Remarks on Muckenhoupt weights with variable exponent

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概 要

本講演の目的は，変動指数によって一般化された Muckenhoupt の $A_{p(\cdot)}$ 条件に対して，弱正値性という特殊な性質を持った作用素の族を用いて同値な条件を与える事である．

1 変動指数 Lebesgue 空間

定義 1.1. 可測関数 $p(\cdot)$: $\mathbb{R}^n \to [1, \infty]$ に対し，変動指数 Lebesgue 空間を

$$L^{p(\cdot)}(\mathbb{R}^n) := \{f \,:\, \text{定数}\lambda > 0 \text{ が存在し，} \rho_p(f/\lambda) < \infty\}$$

によって定める．但し，

$$\rho_p(f) := \int_{\{p(x)<\infty\}} |f(x)|^{p(x)}\, dx + \|f\|_{L^\infty(\{p(x)=\infty\})}.$$

さらに，

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\{\lambda > 0 \,:\, \rho_p(f/\lambda) \le 1\}$$

と定めると，これは $L^{p(\cdot)}(\mathbb{R}^n)$ のノルムとなる．

定義 1.2.

1. 可測関数 $p(\cdot)$: $\mathbb{R}^n \to [1, \infty]$ に対し，

$$p_+ := \|p\|_{L^\infty(\mathbb{R}^n)},\ p_- := \left\{ \left(\frac{1}{p}\right)_+ \right\}^{-1}$$

と定める．

2. 可測関数 $p(\cdot)$: $\mathbb{R}^n \to [1, \infty]$ に対し，$p'(\cdot)$ は共役指数，すなわち $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ を満たす関数とする．

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3. 以下の 2 条件を満たす可測関数 $p(\cdot):\mathbb{R}^n\to(0,\infty)$ を大域的 log-Hölder 連続であるという：

$$|p(x)-p(y)|\le\frac{C}{-\log(|x-y|)}\qquad(|x-y|\le 1/2),$$
$$|p(x)-p_\infty|\le\frac{C}{\log(e+|x|)}\qquad(x\in\mathbb{R}^n),$$

但し，p_∞ は定数である．大域的 log-Hölder 連続な可測関数全体の集合を $LH(\mathbb{R}^n)$ と定める．

2 弱正値核を持つ作用素の族

以下の条件を満たす $L^2(\mathbb{R}^n)$ の閉部分空間の族 $\{V_j\}_{j\in\mathbb{Z}}$ を多重解像度解析（MRA）と呼ぶ：

1. 全ての $j\in\mathbb{Z}$ に対し，$V_j\subset V_{j+1}$.
2. $\cup_{j\in\mathbb{Z}}V_j$ は $L^2(\mathbb{R}^n)$ において稠密である．
3. $\cap_{j\in\mathbb{Z}}V_j=\{0\}$.
4. 各 $j\in\mathbb{Z}$ に対し，$f\in V_j$ と $f(2x)\in V_{j+1}$ は同値である．
5. $f\in V_0$ ならば, 全ての $k\in\mathbb{Z}^n$ に対して $f(x-k)\in V_0$ である．
6. ある関数 $\varphi\in L^2(\mathbb{R}^n)$ が存在し，$\{\varphi(x-k)\}_{k\in\mathbb{Z}^n}$ は V_0 の正規直交基底となる．

関数 φ を MRA $\{V_j\}_{j\in\mathbb{Z}}$ のスケーリング関数という．ウェーブレットの基本理論より，C^1 級の滑らかさとコンパクトな台を持つスケーリング関数を構成する事ができる ([4, 10] 参照)．以後，C^1 級の滑らかさとコンパクトな台を持つスケーリング関数 φ を持つ MRA $\{V_j\}_{j\in\mathbb{Z}}$ について考える．

各 $j\in\mathbb{Z}$ に対し，直交射影 $P_j:L^2(\mathbb{R}^n)\to V_j$ を

$$P_jf(x)=\sum_{k\in\mathbb{Z}^n}\langle f,\varphi_{j,k}\rangle\varphi_{j,k}(x)=\int_{\mathbb{R}^n}K_j(x,y)f(y)\,dy$$

によって与える事ができる．但し，

$$\begin{aligned}K_j(x,y)&:=\sum_{k\in\mathbb{Z}^n}\varphi_{j,k}(x)\overline{\varphi_{j,k}(y)},\\ \varphi_{j,k}(x)&:=2^{jn/2}\varphi(2^jx-k),\\ \langle f,\varphi_{j,k}\rangle&:=\int_{\mathbb{R}^n}f(x)\overline{\varphi_{j,k}(x)}\,dx.\end{aligned}$$

以下のような直交射影の族 $\{P_j\}_{j\in\mathbb{Z}}$ の興味深い性質が Aimar–Bernardis–Martín-Reyes [1] によって示されている．

補題 2.1.

1. 全ての $j \in \mathbb{Z}$ に対し，$|P_j f(x)| \le C\, Mf(x)$．ここで，M は

$$Mf(x) := \sup_{r>0,\, y\in\mathbb{R}^n\,;\, x\in y+(-r,r)^n} \frac{1}{(2r)^n}\int_{y+[-r,r]^n} |f(z)|\, dz$$

によって定義される *Hardy–Littlewood* の極大作用素である．

2. 核の族 $\{K_j(x,y)\}_{j\in\mathbb{Z}}$ は，弱正値と呼ばれる次の性質を持つ：定数 $\mathcal{C} > 0$ と正数列 $\{\ell_j\}_{j\in\mathbb{Z}}$ が存在し，以下を満たす：

   (a) 全ての $j \in \mathbb{Z}$ に対し，$0 < \ell_{j+1} < \ell_j < \infty$,

   (b) $\lim_{j\to\infty} \ell_j = 0$ かつ $\lim_{j\to-\infty} \ell_j = \infty$,

   (c) $|x-y| < \ell_j$ ならば $K_j(x,y) > \mathcal{C}(\ell_{j+1})^{-n}$.

3 変動指数 Muckenhoupt ウェイトと重み付き空間

定義 3.1. $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ とし，関数 w は $0 < w < \infty$ a.e. $\mathbb{R}^n$，および任意の可測集合 E に対し，$w^{1/p(\cdot)}\chi_E \in L^{p(\cdot)}(\mathbb{R}^n)$, $w^{-1/p(\cdot)}\chi_E \in L^{p'(\cdot)}(\mathbb{R}^n)$ を満たすとする．

1. 重み付き変動指数 Lebesgue 空間 $L^{p(\cdot)}_w(\mathbb{R}^n)$ を

$$\begin{aligned} L^{p(\cdot)}_w(\mathbb{R}^n) &:= \{f \,:\, \|f\|_{L^{p(\cdot)}_w(\mathbb{R}^n)} < \infty\}, \\ \|f\|_{L^{p(\cdot)}_w(\mathbb{R}^n)} &:= \|f w^{1/p(\cdot)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \end{aligned}$$

によって定義する．

2. 関数 w が

$$\sup_{Q:\text{cube}} \frac{1}{|Q|} \|w^{1/p(\cdot)}\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|w^{-1/p(\cdot)}\chi_Q\|_{L^{p'(\cdot)}(\mathbb{R}^n)} < \infty \tag{3.1}$$

を満たす時，$A_{p(\cdot)}$ ウェイトであるという．また，$A_{p(\cdot)}$ ウェイト全体の集合を $A_{p(\cdot)}$ と書く．

Diening–Hästö [8] は次の同値性を $p_- > 1$ の場合について最初に証明した．しかしながら，その証明は Diening の仕事 ([5, 6, 7]) に大きく依存している．関連した結果については，[2] も参照されたい．$p_- = 1$ の場合も含めた自己完結した証明が Cruz-Uribe–Fiorenza–Neugebauer [3] によって与えられている：

定理 3.1. $p(\cdot) \in LH(\mathbb{R}^n)$，$p_+ < \infty$ と仮定する．もし $p_- > 1$ ならば，次の 3 条件は同値である：

1. $w \in A_{p(\cdot)}$.

2. *Hardy–Littlewood* の極大作用素 M は $L^{p(\cdot)}_w(\mathbb{R}^n)$ 上有界である．

3. M は $L^{p(\cdot)}_w(\mathbb{R}^n)$ において弱 $(p(\cdot),p(\cdot))$ 型である，すなわち，任意の $\lambda>0$ と $f\in L^{p(\cdot)}_w(\mathbb{R}^n)$ に対して，

$$\|\chi_{\{Mf(x)>\lambda\}}w^{1/p(\cdot)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\lambda^{-1}\|fw^{1/p(\cdot)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

が成り立つ．

もし $p_-\ge 1$ ならば，２条件 *1.* と *3.* は同値である．

4 主結果

定理 3.1 において，M を作用素の族 $\{P_j\}_{j\in\mathbb{Z}}$ に置き換える事ができる．次の定理は，変動指数 $p(\cdot)$ が定数の場合に Aimar–Bernardis–Martín-Reyes [1] によって証明された．その結果を，出来 [9] が変動指数の場合へ一般化した．

定理 4.1. $p(\cdot)\in LH(\mathbb{R}^n)$，$p_+<\infty$ と仮定する．もし $p_->1$ ならば，次の３条件 *(M1), (M2), (M3)* は同値である：

(M1) $w\in A_{p(\cdot)}$.

(M2) 作用素の族 $\{P_j\}_{j\in\mathbb{Z}}$ は，$L^{p(\cdot)}_w(\mathbb{R}^n)$ において一様に有界である，すなわち，全ての $j\in\mathbb{Z}$ および $f\in L^{p(\cdot)}_w(\mathbb{R}^n)$ に対して，

$$\|(P_jf)w^{1/p(\cdot)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\,\|fw^{1/p(\cdot)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

(M3) 族 $\{P_j\}_{j\in\mathbb{Z}}$ は，$L^{p(\cdot)}_w(\mathbb{R}^n)$ において一様に弱 $(p(\cdot),p(\cdot))$ 型である，すなわち，全ての $j\in\mathbb{Z}$，$\lambda>0$ および $f\in L^{p(\cdot)}_w(\mathbb{R}^n)$ に対して

$$\|\chi_{\{P_jf(x)>\lambda\}}w^{1/p(\cdot)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\lambda^{-1}\|fw^{1/p(\cdot)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

もし $p_-\ge 1$ ならば，２条件 *(M1)* と *(M3)* は同値である．

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Duality between Herz-Morrey spaces of variable exponent

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Let G be a boundded open set in R^n. For $x_0 \in G$ and $1 \le q \le \infty$, we consider the small Herz-Morrey space $\underline{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ of variable exponent consisting of all measurable functions f on G satisfying

$$\|f\|_{\underline{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \big(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r))}\big)^q \, dr/r \right)^{1/q} < \infty,$$

where $p(\cdot)$ is a variable exponent and $\omega(x_0,\cdot)$ is a positive monotone function. We also consider the grand Herz-Morrey space $\overline{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ of variable exponent consisting of all measurable functions f on G satisfying

$$\|f\|_{\overline{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \big(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,r))}\big)^q \, dr/r \right)^{1/q} < \infty.$$

Our aim in this talk is to discuss duality between these Herz-Morrey spaces. For example, we show the following result under some conditions on weights.

THEOREM. *Let $x_0 \in G$ and $1 < q \le \infty$. Suppose there exist constants $a, b, Q > 0$ such that*

(1) $$\int_0^{2d_G} \omega(x_0,s)^q \frac{ds}{s} < \infty \ \ (q<\infty) \quad \textit{and} \quad \omega(x_0,0) = \inf_{t>0} \omega(x_0,t) = 0 \ (q=\infty);$$

(2) $$\int_0^t s^b \omega(x_0,s)^{-q'} \frac{ds}{s} \le Q t^b \eta(x_0,t)^{q'}; \textit{ and}$$

(3) $$\int_t^{2d_G} s^a \eta(x_0,s)^{q'} \frac{ds}{s} \le Q t^a \omega(x_0,t)^{q'} \qquad \textit{for all } 0<t<d_G.$$

Then $\left(\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)' = \underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)$.

The principal inverse of the gamma function

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Abstract

Let $\Gamma(x)$ be the gamma function in the real axis and α the maximal zero of $\Gamma'(x)$. We call the inverse function of $\Gamma(x)|_{(\alpha,\infty)}$ the principal inverse and denote it by $\Gamma^{-1}(x)$. We show that $\Gamma^{-1}(x)$ has the holomorphic extension $\Gamma^{-1}(z)$ to $\mathbf{C}\setminus(-\infty,\Gamma(\alpha)]$, which maps the upper half plane into itself, namely a Pick function, and that $\Gamma(\Gamma^{-1}(z))=z$ on $\mathbf{C}\setminus(-\infty,\Gamma(\alpha)]$.

Remarks on analytic projection on certain compact groups

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$\mathbb{T}$ を 一次元 circle group とし、$Trig(\mathbb{T})$ を $\mathbb{T}$ 上の三角多項式全体の空間とする．　Φ を $\mathbb{T}$ 上の解析的射影作用素とする．すなわち、

$$\Phi(\sum_n a_n e^{inx}) = \sum_{n \geqq 0} a_n e^{inx}.$$

$1 < p < \infty$ のとき、Φ は $(Trig(\mathbb{T}), \|\cdot\|_p)$ 上の有界線形作用素になるが、$(Trig(\mathbb{T}), \|\cdot\|_1)$ 上の有界線形作用素にならないことが知られている．　ところが、$0 < p < 1$ に対し、$C = C(p) > 0$ が存在し、

$$\|\Phi(f)\|_p \leqq C\|f\|_1 \quad (\forall f \in Trig(\mathbb{T}))$$

が成り立つ．但し、$\|\Phi(f)\|_p = (\int_{\mathbb{T}} |\Phi(f)(x)|^p\, dx)^{\frac{1}{p}}$.

ここでは、ある種のコンパクト群上に解析的射影作用素を定義し、その有界性を調べる．

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Concavity of an auxiliary mean function and its application

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Theorem 1 *For $x > 0$, we set the function of y as*

$$F(y) \equiv \left(\frac{1+x^y}{2}\right)^{1/y}.$$

Then $F(y)$ has following properties.

(i) $F(y)$ *is monotone increasing for* $y \in \mathbb{R}$.

(ii) $F(y)$ *is convex for* $y < 0$.

(iii) $F(y)$ *is concave for* $y \geq 1/2$.

Proof of Thorem 1:

(i) Since $F(y) > 0$ for $x > 0$ and $y \in \mathbb{R}$, it is sufficient to prove $\frac{d}{dy}\log F(y) > 0$ for the proof of $F'(y) > 0$. We have

$$\frac{d}{dy}\log F(y) = \frac{1}{y^2}\left(\log 2 + \frac{x^y \log x^y}{1+x^y} - \log(1+x^y)\right).$$

Then we put

$$G(r) \equiv (r+1)\log 2 + r\log r - (r+1)\log(r+1), (r > 0),$$

where we put $x^y \equiv r > 0$. From elementary calculations, we have $G(r) \geq G(1) = 0$ which implies $\frac{d}{dy}\log F(y) > 0$.

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(ii) We firstly set $f(y) \equiv \log F(y)$. Since $F(y) > 0$, we have only to prove $f''(y) \geq 0$ for the proof of $F''(y) \geq 0$. We set again $g(y) \equiv \frac{1+x^y}{2}, (x > 0, y < 0)$. Then we have $\frac{d^2}{dy^2} \log g(y) \equiv \frac{x^y (\log x)^2}{(1+x^y)^2} > 0$. In addition, by $f(y) = \frac{1}{y} \log g(y)$, we have

$$f'(y) = \frac{1}{y}\frac{g'(y)}{g(y)} - \frac{1}{y^2} \log g(y) > 0.$$

By $\frac{d^2}{dy^2} \log g(y) = \frac{g(y)g''(y) - \{g'(y)\}^2}{g(y)^2}$, we have

$$f''(y) = \frac{1}{y}\frac{g(y)g''(y) - \{g'(y)\}^2}{g(y)^2} - \frac{2}{y^2}\frac{g'(y)}{g(y)} + \frac{2}{y^3}\log g(y) = \frac{1}{y}\frac{d^2}{dy^2}\log g(y) - \frac{2}{y} f'(y).$$

We prove $f''(y) \geq 0$ for $y < 0$. We calculate

$$\begin{aligned} f''(y) &= \frac{1}{y}\frac{x^y (\log x)^2}{(1+x^y)^2} - \frac{2}{y}\frac{1}{y^2}\left(\log 2 + \frac{x^y \log x^y}{1+x^y} - \log(1+x^y)\right) \\ &= \frac{1}{y^3(1+x^y)^2}\left\{-2x^y(1+x^y)\log x^y + x^y(\log x^y)^2 + 2(1+x^y)^2 \log\frac{1+x^y}{2}\right\}. \end{aligned}$$

Thus, if we put

$$h(y) \equiv -2x^y(1+x^y)\log x^y + x^y(\log x^y)^2 + 2(1+x^y)^2\log\frac{1+x^y}{2},$$

then we have only to prove $h(y) \leq 0$ for $y < 0$. Since we have $h(0) = 0$, we have only to prove $h'(y) \geq 0$ for $y < 0$. Here we have

$$h'(y) = -x^y \log x \left\{4x^y \log x^y - (\log x^y)^2 - 4(1+x^y)\log\frac{1+x^y}{2}\right\}.$$

If we set again

$$l(t) \equiv 4t\log t - (\log t)^2 - 4(t+1)\log\frac{t+1}{2},$$

where we put $x^y \equiv t > 0$, then we prove the following cases.

(a) If $x \leq 1\,(i.e., t \geq 1)$, then $l(t) \geq 0$.

(b) If $x \geq 1\,(i.e., 0 < t \leq 1)$, then $l(t) \leq 0$.

For the case (a), we calculate

$$l'(t) = \frac{1}{t}\left(4t\log 2 + (4t-2)\log t - 4t\log(t+1)\right)$$

and

$$l''(t) = \frac{2\{(t+1)\log t + t - 1\}}{t^2(t+1)} \geq 0, (t \geq 1).$$

Thus we have $l'(t) \geq l'(1) = 0$, and then we have $l(t) \geq l(1) = 0$. For the case (b), we easily find that

$$l''(t) = \frac{2\{(t+1)\log t + t - 1\}}{t^2(t+1)} \leq 0, (0 < t \leq 1).$$

Thus we have $l'(t) \geq l'(1) = 0$, and then we have $l(t) \leq l(1) = 0$. Therefore we have $h'(y) \geq 0$ for $y < 0$.

(iii) We calculate

$$\frac{d^2}{dy^2}F(y) = \frac{1}{y^4}\left(\frac{1+x^y}{2}\right)^{1/y} h(x,y),$$

where

$$h(x,y) = (\log 2 - 2y)\log 2 + \frac{2\log 2}{1+x^y}\{x^y \log x^y - (1+x^y)\log(1+x^y)\}$$
$$+\frac{1}{(1+x^y)^2}\{x^y y^2(x^y+y)(\log x)^2\} - \frac{1}{(1+x^y)^2}\{2x^y(1+x^y)(y+\log(1+x^y))\log x^y\}$$
$$+\{2y+\log(1+x^y)\}\log(1+x^y).$$

We prove $h(x,y) \leq 0$ for $x > 0$ and $y \geq 1/2$. Then we have

$$\frac{dh(x,y)}{dx} = -\frac{x^{-1+y}y^2\log x}{(1+x^y)^3}\left\{(x^y(y-2)-y)\log x^y + 2(1+x^y)\log\left(\frac{1+x^y}{2}\right)\right\}.$$

Here we note that $\frac{dh(1,y)}{dx} = 0$. We also put

$$g(x,y) = \{x^y(-2+y) - y\}\log x^y + 2(1+x^y)\log\left(\frac{1+x^y}{2}\right).$$

If we have $g(x,y) \geq 0$ for $x > 0$ and $y \geq 1/2$, then we have $\frac{dh(x,y)}{dx} \geq 0$ for $0 < x \leq 1$, and $\frac{dh(x,y)}{dx} \leq 0$ for $x \geq 1$. Thus we then obtain $h(x,y) \leq h(1,y) = 0$ for $y \geq 1/2$, due to $\frac{dh(1,y)}{dx} = 0$. Therefore, we have only to prove $g(x,y) \geq 0$ for $x > 0$ and $y \geq 1/2$.

(a) For the case $0 < x \leq 1$, we have

$$\frac{dg(x,y)}{dx} = \frac{y}{x}\left\{y(x^y-1) + (y-2)x^y\log x^y + 2x^y\log\left(\frac{x^y+1}{2}\right)\right\}.$$

Since $g(1,y) = 0$, if we prove $\frac{dg(x,y)}{dx} \leq 0$, then we can prove $g(x,y) \geq g(1,y) = 0$ for $y \geq 1/2$ and $0 < x \leq 1$. Since we have the relations

$$\frac{x-1}{\sqrt{x}} \leq \log x \leq \frac{2(x-1)}{x+1} \leq 0$$

for $0 < x \leq 1$, we calculate

$$y(x^y-1) + (y-2)x^y\log x^y + 2x^y\log\left(\frac{x^y+1}{2}\right)$$
$$\leq y(x^y-1) + (y-2)x^y\frac{(x^y-1)}{x^{y/2}} + 2x^y\frac{2\left(\frac{x^y+1}{2}-1\right)}{\frac{x^y+1}{2}+1}$$
$$= \frac{x^y-1}{x^y+3}\left\{3(y-2)x^{y/2} + (y-2)x^{3y/2} + 3y + (y+4)x^y\right\}.$$

Thus we have only to prove

$$k(y) \equiv 3\,(y-2)\,x^{y/2} + (y-2)\,x^{3y/2} + 3y + (y+4)\,x^{y} \geq 0$$

for $0 < x \leq 1$ and $y \geq 1/2$. Since it is trivial $k(y) \geq 0$ for $y \geq 2$, we assume $1/2 \leq y < 2$ from here. To this end, we prove that $k_1(y) \equiv 3\,(y-2)\,x^{y/2} + (y-2)\,x^{3y/2}$ is monotone increasing for $1/2 \leq y < 2$, and $k_2(y) \equiv 3y + (y+4)\,x^{y}$ is also monotone increasing for $1/2 \leq y < 2$. We easily find that

$$\frac{dk_1(y)}{dy} = \frac{1}{2}x^{y/2}\left\{2(x^y+3) + 3(x^y+1)(y-2)\log x\right\} > 0,$$

for $0 < x \leq 1$ and $1/2 \leq y < 2$.
We also have

$$\frac{dk_2(y)}{dy} = x^y + 3 + (y+4)x^y \log x.$$

Here we prove $\frac{dk_2(y)}{dy} \geq 0$ for $0 < x \leq 1$ and $1/2 \leq y < 2$. We put again

$$k_3(x) \equiv x^y + 3 + (y+4)x^y \log x,$$

then we have

$$\frac{dk_3(x)}{dx} = x^{-1+y}\left\{2(y+2) + y(y+4)\log x\right\}.$$

Thus we have

$$\frac{dk_3(x)}{dx} = 0 \Leftrightarrow x = \mathrm{e}^{-\frac{2(y+2)}{y(y+4)}} \equiv \alpha_y.$$

Since $\frac{dk_3(x)}{dx} < 0$ for $0 < x < \alpha_y$ and $\frac{dk_3(x)}{dx} > 0$ for $\alpha_y < x \leq 1$, we have

$$k_3(x) \geq k_3(\alpha_y) = 3 - \frac{(y+4)\mathrm{e}^{-\frac{2(y+2)}{y+4}}}{y} \equiv k_4(y).$$

Since we have $\frac{dk_4(y)}{dy} = \frac{8(y+2)\mathrm{e}^{-\frac{2(y+2)}{y+4}}}{y^2(y+4)} > 0$, the function $k_4(y)$ is monotone increasing for y. Thus we have

$$k_3(x) \geq k_3(\alpha_y) = 3 - \frac{(y+4)\mathrm{e}^{-\frac{2(y+2)}{y+4}}}{y} \equiv k_4(y) \geq k_4(1/2) = 3 - \frac{9}{\mathrm{e}^{10/9}} > 0,$$

since $e^{10/9} \simeq 3.03773$. Therefore $k_2(y)$ is also monotone increasing function of y for $0 < x \leq 1$ and $1/2 \leq y < 2$. Thus $k(y)$ is monotone increasing for $y \geq 1/2$, then we have

$$k(y) \geq k(1/2) = -\frac{3}{2}\left(x^{1/4} - 1\right)^3 \geq 0.$$

(b) For the case $x \geq 1$, we firstly calculate

$$\frac{dg\left(x,y\right)}{dy} = \left(x^{y}-1\right)\log x^{y} + \left\{y\left(x^{y}-1\right) + \left(y-2\right)x^{y}\log x^{y} + 2x^{y}\log\left(\frac{1+x^{y}}{2}\right)\right\}\log x.$$

We put

$$p\left(x,y\right) \equiv (x^{y}-1)y + x^{y}(y-2)\log x^{y} + 2x^{y}\log\left(\frac{1+x^{y}}{2}\right).$$

Then we calculate

$$\frac{dp\left(x,y\right)}{dx} = \frac{y}{x+x^{1-y}}\left\{(1+x^{y})(y-2)\log x^{y} + 2\left(y(1+x^{y}) - 1 + (1+x^{y})\log\left(\frac{1+x^{y}}{2}\right)\right)\right\}.$$

Then we put

$$q\left(x,y\right) = (y-2)\log x^{y} + 2\log\left(\frac{1+x^{y}}{2}\right) + 2y - \frac{2}{1+x^{y}}.$$

We have

$$\frac{dq\left(x,y\right)}{dy} = \frac{\left((1+x^{y})^{2}y-2\right)\log x + (1+x^{y})^{2}(\log x^{y}+2)}{(1+x^{y})^{2}} > 0$$

and then

$$q\left(x,y\right) \geq q\left(x,1/2\right) = 1 - \frac{2}{\sqrt{x}+1} + 2\log\left(\frac{1+\sqrt{x}}{2}\right) - \frac{3}{4}\log x.$$

Since we find

$$\frac{dq\left(x,1/2\right)}{dx} = \frac{(\sqrt{x}+3)(\sqrt{x}-1)}{4x(\sqrt{x}+1)^{2}} \geq 0$$

for $x \geq 1$, we have $q\left(x,y\right) \geq q\left(x,1/2\right) \geq q\left(1,1/2\right) = 0$. Therefore we have $\frac{dp(x,y)}{dx} \geq 0$, which implies $p\left(x,y\right) \geq p\left(1,y\right) = 0$. Thus we have $\frac{dg(x,y)}{dy} \geq 0$, and then we have $g(x,y) \geq g(x,1/2)$, where

$$g(x,1/2) = -\frac{1}{2}(3x^{1/2}+1)\log x^{1/2} + 2(x^{1/2}+1)\log\left(\frac{x^{1/2}+1}{2}\right).$$

To prove $g(x,1/2) \geq 0$ for $x \geq 1$ and $y \geq 1/2$, we put $x^{1/2} \equiv z \geq 1$ and

$$r(z) \equiv -\frac{1}{2}(3z+1)\log z + 2(z+1)\log\left(\frac{z+1}{2}\right).$$

Since we have $r''(z) = \frac{(z-1)^{2}}{2z^{2}(z+1)} \geq 0$ and

$$r'(z) = \frac{1}{2z}\left\{z - 1 - 3z\log z + 4z\log\left(\frac{z+1}{2}\right)\right\},$$

we have $r'(1) = 0$ and then we have $r'(z) \geq 0$ for $z \geq 1$. Thus we have $r(z) \geq 0$ for $z \geq 1$ by $r(1) = 0$. Finally we have $g(x,y) \geq g(x,1/2) \geq 0$, for $x \geq 1$ and $y \geq 1/2$. $\square$