Optimization Theory (DS2) HW#2
Certificates, Standard Equality Form and Simplex Iteration

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Problems and page numbers from the paperback edition of *A Gentle Introduction to Optimization*, by B. Guenin *et al.*

1. (Problem 1 in Sec. 2.1 of the text, p. 50.)

   (a) Prove that the following LP problem is infeasible:

   $$\max\{3x_1 + 4x_2 + 6x_3\} \tag{1}$$

   subject to

   $$3x_1 + 5x_2 - 6x_3 = 4 \tag{2}$$
   $$x_1 + 3x_2 - 4x_3 = 2 \tag{3}$$
   $$-x_1 + x_2 - x_3 = -1 \tag{4}$$
   $$x_1, x_2, x_3 \geq 0. \tag{5}$$

   (b) Prove that the following LP problem is unbounded:

   $$\max\{-x_3 + x_4\} \tag{6}$$

   subject to

   $$x_1 + x_3 - x_4 = 1 \tag{7}$$
   $$x_2 + 2x_3 - x_4 = 2 \tag{8}$$
   $$x_1, x_2, x_3, x_4 \geq 0. \tag{9}$$

   (c) **Optional:** Prove that the LP Problem

   $$max\{\vec{c}^\intercal \vec{x} : A\vec{x} = \vec{b}, \vec{x} \geq \vec{0}\} \tag{10}$$

   is unbounded, where

   $$A = \begin{pmatrix} 4 & 2 & 1 & -6 & -1 \\ -1 & 1 & -4 & 1 & 3 \\ 3 & -6 & 5 & 3 & -5 \end{pmatrix}, \tag{11}$$

$$\vec{b} = \begin{pmatrix} 11 \\ -2 \\ -8 \end{pmatrix}, \tag{12}$$

$$\vec{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{13}$$

Hint: Consider the vectors

$$\vec{x} = (1, 3, 1, 0, 0)^\intercal \text{ and } \vec{d} = (1, 1, 1, 1)^\intercal \tag{14}$$

(d) **Optional:** For each of the problems in parts (b) and (c), give a feasible solution having object value exactly 5000.

2. (Problem 1 in Sec. 2.2 of the text, p. 54.)

(a) Convert the following LPs into SEF:

$$\min\{(2, -1, 4, 2, 4)(x_1, x_2, x_3, x_4, x_5)^\intercal\} \tag{15}$$

subject to

$$\begin{pmatrix} 1 & 2 & 4 & 7 & 3 \\ 2 & 8 & 9 & 0 & 0 \\ 1 & 1 & 0 & 2 & 6 \\ -3 & 4 & 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \begin{matrix} \leq \\ = \\ \geq \\ \geq \end{matrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}. \tag{16}$$

$$x_1, x_2, x_4 \geq 0. \tag{17}$$

(Careful, note that not all of the x_i have a non-negativity constraint.)

(b) **Optional:** Let A, B, D be matrices and $\vec{b}, \vec{c}, \vec{d}, \vec{f}$ vectors (all of suitable dimensions). Convert the following LP with variables $\vec{x}$ and $\vec{y}$ (where $\vec{x}$, $\vec{y}$ are vectors) into SEF:

$$\min\{\vec{c}^\intercal \vec{x} + \vec{d}^\intercal \vec{y}\} \tag{18}$$

subject to

$$A\vec{x} \geq \vec{b} \tag{19}$$

$$B\vec{x} + D\vec{y} = \vec{f} \tag{20}$$

$$\vec{x} \geq \vec{0}. \tag{21}$$

3. (Problem 1 in Sec. 2.3 of the text, p. 56.)
In this exercise, you are asked to repeat the argument in Section 2.3 with different examples.

(a) Consider the following LP:

$$\max\{(-1,0,0,2)\vec{x}\} \tag{22}$$

Subject to

$$\begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -3 \end{pmatrix} \vec{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{23}$$

$$\vec{x} \geq \vec{0}. \tag{24}$$

Observe that $\vec{x} = (0,2,3,0)^\intercal$ is a feasible solution. Starting from $\vec{x}$, construct a feasible solution $\vec{x'}$ with value larger than that of $\vec{x}$ by increasing as much as possible the value of exactly one of x_1 or x_4 (keeping the other variable unchanged).

(b) Consider the following LP:

$$\max\{(0,0,4,-6)\vec{x}\} \tag{25}$$

subject to

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{26}$$

$$\vec{x} \geq \vec{0}. \tag{27}$$

Observe that $\vec{x} = (2,1,0,0)^\intercal$ is a feasible solution. Starting from $\vec{x}$, construct a feasible solution $\vec{x'}$ with value larger than that of $\vec{x}$ by increasing as much as possible the value of exactly one of x_1 or x_4 (keeping the other variable unchanged).