1 Real Numbers

(1) $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$ — 0

(2) $\lim_{n\to\infty}\frac{n^2+5n+10}{2n^2-2n+1}$ — $\frac{1}{2}$

(3) $\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n$ — $\frac{1}{e}$

(4) $\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n$ — 1

(5) $\lim_{n\to\infty}\frac{(n+1)^4-(n-1)^4}{(n+1)^4+(n-1)^4}$ — 0

(6) $\lim_{n\to\infty}\frac{\sqrt[3]{n^3+2n-1}}{n+2}$ — 1

(7) $\lim_{n\to\infty}\frac{\sqrt[3]{n^2+n}}{n+1}$ — 0

(8) $\lim_{n\to\infty}\frac{(\sqrt{n^2+1}+n)^2}{\sqrt[3]{n^6+1}}$ — 4

(9) $\lim_{n\to\infty}\frac{n!}{(n+1)!-n!}$ — 0

(10) $\lim_{n\to\infty}\frac{(n+2)!+(n+1)!}{(n+3)!}$ — 0

(11) $\lim_{n\to\infty}\frac{n+1}{n}$ — 1

(12) $\lim_{n\to\infty}\frac{(n+1)^2}{2n^2}$ — $\frac{1}{2}$

(13) $\lim_{n\to\infty}\frac{(n+1)^3-(n-1)^3}{(n+1)^2+(n-1)^2}$ — 3

(14) $\lim_{n\to\infty}\frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$ — 1

(15) $\lim_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^n}}$ — $\frac{3}{4}$

(16) $\lim_{n\to\infty}\frac{1}{n^2}(1+2+\cdots+n)$ — $\frac{1}{2}$

(17) $\lim_{n\to\infty}\left(\frac{1+2+\cdots+n}{n+2}-\frac{n}{2}\right)$ — $-\frac{1}{2}$

(18) $\lim_{n\to\infty}\frac{1-2+3-4+\cdots-2n}{\sqrt{n^2+1}}$ — -1

(19) $\lim_{n\to\infty}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)n}\right)$ — 1

(20) $\lim_{n\to\infty}\frac{2^n-1}{2^n+1}$ — 1

2 Limits of Functions

(1) $\sin^{-1}\dfrac{1}{\sqrt{2}}$ $\qquad \dfrac{\pi}{4}$

(2) $\cos^{-1}\dfrac{\sqrt{3}}{2}$ $\qquad \dfrac{\pi}{6}$

(3) $\cos^{-1}x = \tan^{-1}\sqrt{5} \Rightarrow x = ?$ $\qquad 1/\sqrt{6}$

(4) $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x^3-3}$ $\qquad 0$

(5) $\displaystyle\lim_{x\to 0}\left(\frac{x^3-3x+1}{x-4}+1\right)$ $\qquad \dfrac{3}{4}$

(6) $\displaystyle\lim_{x\to 1}\frac{x^2-2x+1}{x^3-x}$ $\qquad 0$

(7) $\displaystyle\lim_{x\to 1}\frac{x^3-3x+2}{x^3-x^2-x+1}$ $\qquad \displaystyle= \lim_{x\to 1}\frac{x+2}{x+1} = \frac{3}{2}$

(8) $\displaystyle\lim_{x\to 1}\left(\frac{x+2}{x^2-5x+4}+\frac{x-4}{3(x^2-3x-2)}\right)$ $\qquad 0$

(9) $\displaystyle\lim_{x\to \infty}\frac{x^3+x}{2x^2+1}$ $\qquad \infty$

(10) $\displaystyle\lim_{x\to \infty}\frac{x^4-5x}{x^2-3x+1}$ $\qquad \infty$

(11) $\displaystyle\lim_{x\to \infty}\left(\frac{x^3}{x^2+1}-x\right)$ $\qquad 0$

(12) $\displaystyle\lim_{x\to \infty}\left(\frac{3x^2}{2x+1}-\frac{(2x-1)(3x^2+x+2)}{4x^2}\right)$ $\qquad -1/2$

(13) $\displaystyle\lim_{x\to \infty}\left(\sqrt{x^2+x+1}-x\right)$ $\qquad \displaystyle= \lim_{x\to \infty}\frac{x+1}{\sqrt{x^2+1}+x} = \frac{1}{2}$

(14) $\displaystyle\lim_{x\to \infty}\frac{\sqrt{x^2+1}}{\sqrt[4]{x^4+x}-x}$ $\qquad \infty$

(15) $\displaystyle\lim_{x\to \infty}\frac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$ $\qquad 1$

(16) $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x}$ $\qquad 0$

(17) $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x^2}$ $\qquad \dfrac{1}{2}$

(18) $\displaystyle\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$ $\qquad 4$

(19) $\displaystyle\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}$ $\qquad \dfrac{1}{4}$

(20) $\displaystyle\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$ $\qquad 3$

3 Ordinary Derivatives

(1)	$x^4 + 2x^2 - 1$	$4x^3 + 4x$
(2)	$x^5 - \dfrac{1}{x^3}$	$5x^4 + \dfrac{3}{x^4}$
(3)	$\dfrac{x^3 + 2x - 1}{x^2 - 1}$	$\dfrac{x^4 + x^2 + 2x + 2}{(x^2+1)^2}$
(4)	$x \sin x + \cos x$	$x \cos x$
(5)	$\dfrac{1}{\tan x}$	$-\mathrm{cosec}^2 x$
(6)	$e^x \cos x$	$e^x(\cos x - \sin x)$
(7)	$x \log x - x$	$\log x$
(8)	$\dfrac{x}{\log x}$	$\dfrac{\log x - 1}{(\log x)^2}$
(9)	$(x^2+1)^7$	$14x(x^2+1)^6$
(10)	$\cos 5x^2$	$-10x \sin 5x^2$
(11)	$(\sin 3x)^2$	$2 \sin 6x$
(12)	$\dfrac{1}{(x^2+1)^2}$	$-\dfrac{4x}{(x^2+1)^3}$
(13)	$x\sqrt{x^2+a^2}$	$\dfrac{2x^2+a^2}{x^2+a^2}$
(14)	$xe^{\cos 2x}$	$e^{\cos 2x}(1 - 2x \sin 2x)$
(15)	$\sqrt{1 + 2\log x}$	$\dfrac{1}{x\sqrt{1+2\log x}}$
(16)	$\log(\log x)$	$\dfrac{1}{x \log x}$
(17)	x^x	$x^x(\log x + 1)$
(18)	$\sqrt{\dfrac{1-x^2}{1+x^2}}$	$\dfrac{2x}{x^4-1}\sqrt{\dfrac{1-x^2}{1+x^2}}$
(19)	$\sin^{-1}\dfrac{x}{a} \quad (a > 0)$	$\dfrac{1}{\sqrt{a^2-x^2}}$
(20)	$(\tan^{-1} 2x)^3$	$\dfrac{6(\tan^{-1} 2x)^2}{1+4x^2}$
(21)	$\cos^{-1}\dfrac{1}{x}$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$
(22)	$(x^2 - 3x + 3)(x^2 + 2x - 1)$	$4x^3 - 3x^2 - 8x + 9$
(23)	$(x^3 - 3x + 2)(x^4 + x^2 - 1)$	$7x^6 - 10x^4 + 8x^3 - 12x^2 + 4x + 3$
(24)	$(\sqrt{x}+1)\left(\dfrac{1}{\sqrt{x}} - 1\right)$	$-\dfrac{x+1}{2x\sqrt{x}}$

(25)	$\left(\frac{2}{\sqrt{x}}-\sqrt{3}\right)\left(4x\sqrt[3]{x}+\frac{\sqrt[3]{x^2}}{3x}\right)$	$\frac{1}{9}\left(\frac{60}{\sqrt[6]{x}}-\frac{5}{x\sqrt[6]{x^5}}+\frac{\sqrt{3}}{x\sqrt[3]{x}}-48\sqrt[6]{27x^2}\right)$
(26)	$(x^2-1)(x^2-4)(x^2-9)$	$\frac{1-x^2}{(1+x^2)^2}$
(27)	$(1+\sqrt{x})(1+\sqrt{2x})(1+\sqrt{3x})$	$\frac{1+\sqrt{2}+\sqrt{3}+2\sqrt{2x}+2\sqrt{3x}+2\sqrt{6x}+3x\sqrt{6}}{2\sqrt{x}}$
(28)	$\frac{x+1}{x-1}$	$-\frac{2}{(x-1)^2}$
(29)	$\frac{x}{x^2+1}$	$\frac{1-x^2}{(1+x^2)}$
(30)	$\frac{3x^2+1}{x-1}$	$\frac{3x^2-6x-1}{(x-1)^2}$
(31)	$\frac{x^3-2x}{x^2+x+1}$	$\frac{x^4+2x^3+5x^2-2}{(x^2+x+1)^2}$
(32)	$\frac{ax+b}{cx+d} \quad (ad-bc\neq 0)$	$\frac{ad-bc}{(cx+d)^2}$
(33)	$\frac{x^2+1}{3(x^2-1)}+(x^2-1)(1-x)$	$-\frac{4x}{3(x^2-1)^2+1+2x-3x^2}$
(34)	$\frac{x^5}{x^3-2}$	$\frac{2x^4(x^2-5)}{(x^2-2)^2}$
(35)	$\frac{1-x^3}{1+x^3}$	$-\frac{6x^2}{(x^3+1)^2}$
(36)	$\frac{2}{x^3-1}$	$-\frac{6x^2}{x^3-1}{}^{2}$
(37)	$\frac{1}{x^3+x+1}$	$-\frac{3x^2+1}{x^3+x+1}{}^{2}$
(38)	$\frac{1}{x^2-3x+6}$	$-\frac{2x-3}{x^2-3x+6}{}^{2}$
(39)	$\frac{2x^4}{b^2-x^2}$	$\frac{4x^3(2b^2-x^2)}{b^2-x^2}{}^{2}$
(40)	$\frac{x^2+x-1}{x^3+1}$	$\frac{1+2x+3x^2-2x^3-x^4}{(1+x^3)^2}$
(41)	$\sin x+\cos x$	$\cos x-\sin x$
(42)	$\frac{x}{1-\cos x}$	$\frac{1-\cos x-x\sin x}{(1-\cos x)^2}$
(43)	$\frac{\tan x}{x}$	$\frac{x-\sin x\cos x}{x^2\cos^2 x}$
(44)	$\frac{\sin x}{x}+\frac{x}{\sin x}$	$(x\cos x-\sin x)\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$
(45)	$\frac{\sin x}{1+\cos x}$	$\frac{1}{1+\cos x}$

(46) $\dfrac{x}{\sin x + \cos x}$ $\qquad$ $\dfrac{\sin x + \cos x + x(\sin x - \cos x)}{1 + \sin 2x}$

(47) $\dfrac{x \sin x}{1 + \tan x}$ $\qquad$ $\dfrac{(1 + \tan x)(\sin x + x \cos x) - x \sin x \sec^2 x}{(1 + \tan x)^2}$

(48) $\dfrac{1}{4} \tan^4 x$ $\qquad$ $\tan^3 x \sec^2 x$

(49) $\cos x - \dfrac{1}{3} \cos^3 x$ $\qquad$ $-\sin^3 x$

(50) $3 \sin^2 x - \sin^3 x$ $\qquad$ $\dfrac{3}{2} \sin 2x(2 - \sin x)$

(51) $\dfrac{1}{3} \tan^3 x - \tan x + x$ $\qquad$ $\tan^4 x$

(52) $3 \sin(3x + 5)$ $\qquad$ $9 \cos(3x + 5)$

(53) $\sin \dfrac{1}{x^2}$ $\qquad$ $-2x^3 \cos \dfrac{1}{x^2}$

(54) $(1 + \sin^2 x)^4$ $\qquad$ $4(1 + \sin^2 x)^3 \sin 2x$

(55) $x \sin^{-1} x$ $\qquad$ $\sin^{-1} x + \dfrac{x}{\sqrt{1 - x^2}}$

(56) $\dfrac{\cos^{-1} x}{x^2}$ $\qquad$ $-\dfrac{1}{x^2\sqrt{1 - x^2}} - \dfrac{2x \cos^{-1} x}{x^3}$

(57) $\dfrac{x}{1 + x^2} - \tan^{-1} x$ $\qquad$ $-\dfrac{2x^2}{(1 + x^2)^2}$

(58) $\dfrac{x^2}{\tan^{-1} x}$ $\qquad$ $\dfrac{2x}{\tan^{-1} x} - \dfrac{x^2}{(1 + x^2)(\tan^{-1} x)^2}$

(59) $\log \sin x$ $\qquad$ $\cot x$

(60) $\tan^{-1}(\log x^2)$ $\qquad$ $\dfrac{2}{x(1 + (\log x^2)^2)}$

4 Taylor's Theorem

(1) $\log \dfrac{1 + x}{1 - x}$ $\qquad$ $2 \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{2n + 1}$

(2) $\dfrac{1}{(1 + x)^2}$ $\qquad$ $\sum_{n=0}^{\infty} (-1)^n (n + 1) x^n$

(3) $\sqrt{1 + x}$ $\qquad$ $1 + \dfrac{1}{2}x + \sum_{n=2}^{\infty} (-1)^{n-1} \dfrac{1 \cdot 3 \cdots (2n - 3)}{2 \cdot 4 \cdots (2n)} x^n$

5 Application

(1) Show $\dfrac{x}{1 + x} < \log(1 + x)$.

Set $f(x) = \log(1 + x) - \dfrac{x}{1 + x}. f(0) = 0, f'(x) = (1 + x)^{-2} > 0 \ (x > 0)$

(2) $\dfrac{x}{1 + x^2} < \tan^{-1} x < x$ $\qquad$ $\left(\tan^{-1} x - \dfrac{x}{1 + x^2}\right)' = \dfrac{2x^2}{(1 + x^2)^2} \geq 0$

(3) $x\log x \geq x-1 \quad (x>0)$ $\qquad (x\log x - x + 1)' = \log x = 0 \ (x=0)$

(4) $\frac{2}{\pi}x < \sin x < x \quad (0<x<\frac{\pi}{2})$ $\qquad (\sin x - \frac{2}{\pi}x)'' = -\sin x < 0$

(5) $\alpha(x-1) < x^\alpha - 1 < \alpha x^{\alpha-1}(x-1) \quad (\alpha>1, x>1)$

$(x^\alpha - 1 - \alpha(x-1))' = \alpha(x^{\alpha-1}-1) > 0, f(0) = \alpha - 1 > 0$

(6) $e^x > 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} \quad (x>0)$

Use induction and $(lhs - rhs)' = e^x - (1 + x + \cdots + \frac{x^{n-1}}{(n-1)!}) > 0$

(7) $\sqrt[3]{3} > \sqrt[4]{4} > \ldots > \sqrt[n]{n}$ $\qquad \frac{x^{x+1}}{(x+1)^x} = x(1+x^{-1})^{-x} < x/3 < 1$

(8) Find extrema of $\frac{\log x}{x}$.

Use $\left(\frac{\log x}{x}\right)' = \frac{1-\log x}{x^2}$. $\quad e^{-1}$(maximum)

(9) Find extrema of $\sin^2 x - \sqrt{3}\cos x \quad (0 < x < 2\pi)$. $\qquad \frac{7}{4}$ *(maximum)*, $\sqrt{3}$ *(minimum)*

(10) Find extrema of $x^2 \log x$. $\qquad -\frac{1}{2e}$ *(minimum)*

(11) Examine extrema of $2(e^x + e^{-x}\cos x) - x^3 - x^2$ around $x=0$ $\qquad$ 4 *(maximum)*

L'Hospital's Method

(1) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $\qquad 1/2$

(2) $\lim_{x\to 1} \frac{\log x}{1-x}$ $\qquad -1$

(3) $\lim_{x\to\infty} \frac{x^2}{e^{2x}}$ $\qquad 0$

(4) $\lim_{x\infty} x^{1/x}$ $\qquad 1$

(5) $\lim_{x\to\pi/2-0} \left(\tan x - \frac{1}{\cos x}\right)$ $\qquad 0$

(6) $\lim_{x\to\infty} \frac{(\log x)^2}{\sqrt{x}}$ $\qquad 0$

(7) $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ $\qquad 2$

(8) $\lim_{x\to +0} x\log x$ $\qquad 0$

(9) $\lim_{x\to\infty} x\left(\frac{\pi}{2} - \tan^{-1} x\right)$ $\qquad 1$

(10) $\lim_{x\to\infty} (1+x)^{1/x}$ $\qquad 1$

(11) $\lim_{x\to 1} (1-\log x)^{1/\log x}$ $\qquad 1/e$

(12) $\lim_{x\to 0} \left(\frac{a^x + b^x}{x}\right)^{1/x} \quad (a, b > 0)$ $\qquad \sqrt{ab}$

6 Fucntions of 2 Vairables

1) $\lim_{(x,y)\to(0,0)} \dfrac{\sin xy}{\sqrt{x^2+y^2}}$ 0

2) $\lim_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^4}$ 0

3) $\lim_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2}$ 0

4) $\lim_{(x,y)\to(0,0)} xy\log(x^2+y^2)$ 0

5) $f(x,y) = \dfrac{x^2}{x^2+y^2}$ $((x,y)\neq(0,0))$, $f(0,0)=0$ is continuous at $(0,0)$

False since $f(x,0)=1(x\neq 0)$

6) $f(x,y) = \dfrac{xy}{\sin(x^2+y^2)}$ $((x,y)\neq(0,0))$, $f(0,0)=0$ is not continuous at $(0,0)$ True

7) $f(x,y) = \dfrac{xy^2}{x^2+y^2}$ $((x,y)\neq(0,0))$, $f(0,0)=0$ is continuous at $(0,0)$ True

7 Partial Derivatives

A

(1) $z = x^3+y^3-3axy$ $z_x = 3x^2-3ay, z_y = 3y^2-3ax$

(2) $z = \sqrt{x^2+y^2}$ $z_x = x/z, z_y = y/z$

(3) $z = e^{ax}\cos by$ $z_x = ae^{ax}\cos by, z_y = -be^{ax}\sin by$

(4) $z = \log(x^2+y^2)$ $z_x = 2x/(x^2+y^2), z_y = 2y/(x^2+y^2)$

(5) $z = x^y$ $z_x = yx^{y-1}, z_y = x^y\log x$

(6) $z = \sin^{-1}(x/y)$ $z_x = \dfrac{1}{\sqrt{y^2-x^2}}, z_y = -\dfrac{x}{y\sqrt{y^2-x^2}}$

B

(1) $z = xy(2x+3y)$ $4x+6y$

(2) $z = e^{xy}$ $e^{xy}(1+xy)$

(3) $z = \cos(x-2y)$ $2\cos(x-2y)$

(4) $z = \log(e^x+e^y)$ $-\dfrac{e^{x+y}}{(e^x+e^y)^2}$

(5) $z = \sin^{-1}(xy)$ $x^2y^2(1-x^2y^2)^{-3/2}+(1-x^2y^2)^{-1/2}$

C

(1) $z = \log(x^2+y^2)$ 0

(2) $z = e^x\cos y$ 0

(3) $z = \tan^{-1}(x/y)$ 0

8 Composite and Implicit Functions

Find partial derivatives z_u, z_v

(1) $z = \log(x^2 + y^2)$, $x = u - v, y = u + v$ $\qquad z_u = \dfrac{2u}{u^2+v^2}, z_v = \dfrac{2v}{u^2+v^2}$

(2) $z = e^{x+y}$, $x = \log(u + v)$,$y = \log(u - v)$ $\qquad z_u = 2u, z_v = -2v$

Calculate dy/dx.

(1) $x^2 + xy - y^2 = 1$ $\qquad \dfrac{2x+y}{2y-x}$

(2) $x^3 - 3axy + y^3 = 0$ $\qquad \dfrac{x^2-ay}{ax-y}$

(3) $e^x + e^y = e^{x+y}$ $\qquad \dfrac{e^x(e^y-1)}{e^y(1-e^x)}$

9 Application

Find the extrema.

(1) $x^2 - xy + y^2 - 4x - y$ $\qquad$ local minimum -7, at $(3, 2)$

(2) $xy(2 - x - y)$ $\qquad$ local minimum $\dfrac{8}{27}$, at $(\dfrac{2}{3}, \dfrac{2}{3})$

(3) $xy(x^2 + y^2 + 1)$ $\qquad$ no extrema

(4) $(x^2 + y^2)e^{x-y}$ $\qquad$ local minimum 0, at $(0, 0)$

(5) $x^2 + 4xy + 2y^2 - 6x - 8y - 1$ $\qquad$ no extrema

(6) $x^3 - 9xy + y^3 + 1$ $\qquad$ local minimum -26, at $(3, 3)$

Find the extrema of implicit functions.

(1) $x^2 - xy + y^2 - 3 = 0$

$(x - 2y)y' = 2x - y$, $(x - 2y)y'' + (1 - 2y')y' = 2 - y'$
local mximum 2 at $x = 1$, local minimum -2 at $x = -1$

(2) $xy(y - x) - 16 = 0$ $\qquad$ local minimum 4 at $x = 2$

(3) $x^3 - 3xy + y^3$ $\qquad$ local maximum $\sqrt[3]{4}$ at $x = \sqrt[3]{2}$

(4) $x^4 + 3x^2 + y^3 - y = 0$ $\qquad$ local maximum 4 at ± 1, local minimum 0 at $x = 0$

Use Lagrange's method.

(1) $x + y$ with $x^2 + y^2 - 8 = 0$ $\qquad$ minimum -4, maximum 4

(2) $x^2 + y^2$ with $xy - 1 = 0$ $\qquad$ minimum 1, no maximum

(3) xy with $x^2 + xy + y^2 = 1$ $\qquad$ minimum -1, maximum $\frac{1}{3}$

(4) x^2+y^2 with $x^3-6xy+y^3=0$ local minimum 0, local maximum 18

Find the maxima and minima.

(1) $z=\dfrac{1}{x}+\dfrac{1}{y}$ with $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{1}{a^2}$ $(a>0)$ minimum $-\dfrac{\sqrt{2}}{a}$, maxmum $\dfrac{\sqrt{2}}{a}$

(2) $u=x+y+z$ with $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=1$ minimum 1, no maximum

(3) $u=y^2+4z^2-4yz-2xz-2xy$ with $2x^2+3y^2+6z^2=1$ minimum $-\dfrac{1}{2}$, maximum 1

10 Infinite Series

Calculate the sum.

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{3}{(n+1)(n+2)}$ $\displaystyle =3\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{3}{2}$

(2) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$ $\displaystyle =\sum_{n=1}^{\infty}\left(\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\right)=\frac{1}{4}$

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n-2^n}{5^n}$ $\displaystyle =\sum_{n=1}^{\infty}\frac{3^n}{5^n}-\sum_{n=1}^{\infty}\frac{2^n}{5^n}=\frac{5}{6}$

Does the sum converge or not?

(4) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$ 収束 $p=1$ case in $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$

(5) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ 発散 $p=\dfrac{1}{2}$ case in $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$

(6) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\log n}$ 発散 $p=1$ case in $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}$

(発散であることはつぎのようにして直接調べることができる。分割 $2^k \leq n < 2^{k+1}$ $(k\geq 1)$ とすれば、このような n は 2^k こあり、$n\log_2 n\leq k2^k$ となるから、$\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\log n}\geq \log_2 e\sum_{n=1}^{\infty}\frac{1}{k}=\infty$)

(7) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+2}$ 収束 $\displaystyle\frac{\sqrt{n}}{n^2+2}\frac{1}{n^{-3/2}}\to 1\ (n\to\infty)$

(8) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\log n}$ 発散 $\displaystyle\frac{1}{\log n}\frac{1}{n^{-1}}\to\infty\ (n\to\infty)$

(9) $\displaystyle\sum_{n=1}^{\infty}\sin^2\frac{\pi}{n}$ 収束 $\displaystyle\sin^2\frac{\pi}{n}\frac{1}{n^{-2}}\to\pi^2\ (n\to\infty)$

(10) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n-2^n}$ 収束 $\displaystyle\frac{1}{3^n-2^n}\frac{1}{3^n}\to 1\ (n\to\infty)$

(11) $\sum_{n=1}^{\infty} \frac{2n+3}{3n^3-n-1}$ 収束 $\frac{2n+3}{3n^3-n-1}\frac{1}{n^{-2}} \to \frac{2}{3}\ (n\to\infty)$

(12) $\sum_{n=1}^{\infty} \sin\frac{\pi}{n}$ 発散 $\sin\frac{\pi}{n}\frac{1}{n^{-1}} \to \pi\ (n\to\infty)$

(13) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ 収束 $\frac{n+1}{2^{(n+1)}}\frac{2^n}{n} \to \frac{1}{2} < 1\ (n\to\infty)$

(14) $\sum_{n=1}^{\infty} \left(\frac{3n-2}{2n+3}\right)^n$ 発散 $\sqrt[n]{\left(\frac{3n-2}{2n+3}\right)^n} = \frac{3n-2}{2n+3} \to \frac{3}{2} > 1\ (n\to\infty)$

(15) $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$ 収束 $\sqrt[n]{\frac{1}{(\log n)^n}} = \frac{1}{\log n} \to 0\ (n\to\infty)$

11 Power Series

Does the sum absolutely converge or not ?

(1) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$ absolutely convergent

(2) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ not absolutely convergent

(3) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$ not absolutely convegent

Find the convergence region.

(4) $\sum_{n=1}^{\infty} nx^n$ $(-\infty,\infty)$ since $\frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0\ (n\to\infty)$

(5) $\sum_{n=1}^{\infty} n^n x^n$ $\{0\}$ since $\sqrt[n]{n^n} = n \to \infty\ (n\to\infty)$

(6) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}x^n$ $[-\frac{1}{2},\frac{1}{2}]$ since $\frac{2^n}{n^2}x^n = \frac{1}{n^2}(2x)^n$

Prove.

(7) $\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \quad (|x|<1)$ $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ より

(8) $\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} \quad (|x|<1)$ $(1+x)^{1/2} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n$ より

(9) $\sin^{-1}x = x + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\frac{x^{2n+1}}{2n+1} \quad (|x|<1)$

$(1-x^2)^{-1/2} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n x^{2n}$ より