$ and $\Re\gamma < 0$.

Proof. Without loss of generality we may suppose that $f \in C^\infty_{c,e}(\mathbb{R})$. Since $(-Lf)^\wedge(\lambda) = \hat{f}(\lambda)(\lambda^2 + \rho^2) = \hat{f}(\lambda)w(\lambda)^2$ (see (8)) and $w(\lambda)$ is positive on $\mathbb{R}_+ \cap D_{\alpha,\beta}$, the Parseval formula (14) yields that

$$\begin{aligned} \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu &= \int_0^\infty (-Lf)(x)\overline{f(x)}\Delta(x)dx \\ &= \int_0^\infty |f'(x)|^2 \Delta(x)dx. \end{aligned}$$

Hence it follows that

$$\begin{aligned} &\int_0^\infty |f(x)|^2 v(x)^2 \Delta(x)dx \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \\ = &\int_0^\infty |f(x)|^2 v(x)^2 \Delta(x)dx \int_0^\infty |f'(x)|^2 \Delta(x)dx \\ \geq &\Big(\int_0^\infty \Re(f(x)f'(x))v(x)\Delta(x)dx\Big)^2 \\ = &\frac{1}{4}\Big(\int_0^\infty (|f(x)|^2)'V(x)dx\Big)^2 = \frac{1}{4}\Big(\int_0^\infty |f(x)|^2\Delta(x)dx\Big)^2. \end{aligned}$$

Here we used the fact that $V' = \Delta$ (see (16)). Clearly, the equality holds if and only if $fv = cf'$ for some $c \in \mathbb{C}$, that is, $f'/f = c^{-1}v$. This means that $\log(f) = c^{-1}\int_0^x v(t)dt + C$ and thus, the desired result follows. ■

Since $w^2(\lambda) = \lambda^2 + \rho^2$, (21) and the Parseval formula (15) yield the following.

Corollary 5.2. *Let f be the same as in Theorem 5.1.*

$$\|fv\|^2_{L^2(\Delta)} \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \nu^2 d\nu \geq \frac{1}{4} \|f\|^2_{L^2(\Delta)} \int_0^\infty |f(x)|^2 (1 - 4\rho^2 v(x)^2) \Delta(x) dx.$$

We shall estimate v and $1 - 4\rho^2 v^2$. Since $\alpha > -1$, it follows that

$$\begin{aligned} V(x) &= \int_0^x (2 \sinh s)^{2\alpha+1} (2 \cosh s)^{2\beta+1} ds \\ &= 2^{2\rho} \int_0^{\sinh x} t^{2\alpha+1} (1 + t^2)^\beta dt \\ &= 2^{2\rho} (\sinh x)^{2\alpha+2} \int_0^1 t^{2\alpha+1} (1 + (\sinh x)^2 t^2)^\beta dt \\ &= 2^{2\rho-1} (\sinh x)^{2\alpha+2} (\cosh x)^{2\beta} \int_0^1 (1 - s)^\alpha (1 - (\tanh x)^2 s)^\beta ds \\ &= 2^{2\rho-1} (\sinh x)^{2\alpha+2} (\cosh x)^{2\beta} \frac{1}{\alpha+1} F(1, -\beta, 2+\alpha; (\tanh x)^2) \end{aligned}$$

and thus,

$$v(x) = \frac{1}{2(\alpha+1)} F(1, -\beta, 2+\alpha; (\tanh x)^2) \tanh x. \tag{22}$$

Lemma 5.3. *Let notation be as above. If $\beta \leq 0$ or $\beta \leq \alpha$, then*

$$0 \leq v(x) \leq \frac{1}{2\rho}$$

and if $\beta \geq 0$, then

$$0 \leq v(x) \leq \frac{1}{2(\alpha+1)},$$

and if $\beta > 0, \alpha \geq 0$, then

$$0 \leq v(x) \leq \frac{1}{\sqrt{2\rho - 1}}.$$

Proof. We recall Euler's integral expression of the hypergeometric function:

$$F(1,-\beta,2+\alpha,x^2)=(\alpha+1)\int_0^1(1-t)^\alpha(1-tx^2)^\beta dt. \tag{23}$$

Thereby, $v(x)\geq 0$. If $\beta\leq 0$, then it is easy to see that $F(1,-\beta,2+\alpha;x)$ is increasing on $0\leq x\leq 1$. Hence $H(x)=xF(1,-\beta,2+\alpha;x^2)$ is dominated by $H(1)=\Gamma(2+\alpha)\Gamma(\rho)/\Gamma(1+\alpha)\Gamma(\rho+1)=(\alpha+1)/\rho$ and thus $v(x)\leq 1/2\rho$. Let $0<\beta\leq\alpha$. We shall prove that $H(x)$ is also increasing and $H(x)\leq H(1)$ as before. In order to prove that $H(x)$ is increasing, we shall show that its derivative is positive. We put $H_k(\alpha,\beta,x)=x^{2k+1}F(k+1,k-\beta,k+2+\alpha;x^2)$ and we note that

$$\begin{aligned}H'(x) &= x^{-1}H_0(\alpha,\beta,x)-\frac{2\beta}{2+\alpha}x^{-1}H_1(\alpha,\beta,x)\\ &= x^{-1}H_0(\alpha,\beta,x)+2(1+\alpha)x^{-1}\Big(H_0(\alpha-1,\beta,x)-H_0(\alpha,\beta,x)\Big)\\ &= K(x),\end{aligned}$$

where $K(x)=F(1,-\beta,2+\alpha;x^2)+2(1+\alpha)(F(1,-\beta,1+\alpha;x^2)-F(1,-\beta,2+\alpha,x^2))$. Then

$$\begin{aligned}K'(x) &= -2\beta x^{-2}\Big(\frac{1}{2+\alpha}H_1(\alpha,\beta,x)\\ &\quad +2(1+\alpha)\Big(\frac{H_1(\alpha-1,\beta,x)}{1+\alpha}-\frac{H_1(\alpha,\beta,x)}{2+\alpha}\Big)\Big).\end{aligned}$$

Since $\beta>0$, $H_1(\alpha,\beta,x)=x^3F(2,1-\beta,3+\alpha;x)\leq x^3F(2,1-\beta,2+\alpha;x)=H_1(\alpha-1,\beta,x)$ and $1/(1+\alpha)-1/(2+\alpha)>0$, it follows that $K'(x)<0$. Therefore, $H'(x)=K(x)$ is decreasing and

$$H'(x)\geq H'(1)=\frac{(\alpha-\beta)(\alpha+1)}{\rho(\alpha+\beta)}\geq 0$$

under the assumption on β. Hence $H(x)$ is increasing.

Next let $\beta\geq 0$. Then it follows from (23) that

$$\frac{1}{2(\alpha+1)}xF(1,-\beta,2+\alpha;x^2)\leq\frac{1}{2}\int_0^1(1-t)^\alpha dt=\frac{1}{2(\alpha+1)}.$$

Last let $\beta > 0$ and $\alpha \geq 0$. Then it follows from (23) that

$$\begin{aligned} \frac{1}{2(\alpha+1)} xF(1, -\beta, 2+\alpha; x^2) &\leq \frac{x}{2}\int_0^1 (1-x^2t)^{\alpha+\beta}dt \\ &= \frac{1}{2\rho x}(1-(1-x^2)^\rho). \end{aligned}$$

We suppose that the last function takes the maximum at $x = x_0$. Then $2\rho(1-x_0^2)^{\rho-1}x_0^2 = 1-(1-x_0^2)^\rho$ and thereby, the last function is dominated by $(1-x_0^2)^{\alpha+\beta}x_0$. Since $(1-x^2)^{\alpha+\beta}x$ takes the maximum at $x = 1/\sqrt{2(\alpha+\beta)+1}$ and $\alpha+\beta > 0$, we see that $(1-x^2)^{\alpha+\beta}x$ is dominated by

$$\Big(\frac{2(\alpha+\beta)}{2(\alpha+\beta)+1}\Big)^{\alpha+\beta}\frac{1}{\sqrt{2(\alpha+\beta)+1}} \leq \frac{1}{\sqrt{2\rho-1}}.$$

Hence the desired estimate follows. ■

Lemma 5.4. *Let $\Upsilon(x) = 1-4\rho^2 v(x)^2$. If $\beta \leq 0$ or $\beta \leq \alpha$, then $\Upsilon(x) \geq (\cosh x)^{-2}$. Generally,*

$$\Upsilon(x) = \begin{cases} O((\cosh x)^{-2}) & \text{if } x \to \infty, \\ O(1) & \text{if } x \to 0. \end{cases}$$

Proof. Since $F(1, -\beta, 2+\alpha; 0) = 1$ and $F(1, -\beta, 2+\alpha; 1) = (\alpha+1)/\rho$, the asymptotic behavior easily follows. As in the proof of Lemma 5.3, if $\beta \leq 0$ or $\beta \leq \alpha$, then $F(1, -\beta, 2+\alpha; x)$ is increasing with respect to x. Hence $v(x) \leq F(1, -\beta, 2+\alpha; 1)\tanh x\ /2(\alpha+1) \leq (1/2\rho)\tanh x$ and thus, $\Upsilon(x) \geq (\cosh x)^{-2}$. ■

We put

$$\tau_{\alpha,\beta} = \begin{cases} 1 & \text{if } \beta \leq 0 \text{ or } \beta \leq \alpha, \\ \dfrac{\rho}{\alpha+1} & \text{if } \beta > 0 \text{ and } \alpha < 0, \\ \min\Big(\dfrac{\rho}{\alpha+1}, \dfrac{2\rho}{\sqrt{2\rho-1}}\Big) & \text{if } \beta > \alpha \geq 0. \end{cases} \tag{24}$$

Lemma 5.3 implies that

$$0 \leq v(x) \leq \frac{\tau_{\alpha,\beta}}{2\rho}. \tag{25}$$

The following assertion follows from Theorem 5.1, Corollary 5.2, Lemma 5.3 and Lemma 5.4.

Corollary 5.5. *Let $\rho > 0$ and f be the same as in Theorem 5.1.*

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \geq \rho^2 \tau_{\alpha,\beta}^{-2} \|f\|_{L^2(\Delta)}^2, \tag{26}$$

and if $f = f_P$, then

$$\int_0^\infty |\hat{f}(\lambda)|^2 |C(\lambda)|^{-2} d\lambda \geq \rho^2 \tau_{\alpha,\beta}^{-2} \int_0^\infty |f_P(x)|^2 \Upsilon(x) \Delta(x) dx.$$

The shapes of $v(t)$ and $\Upsilon(t)$, $t = \operatorname{arctanh}\sqrt{x}$, $x \geq 0$, are respectively given as follows.

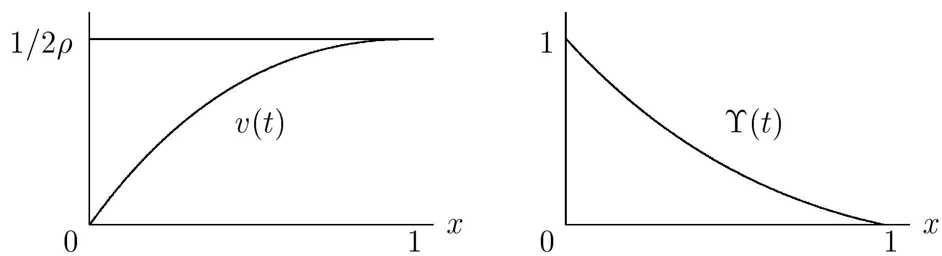

Figure 1: The case of $\beta \leq 0$ or $\beta \leq \alpha$.

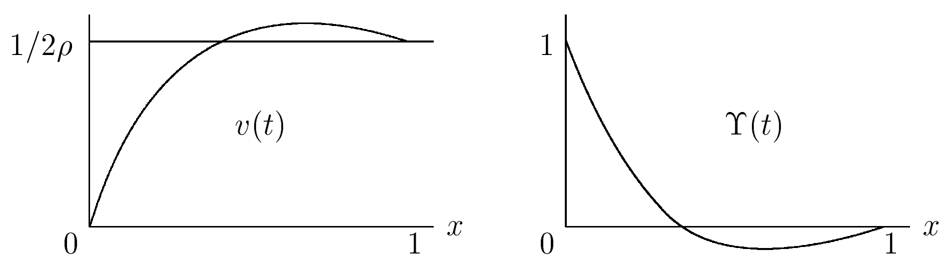

Figure 2: The case of $\beta > 0$ and $\beta > \alpha$.

In (26) we set

$$f(g) = \phi_\mu(g) = \frac{\sqrt{2}}{\Gamma(\alpha+1)} \Big(\frac{\Gamma(\alpha+1)}{\sqrt{2}} d(\mu)^{-1} \Big) \phi_\mu(g) d(\mu)$$

for $\mu \in D_{\alpha,\beta}$. Then it follows from (12) that

$$\|\phi_\mu v\|^2_{L^2(G)}(-|\mu|^2 + \rho^2) \geq \frac{1}{4}\|\phi_\mu\|^2_{L^2(G)}.$$

Especially,

$$\int_0^\infty |\phi_\mu(x)|^2 \Upsilon(x)\Delta(x)dx \leq -4\|\phi_\mu v\|^2_{L^2(G)}|\mu|^2 < 0.$$

Moreover, if we denote the maximum value of v by $v_{\max}$, then for $\mu \in D_{\alpha,\beta}$,

$$v^2_{\max} \geq \frac{1}{4(-|\mu|^2 + \rho^2)}$$

and hence

$$v^2_{\max} \geq \frac{1}{16\beta(\alpha+1)}.$$

6. Uncertainty principles. We shall apply the inequalities obtained in the previous section to deduce some information on the concentration of f and $\hat{f}$. Let f be a non-zero function in $L^2(\Delta)$. We recall that

$$f = f_P + {}^\circ f, \quad {}^\circ f(x) = \frac{2}{\Gamma(\alpha+1)} \sum_{\mu \in D_{\alpha,\beta}} a_\mu \phi_\mu(x) d(\mu)$$

and $\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{a_\mu\})$ (see (10)).

Definition 6.1. *Let $0 < \epsilon < 1/2\rho$ and $M > 0$.*

(1) We say that a function $f(x)$ on $\mathbb{R}_+$ is (v, ϵ)-concentrated at $x = 0$ if

$$\|fv\|_{L^2(\Delta)} \leq \epsilon \|f\|_{L^2(\Delta)} \tag{27a}$$

and is (v, M)-nonconcentrated at $x = 0$ if the reverse replaced ϵ by M holds.

(2) We say that a function $\hat{f}(\lambda)$ on $\mathbb{R}_+$ is (λ, ϵ)-concentrated at $\lambda = 0$ if

$$\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda \leq \epsilon^2 \|f\|^2_{L^2(\Delta)} \tag{27b}$$

and is (λ, M)-nonconcentrated at $\lambda = 0$ if the reverse replaced ϵ by M holds.

(3) We say that a function $f(x)$ on $\mathbb{R}_+$ has an ϵ-small descrete part if

$$\|{}^\circ f\| \leq \epsilon \|f\|_{L^2(\Delta)}. \tag{27c}$$

(4) We say that a function $f(x)$ on $\mathbb{R}_+$ is (Υ, ϵ)-nonconcentrated at $x = 0$ if

$$\Big| \int_0^\infty |f(x)|^2 \Upsilon(x) \Delta(x) dx \Big| \leq \epsilon^2 \|f\|^2_{L^2(\Delta)}.$$

(5) We say that a function $f(x)$ on $\mathbb{R}_+$ is (x_0, ϵ)-bounded if

$$|f(x)| \leq \epsilon e^{-\rho x} \|f\|_{L^2(\Delta)} \text{ if } x \geq x_0.$$

Now we suppose that $f(x)$ is (v, ϵ)-concentrated at $x = 0$. Since

$$\begin{aligned} & \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \\ = & \int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda - \sum_{D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) + \rho^2 \|f\|^2_{L^2(\Delta)} \end{aligned}$$

(see (15)), it follows from (21) and (27a)

$$\begin{aligned} & \int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda \\ \geq & \int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda - \sum_{D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) \\ = & \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu - \rho^2 \|f\|^2_{L^2(\Delta)} \\ \geq & (1/4\epsilon^2 - \rho^2) \|f\|^2_{L^2(\Delta)}. \end{aligned} \tag{28}$$

Therefore, $\hat{f}(\nu)$ is $(\lambda, (1/4\epsilon^2 - \rho^2)^{1/2})$-nonconcentrated at $\lambda = 0$.

Conversely, we suppose that $\hat{f}(\nu)$ is (λ, ϵ)-concentrated at $\lambda = 0$. Since $\Upsilon(x) = 1 - 4\rho^2 v(x)^2 \geq 1 - \tau^2_{\alpha,\beta}$ (see (25)), it follows that

$$\int_0^\infty |f_P(x)|^2 \Upsilon(x) \Delta(x) dx \geq (1 - \tau^2_{\alpha,\beta}) \|f_P\|^2_{L^2(\Delta)}. \tag{29}$$

We recall that $1-\tau_{\alpha,\beta}^2 \leq 0$. Moreover, letting $A = \int_0^\infty |f_P(x)|^2 \Upsilon(x)\ \Delta(x)dx$ and $B = \|f_P\|^2_{L^2(\Delta)}$, we see from Corollary 5.2 for $f = f_P$ and (27b) that

$$(B-A)\epsilon^2 B \geq \rho^2 AB$$

and thus, $A \leq \dfrac{\epsilon^2 B}{\rho^2+\epsilon^2} \leq \dfrac{\epsilon^2}{\rho^2}B$, that is,

$$\int_0^\infty |f_P(x)|^2 \Upsilon(x)\Delta(x)dx \leq \frac{\epsilon^2}{\rho^2}\|f_P\|^2_{L^2(\Delta)}. \tag{30}$$

Therefore, (29) and (30) imply that $f_P(x)$ is (Υ, δ)-nonconcentrated at $x = 0$, where

$$\delta = \max\{(\tau_{\alpha,\beta}^2 - 1)^{1/2}, \rho^{-1}\epsilon\}.$$

Moreover, letting $\delta = 1$ in (5), we see from (10), (3) and (27b) that for $x \geq 1$,

$$\begin{aligned}
|f_P(x)| &\leq c\Big|\int_0^\infty \hat{f}(\lambda)\Phi_\lambda(x)C(\lambda)^{-1}d\lambda\Big| \\
&\leq ce^{-\rho x}K_1\Big(\int_0^\epsilon |\hat{f}(\lambda)||C(-\lambda)|^{-1}d\lambda + \int_\epsilon^\infty |\hat{f}(\lambda)||C(-\lambda)|^{-1}d\lambda\Big) \\
&\leq ce^{-\rho x}K_1\Big(\epsilon^{1/2}\|f_P\|_{L^2(\Delta)} \\
&\qquad + \Big(\int_\epsilon^\infty |\hat{f}(\lambda)|^2\lambda^2|C(\lambda)|^{-2}d\lambda\Big)^{1/2}\Big(\int_\epsilon^\infty \lambda^{-2}d\lambda\Big)^{1/2}\Big) \\
&\leq 2cK_1\epsilon^{1/2}e^{-\rho x}\|f_P\|_{L^2(\Delta)}.
\end{aligned} \tag{31}$$

Hence we have the following.

Theorem 6.2 *Let $\rho > 0$ and $f \in L^2(\Delta)$. If $f(x)$ is (v, ϵ)-concentrated at $x = 0$, then $\hat{f}(\lambda)$ is $(\lambda, (1/4\epsilon^2 - \rho^2)^{1/2})$-nonconcentrated at $\lambda = 0$. Conversely, if $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$, then $f_P(x)$ is (Υ, δ)-nonconcentrated at $x = 0$, where $\delta = \max\{(\tau_{\alpha,\beta}^2 - 1)^{1/2}, \rho^{-1}\epsilon\}$, and there exists a positive constant $c = c_{\alpha,\beta}$ such that $f_P(x)$ is $(1, c\epsilon^{1/2})$-bounded.*

When $\beta \leq \alpha$, we recall that $D_{\alpha,\beta} = \emptyset$, $f = f_P$ and $\tau_{\alpha,\beta} = 1$. Hence, the above theorem implies that, if $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$, then $f(x)$ is $(\Upsilon, \rho^{-1}\epsilon)$-nonconcentrated at $x = 0$ and $(1, c\epsilon^{1/2})$-bounded. Therefore, $f(x)$ is spread if ϵ goes to 0.

When $\beta > \alpha$, then $\tau_{\alpha,\beta} > 1$ and it is not clear that $f(x)$ is spread if ϵ goes to 0. We must pay attention to the discrete part of f. We suppose that $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$ and moreover, $f(x)$ has an ϵ_d-small discrete part. Of course, if $\beta < \alpha + 1$, then we can take $\epsilon_d = 0$, because $D_{\alpha,\beta} = \emptyset$. We shall prove that $f(x)$ is spread if ϵ and ϵ_d go to 0. First we note that (30) replaced f_P by f holds as before:

$$\int_0^\infty |f(x)|^2 \Upsilon(x)\Delta(x)dx \leq \frac{\epsilon^2}{\rho^2}\|f\|_{L^2(\Delta)}. \tag{32}$$

Let $x_0 > 0$ be the point such that $\Upsilon(x_0) = 0$ (see Fig. 2). In (31), replacing $\delta = 1$ in (5) by $\delta = x_0$, we see that for $x \geq x_0$,

$$|f_P(x)| \leq cK_{x_0}\epsilon^{1/2}e^{-\rho x}\|f_P\|_{L^2(\Delta)}.$$

On the other hand, it follows from (11), (15b) and (27c) that

$$\begin{aligned} |{}^\circ f(x)| &\leq c \sum_{\mu \in D_{\alpha,\beta}} |a_\mu||\phi_\mu(x)|d(\mu) \\ &\leq ce^{-\rho x}\Big(\sum_{\mu \in D_{\alpha,\beta}} e^{-2|\mu|x_0}d(\mu)\Big)^{1/2}\|{}^\circ f\|_{L^2(\Delta)} \leq c\epsilon_d e^{-\rho x}\|f\|_{L^2(\Delta)}. \end{aligned}$$

Hence, for $x \geq x_0$, we see that there exists a positive constant c_0 such that

$$|f(x)| \quad \leq \quad c_0 e^{-\rho x}(\epsilon^{1/2} + \epsilon_d)\|f\|_{L^2(\Delta)}. \tag{33}$$

Since $\Upsilon(x) \leq 0$ if $x \geq x_0$, it follows that

$$\begin{aligned} \int_0^\infty |f(x)|^2 \Upsilon(x)\Delta(x)dx &\geq c\int_{x_0}^\infty |f(x)e^{\rho x}|^2 \Upsilon(x)dx \\ &\geq cc_0^2(\epsilon^{1/2} + \epsilon_d)^2\|f\|^2_{L^2(\Delta)}\int_{x_0}^\infty \Upsilon(x)dx \\ &= -c_\Upsilon(\epsilon^{1/2} + \epsilon_d)^2\|f\|^2_{L^2(\Delta)}, \end{aligned} \tag{34}$$

where $c_\Upsilon \geq 0$. Then (32), (33) and (34) imply the following.

Theorem 6.3 *Let $\rho > 0$, $\beta > \alpha$ and $f \in L^2(\Delta)$. We suppose that $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$ and $f(x)$ has an ϵ_d-small discrete part. We take a sufficiently small ϵ such that $\delta^2 = c_\Upsilon(\epsilon^{1/2} + \epsilon_d)^2 \geq \rho^{-2}\epsilon^2$. Then $f(x)$ is*

(Υ, δ)-nonconcentrated at $x = 0$ and there exists a positive constant $c = c_{\alpha,\beta}$ such that $f(x)$ is $(x_0, c\delta)$-bounded.

We suppose that f is supported on $[R, \infty)$. Then there exists a constant $0 < \delta(R) \le 1$ such that

$$0 \le v(x) \le \frac{1}{2\rho\delta(R)}, \ x \ge R$$

and $\delta(R) \to 1$ if $R \to \infty$. Since $1 - 4\rho^2 v(x)^2 \ge 1 - \delta(R)^{-2}$, it follows from Corollary 5.2 that

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \nu^2 d\nu \ge \rho^2(\delta(R)^2 - 1)\|f\|^2_{L^2(\Delta)}.$$

Then we obtain the following.

Proposition 6.4. *Let $\rho > 0$ and suppose that $f \in L^2(\Delta)$ is supported on $[R, \infty)$. Then*

$$\sum_{\mu \in D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) \le \int_0^\infty |\hat{f}_P(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda + \rho^2(1 - \delta(R)^2)\|f\|^2_{L^2(\Delta)}.$$

Remark 6.5. When $\beta = 0$ and $\alpha \ge 0$, it follows from (22) that $v(x) = (2\rho)^{-1} \tanh x$ and $1 - 4\rho^2 v(x)^2 = (\cosh x)^{-2}$. Therefore, the inequalities in Theorem 5.1 and Corollary 5.2 became

$$\|f(x) \tanh x\|^2_{L^2(\Delta)} \|\hat{f}(\lambda)(\lambda^2 + \rho^2)^{1/2}\|^2_{L^2(|C|^{-2})} \ge \rho^2 \|f\|^4_{L^2(\Delta)},$$

where the equality holds if and only if f is of the form $c(\cosh x)^\gamma$, $c, \gamma \in \mathbb{C}$, $\Re\gamma < 0$, and

$$\|f(x) \tanh x\|^2_{L^2(\Delta)} \|\hat{f}(\lambda)\lambda\|^2_{L^2(|C|^{-2})} \ge \rho^2 \|f\|^2_{L^2(\Delta)} \|f(x)(\cosh x)^{-1}\|^2_{L^2(\Delta)}.$$

Since the Jacobi transform of $(\cosh \lambda)^\gamma$ is explicitly calculated in [1], we can directly check the above equality condition for these inequalities.

7. Uncertainty principles on $SU(1,1)$. We briefly give some basic notations to introduce the spherical Fourier transform on $G = SU(1,1)$. For the precise definitions we refer to [6] and [8]. We denote ϕ_λ, $\Delta(x)$ and $C(\lambda)$ in §1 respectively by $\phi_\lambda^{\alpha,\beta}$, $\Delta_{\alpha,\beta}(x)$ and $C_{\alpha,\beta}(\lambda)$.

Let A, K denote the subgroups of G of the matrices

$$a_x = \begin{pmatrix} \cosh x/2 & \sinh x/2 \\ \sinh x/2 & \cosh x/2 \end{pmatrix} \quad \text{and} \quad k_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix},$$

where $x \in \mathbb{R}$ and $0 \leq \phi \leq 4\pi$ respectively. According to the Cartan decomposition of G, each $g \in G$ can be written uniquely as $g = k_\phi a_x k_\psi$ where $0 \leq x$, $0 \leq \phi, \psi \leq 4\pi$. Let $\pi_{j,\lambda}$ ($j = 0, 1/2, \lambda \in \mathbb{R}$) denote the principal series representation of G. Then the (operator-valued) spherical Fourier transform $\pi_{j,\lambda}(f)$ of f on G is defined as $\pi_{j,\lambda}(f) = \displaystyle\int_G f(g)\pi_{j,\lambda}(g)dg$, where dg a Haar measure on G. In the following, we normalize dg as $dg = \Delta_{0,0}(x)dxd\phi d\psi$ and we treat only functions f on G whose K-types are supported on $\mathbb{Z} \times \mathbb{Z}$. Under this restriction, $\pi_{j,\lambda}(f)$ is supported on $j = 0$ and $\lambda > 0$ (cf. [6] and [8, §8]) and

$$f(a_x) = f(a_{-x}), \quad x \in \mathbb{R}.$$

Let $n, m \in \mathbf{N}$ and $\psi_\lambda^{n,m}(g)$ ($\lambda \in \mathbb{R}$, $g \in G$) denote the matrix coefficient of $\pi_{0,\lambda}(g)$ with K-type (n, m). Let f be a compactly supported C^∞ function on G whose K-type is (n, m). Then the scalar-valued spherical Fourier transform $\tilde{f}_{n,m}(\lambda)$ of type (n, m) is defined by

$$\tilde{f}_{n,m}(\lambda) = \int_G f(g)\psi_\lambda^{(n,m)}(g)dg. \tag{35}$$

Since the K-type of $\psi_\lambda^{n,m}(g)$ is of (n, m), this integral is determined on $A_+ \cong \mathbb{R}_+$. We recall that the explicit form of $\psi_\lambda^{n,m}(a_x)$ is given by using the Jacobi function (cf. [4, (4.17)] and [6, (3.4.10)]): For $g = k_\phi a_x k_\psi \in G$,

$$\psi_\lambda^{n,m}(g) = (\cosh x)^{n+m}(\sinh x)^{|n-m|}Q_{n,m}(\lambda)\phi_\lambda^{|n-m|,n+m}(x)e^{in\phi}e^{im\psi}, \tag{36}$$

where

$$Q_{n,m}(\lambda) = \binom{-1/2 - i\lambda/2 \mp m}{|n-m|}$$

and $\mp m$ is equal to $-m$ if $n \geq m$ and m if $n \leq m$. Hence, compared with (7) and (35), we see from (36) that

$$\begin{aligned}\tilde{f}_{n,m}(\lambda) &= 2^{-(|n-m|+n+m)-1/2}\Gamma(|n-m|+1)Q_{n,m}(\lambda) \\ &\quad \times \Big(f(x)(2\sinh x)^{-|n-m|}(2\cosh x)^{-(n+m)}\Big)^{\wedge}_{|n-m|,n+m}(\lambda).\end{aligned}$$

We here fix the K-type of f as (n, m) and we define a compactly supported C^∞ even function F on $\mathbb{R}$ as

$$F(x) = f(x)(2\sinh x)^{-|n-m|}(2\cosh x)^{-(n+m)}.$$

Then it follows that

$$\|f\|^2_{L^2(G)} = \int_0^\infty |f(x)|^2 \Delta_{0,0}(x)dx = \|F\|_{L^2(\Delta_{|n-m|,n+m})}$$

and

$$\tilde{f}_{n,m}(\lambda) = 2^{-(|n-m|+n+m)-1/2}\Gamma(|n-m|+1)Q_{n,m}(\lambda)\hat{F}_{|n-m|,n+m}(\lambda).$$

Therefore, since

$$Q_{n,m}(\lambda)^{-2}|C_{|n-m|,n+m}(\lambda)|^{-2} = 2^{-2(|n-m|+n+m)}\Gamma(|n-m|+1)^2|C_{0,0}(\lambda)|^{-2},$$

the Plamcherel formula for the Jacobi transform for F (see (10) and (15)) implies that

$$\|f\|^2_{L^2(G)} = 2\Big(\int_0^\infty |\tilde{f}_{n,m}(\lambda)|^2|C_{0,0}(\lambda)|^{-2}d\lambda + \sum_{\mu\in D^{n,m}} |\tilde{f}_{n,m}(\mu)|^2 d^{n,m}(\mu)\Big),$$

where $D^{n,m} = D_{|n-m|,n+m}$ in §1 and $d^{n,m}(\mu) = 2^{2(|n-m|+n+m)}\Gamma(|n-m|+1)^{-2}Q_{n,m}(\mu)^{-2}d_{|n-m|,n+m}(\mu)$. This is nothing but the Plancherel formula for the spherical Fourier transform of type (n, m) on G (see [4, (4.21)] and [8, Theorem 8.2]). As before, this transform can be extended to the one for L^2-functions on G with K-type (n, m). According to the decomposition (10) for F, each L^2-function f on G with K-type (n, m) is of the form

$$f = f_P + {}^\circ f,$$

where ${}^\circ f(g) = 2\sum_{\mu\in D^{n,m}} a_\mu \psi^{n,m}_\mu(g)d^{n,m}(\mu)$, and then $\tilde{\boldsymbol{f}} = (\tilde{f}_{\alpha,\beta}, \{a_\mu\})$. We call f_P and ${}^\circ f$ the principal part and the discrete part of f respectively. We here introduce $v_{n,m}, w_{n,m}$ and $\rho_{n,m}$ respectively corresponding to v, w and ρ with $\alpha = |n-m|, \beta = n+m$ in §1. Then for $\tilde{\boldsymbol{f}} = (\tilde{f}_{\alpha,\beta}, \{a_\mu\})$ it follows that

$$\int_{\mathbb{R}_+ \cup D^{\alpha,\beta}} \boldsymbol{f}(\nu) d_{m,n}\nu = \int_{-\infty}^{\infty} \tilde{f}(\lambda)|C_{0,0}(\lambda)|^{-2} d\lambda + \frac{1}{2} \sum_{\mu \in D^{n,m}} a_\mu d^{n,m}(\mu).$$

Hence the inequality in Theorem 5.1 can be rewritten as

$$\|f v_{n,m}\|^2_{L^2(G)} \int_{\mathbb{R}_+ \cup D^{n,m}} |\tilde{\boldsymbol{f}}(\nu)|^2 w_{n,m}(\nu)^2 d_{n,m}\nu \geq \frac{1}{4} \|f\|^4_{L^2(G)}.$$

We now suppose that $f(g)$ is concentrated at $g = e$: There exists a positive constant $\epsilon_{n,m}$ such that

$$\|f v_{n,m}\|^2_{L^2(G)} \leq \epsilon_{n,m} \|f\|^2_{L^2(G)}. \tag{37}$$

As in the same argument in §5 (see (28)), it follows that

$$\int_0^\infty |\tilde{f}_{n,m}(\lambda)|^2 \lambda^2 |C_{0,0}(\lambda)|^{-2} d\lambda \geq \left(\frac{1}{4\epsilon_{n,m}} - \rho^2_{n,m} \right) \|f\|^2_{L^2(G)}. \tag{38}$$

In particular, if $\epsilon_{n,m}$ is of the form

$$\epsilon_{n,m} = \frac{\epsilon}{8\rho^2_{n,m}}.$$

for $0 < \epsilon < 1$, then

$$\epsilon_{n,m} = \frac{\epsilon}{8\rho^2_{n,m}} \leq \frac{\epsilon}{4(1+\epsilon)\rho^2_{n,m}} \leq \frac{\epsilon}{4(1+\epsilon\rho^2_{n,m})}$$

and thus,

$$\left(\frac{1}{4\epsilon_{n,m}} - \rho^2_{n,m} \right) \geq \frac{1}{\epsilon}.$$

Therefore, (37) and (38) are respectively rewritten as

$$\|f \rho_{n,m} v_{n,m}\|^2_{L^2(G)} \leq \frac{\epsilon}{8} \|f\|^2_{L^2(G)}.$$

and

$$\int_0^\infty |\tilde{f}_{n,m}(\lambda)|^2 \lambda^2 |C_{0,0}(\lambda)|^{-2} d\lambda \geq \frac{1}{\epsilon} \|f\|^2_{L^2(G)}.$$

Let $f = \sum_{n,m\in\mathbb{N}} f^{n,m}$ denote the K-type decomposition of an L^2-function f on G whose K-types are supported on $\mathbb{N}\times\mathbb{N}$. Since

$$\|f\|^2_{L^2(G)} = \sum_{n,m\in\mathbb{N}} \|f^{n,m}\|_{L^2(G)}$$

and the Hilbert-Schmidt norm of $\pi_{0,\lambda}(f) = \Big((f^{n,m})^\wedge(\lambda)\Big)_{n,m\in\mathbb{N}}$ is given by

$$\|\pi_{0,\lambda}(f)\|^2_{\mathrm{HS}} = \sum_{n,m\in\mathbb{N}} |f^{n,m}(\lambda)|^2,$$

we can obtain the following.

Theorem 7.1. *Let $\epsilon > 0$ and $f = \sum_{n,m\in\mathbb{N}} f^{n,m}$ be an L^2-function on $SU(1,1)$. We suppose that each $f^{n,m}$ is concentrated at $x = 0$ such as*

$$\|f^{n,m}\rho_{n,m}v_{n,m}\|^2_{L^2(G)} \leq \frac{\epsilon}{8}\|f^{n,m}\|^2_{L^2(G)}. \tag{39}$$

Then

$$\int_0^\infty \|\pi_{0,\lambda}(f)\|^2_{\mathrm{HS}}\lambda^2|C_{0,0}(\lambda)|^{-2}d\lambda \geq \frac{1}{\epsilon}\|f\|^2_{L^2(G)},$$

where $\|\ \|_{\mathrm{HS}}$ is the Hilbert-Schmid norm. In particular, $\|\pi_{0,\lambda}(f)\|_{\mathrm{HS}}$ does not concentrate at $\lambda = 0$.

Remark 7.2. It easily follows from (24) and (25) that

$$\rho_{n,m}v_{n,m} = O\Big(\min\Big(\frac{|n-m|+n+m}{|n-m|+1}, \sqrt{|n-m|+n+m}\Big)\Big).$$

Therefore, if the right or left K-types of f are finite, then $\{\rho_{n,m}v_{n,m}\}$ in (39) are uniformly bounded. However, for example, if $n = m$, then $\{\rho_{n,n}v_{n,n}\}$ are not uniformly bounded.

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Present addresses:
Takeshi Kawazoe
Department of Mathematics, Keio University at Fujisawa,
Endo, Fujisawa, Kanagawa, 252-8520, Japan.
e-mail: kawazoe@sfc.keio.ac.jp