# フラクタル曲線と有向ネットワーク

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Lebesgue の特異関数をそのパラメターで微分すると Takagi 関数になることは, 畑-山口 [1] で指摘された．その後, 高次微分などの改良およびその応用が塩田, 岡田, 小林, 神谷, 我々等 [2],[3],[4],[5],[6],[7] によりなされた．本発表では、これらの結果を有向ネットワーク上に拡張する．また, それを用いて描けるフラクタル曲線についてもふれる．

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# On the generalized $\sigma$-Lipschitz spaces and the generalized fractional integrals

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For $r > 0$, let $Q_r = \{y \in \mathbb{R}^n : |y| < r\}$ or $Q_r = \{y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n : \max_{1\le i\le n} |y_i| < r\}$, and for $x \in \mathbb{R}^n$, let $Q(x, r) = x + Q_r = \{x + y : y \in Q_r\}$. For a measurable set $G \subset \mathbb{R}^n$, we denote the Lebesgue measure of $G$ by $|G|$ and the characteristic function of $G$ by $\chi_G$. Further, for a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a measurable set $G \subset \mathbb{R}^n$ with $|G| > 0$, let $f_G = \fint_G f(y)\,dy = \frac{1}{|G|}\int_G f(y)\,dy$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

First, we recall the definitions of the non-homogeneous central Morrey space $B^{p,\lambda}(\mathbb{R}^n)$ and the $\lambda$-central mean oscillation ($\lambda$-CMO) space $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$.

**Definition 1.** *For $1 \le p < \infty$ and $-n/p \le \lambda < \infty$,*

$$B^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{B^{p,\lambda}} = \sup_{r\ge 1} \frac{1}{r^\lambda} \left( \fint_{Q_r} |f(y)|^p\,dy \right)^{1/p} < \infty \right\}.$$

On the other hand, we introduce the "new" function space, i.e., the generalized $\sigma$-Lipschitz space $\mathrm{Lip}^{(d)}_{\beta,\sigma}(\mathbb{R}^n)$ (see Nakai and Sawano (2012), M. (to appear); cf. Komori-Furuya, M., Nakai and Sawano (2013)).

**Definition 2.** *Let $U = \mathbb{R}^n$ or $U = Q_r$ with $r > 0$. For $d \in \mathbb{N}_0$ and $0 \le \beta \le 1$, the continuous function $f$ will be said to belong to the generalized Lipschitz space on $U$, i.e., $\mathrm{Lip}^{(d)}_\beta(U)$ if and only if*

$$\|f\|_{\mathrm{Lip}^{(d)}_\beta(U)} = \sup_{x, x+h\in U, h\ne 0} \frac{1}{|h|^\beta} |\triangle_h^{d+1} f(x)| < \infty,$$

*where $\triangle_h^k$ is a difference operator, which is defined inductively by*

$$\triangle_h^0 f = f, \quad \triangle_h^1 f = \triangle_h f = f(\cdot + h) - f(\cdot),$$
$$\triangle_h^k f = \triangle_h^{k-1} f(\cdot + h) - \triangle_h^{k-1} f(\cdot), \quad k = 2, 3, \cdots.$$

**Definition 3.** *For $d \in \mathbb{N}_0$, $0 \le \beta \le 1$ and $0 \le \sigma < \infty$, the continuous function $f$ will be said to belong to the generalized $\sigma$-Lipschitz ($\sigma$-Lip) space, i.e., $\mathrm{Lip}^{(d)}_{\beta,\sigma}(\mathbb{R}^n)$ if and only if*

$$\|f\|_{\mathrm{Lip}^{(d)}_{\beta,\sigma}} = \sup_{r\ge 1} \frac{1}{r^\sigma} \|f\|_{\mathrm{Lip}^{(d)}_\beta(Q_r)} < \infty.$$

*In particular,*

$$\mathrm{Lip}_{\beta,\sigma}(\mathbb{R}^n) = \mathrm{Lip}^{(0)}_{\beta,\sigma}(\mathbb{R}^n) \text{ and } \mathrm{BMO}^{(d)}_\sigma(\mathbb{R}^n) = \mathrm{Lip}^{(d)}_{0,\sigma}(\mathbb{R}^n), \mathrm{BMO}_\sigma(\mathbb{R}^n) = \mathrm{BMO}^{(0)}_\sigma(\mathbb{R}^n).$$

Next we recall the definition of modified fractional integral $\tilde{I}_\alpha$.

**Definition 4.** *For* $0 < \alpha < n$,

$$\tilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1-\chi_{Q_1}(y)}{|y|^{n-\alpha}} \right) dy.$$

Recently, in M. and Nakai (2011), from the $B_\sigma$-Morrey-Campanato estimate for $\tilde{I}_\alpha$ we obtained the following as the corollary.

**Theorem 1** (M. and Nakai (2011); cf. Komori-Furuya and M. (2010))**.** *Let* $0 < \alpha < n$, $n/\alpha < p < \infty$ *and* $-n/p \le \lambda < 1-\alpha$. *If* $\beta = \alpha - n/p$ *and* $\sigma = \lambda + n/p$, *then*

$$\tilde{I}_\alpha : B^{p,\lambda}(\mathbb{R}^n) \to \mathrm{Lip}_{\beta,\sigma}(\mathbb{R}^n).$$

Now we define the generalized fractional integral $\tilde{I}_{\alpha,d}$.

**Definition 5.** *For* $0 < \alpha < n$ *and* $d \in \mathbb{N}_0$, *we define the generalized fractional integral (of order* $\alpha$*), i.e.,* $\tilde{I}_{\alpha,d}$, *as follows: For* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\tilde{I}_{\alpha,d} f(x) = \int_{\mathbb{R}^n} f(y) \left\{ K_\alpha(x-y) - \left( \sum_{\{l:|l|\le d\}} \frac{x^l}{l!} (D^l K_\alpha)(-y) \right) (1-\chi_{Q_1}(y)) \right\} dy,$$

*where*

$$K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$$

*and for* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ *and* $l = (l_1, l_2, \cdots, l_n) \in \mathbb{N}_0^n$, $|l| = l_1 + l_2 + \cdots + l_n$, $x^l = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$ *and* $D^l$ *is the partial derivative of order* $l$, *i.e.,*

$$D^l = (\partial/\partial x_1)^{l_1} (\partial/\partial x_2)^{l_2} \cdots (\partial/\partial x_n)^{l_n}.$$

Then as one of the results for a generalized fractional integral $\tilde{I}_{\alpha,d}$ we can get the following estimate on $B^{p,\lambda}(\mathbb{R}^n)$, which extends Theorem 1.

**Theorem 2** ([M])**.** *Let* $0 < \alpha < n$, $n/\alpha < p < \infty$, $d \in \mathbb{N}_0$ *and* $-n/p + \alpha + d \le \lambda + \alpha < d+1$. *If* $\beta = \alpha - n/p$ *and* $\sigma = \lambda + n/p$, *then*

$$\tilde{I}_{\alpha,d} : B^{p,\lambda}(\mathbb{R}^n) \to \mathrm{Lip}^{(d)}_{\beta,\sigma}(\mathbb{R}^n).$$

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# A note on Herz type inequalities

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Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\{\mathcal{F}_n\}_{n\geq 0}$ a nondecreasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. The expectation operalor and the conditional expectation operator relative to $\mathcal{F}_n$ are denoted by $E$ and $E_n$, respectively.

A sequence of integrable random variables $f = (f_n)_{n\geq 0}$ is called a martingale relative to $\{\mathcal{F}_n\}_{n\geq 0}$ if, for every $n$, $f_n$ is $\mathcal{F}_n$ measurable and satisfies

$$E_n[f_m] = f_n \quad (n \leq m).$$

If $f \in L_p$, $p \in [1, \infty)$, then $(f_n)_{n\geq 0}$ with $f_n = E_n f$ is an $L_p$-bounded martingale and converges to $f$ in $L_p$ ([6]). For this reason a function $f \in L_1$ and the corresponding martingale $(f_n)_{n\geq 0}$ with $f_n = E_n f$ will be denoted by the same symbol $f$.

We now introduce two martingale Hardy spaces. Let $\mathcal{M}$ be the set of all martingale $f = (f_n)_{n\geq 0}$ relative to $\{\mathcal{F}_n\}_{n\geq 0}$ such that $f_0 = 0$. Then the maximal function of a martingale $f$ are defined by

$$f_n^* = \sup_{0\leq m\leq n} |f_m|, \quad f^* = \sup_{n\geq 0} |f_n|.$$

Denote by $\Lambda$ the collection of all sequences $(\lambda_n)_{n\geq 0}$ of nondecreasing, nonnegative and adapted functions, and set $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. For $f \in \mathcal{M}$ and $0 < p < \infty$, let

$$\Lambda[P_p](f) = \{(\lambda_n)_{n\geq 0} \in \Lambda : |f_n| \leq \lambda_{n-1}, \ \ \lambda_\infty \in L_p\}.$$

We define two martingale spaces by

$$\begin{aligned} H_p^* &= \{f \in \mathcal{M} : \|f\|_{H_p^*} = \|f^*\|_p < \infty\}, \\ P_p &= \{f \in \mathcal{M} : \|f\|_{P_p} = \inf_{(\lambda_n)_{n\geq 0} \in \Lambda[P_p](f)} \|\lambda_\infty\|_p < \infty\}. \end{aligned}$$

We next introduce two martingale BMO spaces. For $f \in L_1$, let

$$\|f\|_{\mathrm{BMO}^-} = \sup_n \|E_n|f - E_{n-1}f|\|_\infty, \quad \|f\|_{\mathrm{BMO}} = \sup_n \|E_n|f - E_n f|\|_\infty.$$

Then, we define two martingale BMO spaces:

$$\begin{aligned} \mathrm{BMO}^- &= \{f \in L_1 : \|f\|_{\mathrm{BMO}^-} < \infty\}, \\ \mathrm{BMO} \ &= \{f \in L_1 : \|f\|_{\mathrm{BMO}} < \infty\}. \end{aligned}$$

In [3], Herz discussed the duality between $H_1^*$ and $\mathrm{BMO}^-$ and proved the following inequality for martingales:

$$|E[f\varphi]| \leq 12\|f\|_{P_1}\|\varphi\|_{\mathrm{BMO}} \quad (f \in L_\infty, \varphi \in \mathrm{BMO}).$$

This inequality is generalized by many authors.

In this talk, we give an extension of Herz type inequality with a different proof.

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# $B_w^u(E)$-関数空間の補間定理とその応用

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The purpose of this talk is to introduce $B_w^u(E)$-funciton spaces which unify many function spaces, Lebesgue, Morrey-Campanato, Lipschitz, $B^p$, CMO, local Morrey-type spaces, etc. We investigate the interpolation property of $B_w^u(E)$-funciton spaces and apply it to the boundedness of linear and sublinear operators, for example, the Hardy-Littlewood maximal operator, singular and fractional integral operators, and so on, which contains previous results and extends them to $B_w^u(E)$-funciton spaces.