

# 論理学

## 第5回「証明(演習)」

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# ボトムアップとトップダウンの証明

## ボトムアップ証明

- 正しいと証明された式からはじめて、目的の式まで推論規則を適用していく。
- 始式から始める通常の証明。
- 途中段階も、すべて証明。

$$\frac{\frac{}{A \vdash A} \text{ (I)} \quad \frac{}{B \rightarrow C \vdash B \rightarrow C} \text{ (I)}}{A \rightarrow (B \rightarrow C), A \vdash B \rightarrow C} \text{ (}\rightarrow\text{L)}$$

## トップダウン証明

- 終式から始めて、その式に至る推論規則を探す。
- 最終的に始式に到達すると証明が完成する。
- 途中段階では補題でしかない。

$$\frac{\frac{}{B \vdash B} \text{ (I)} \quad \frac{}{C \vdash C} \text{ (I)}}{B \rightarrow C, B \vdash C} \text{ (}\rightarrow\text{L)}$$

ボトムアップ

$$\frac{\frac{\frac{A, A \rightarrow B, A \rightarrow (B \rightarrow C) \vdash C}{A \rightarrow B, A \rightarrow (B \rightarrow C) \vdash A \rightarrow C} \text{ (}\rightarrow\text{R)}}{A \rightarrow (B \rightarrow C) \vdash (A \rightarrow B) \rightarrow (A \rightarrow C)} \text{ (}\rightarrow\text{R)}}{\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))} \text{ (}\rightarrow\text{R)}$$

トップダウン

# 宿題(証明)

## 問題

- 次の論理式を証明しなさい。

$$\vdash \neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$$

## 提出

### 注意:

- 証明図を書きなさい
- どの推論規則を使ったかが分かるようにしなさい
- 始式から始めて終式が上記の論理式になるようにしなさい

$$\frac{\frac{\frac{}{A \vdash A} \text{(I)}}{\dots} \quad \frac{\frac{}{B \vdash B} \text{(I)}}{\dots}}{\dots}}{\neg(A \wedge B) \vdash (\neg A \vee \neg B)} \quad \frac{}{\vdash \neg(A \wedge B) \rightarrow (\neg A \vee \neg B)} \text{(\rightarrow R)}$$

$$\vdash \neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$$

$$\begin{array}{c}
 \begin{array}{c}
 \text{(I)} \frac{}{A \vdash A} \\
 (\neg R) \frac{A \vdash A}{\vdash A, \neg A} \\
 (\vee R_1) \frac{\vdash A, \neg A}{\vdash A, \neg A \vee \neg B} \\
 (\text{ER}) \frac{\vdash A, \neg A \vee \neg B}{\vdash \neg A \vee \neg B, A}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{}{B \vdash B} \text{(I)} \\
 (\neg R) \frac{B \vdash B}{\vdash B, \neg B} \\
 (\vee R_2) \frac{\vdash B, \neg B}{\vdash B, \neg A \vee \neg B} \\
 (\text{ER}) \frac{\vdash B, \neg A \vee \neg B}{\vdash \neg A \vee \neg B, B}
 \end{array} \\
 \hline
 \vdash \neg A \vee \neg B, A \wedge B \quad (\wedge R) \\
 \hline
 \neg(A \wedge B) \vdash \neg A \vee \neg B \quad (\neg L) \\
 \hline
 \vdash \neg(A \wedge B) \rightarrow (\neg A \vee \neg B) \quad (\rightarrow R)
 \end{array}$$

# 練習問題

- 次の命題を証明しなさい.
  - $A \rightarrow (B \rightarrow A)$
  - $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
  - $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
  - $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
  - $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$
  - $(A \wedge B) \wedge C \rightarrow A \wedge (B \wedge C)$
  - $(A \vee B) \vee C \rightarrow A \vee (B \vee C)$
  - $\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$
  - $(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$
  - $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$
  - $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

$$\vdash A \rightarrow (B \rightarrow A)$$

$$\frac{\frac{\frac{\frac{}{A \vdash A} \text{ (I)}}{B, A \vdash A} \text{ (WL)}}{A \vdash B \rightarrow A} \text{ (\rightarrow R)}}{\vdash A \rightarrow (B \rightarrow A)} \text{ (\rightarrow R)}$$

$$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

$$\begin{array}{c}
 \text{(I) } \frac{}{A \vdash A} \qquad \frac{\text{(I) } \frac{}{B \vdash B} \quad \frac{}{C \vdash C} \text{(I)}}{B \rightarrow C, B \vdash C} \text{ } (\rightarrow L) \\
 \frac{\text{(I) } \frac{}{A \vdash A} \quad \frac{}{B \rightarrow C, B \vdash C}}{A \rightarrow (B \rightarrow C), A, B \vdash C} \text{ } (\rightarrow L) \\
 \frac{\text{(I) } \frac{}{A \vdash A} \quad \frac{}{A \rightarrow (B \rightarrow C), A, B \vdash C}}{B, A, A \rightarrow (B \rightarrow C) \vdash C} \text{ } (\text{EL} \times 3) \\
 \frac{}{A \vdash A} \quad \frac{}{B, A, A \rightarrow (B \rightarrow C) \vdash C} \text{ } (\rightarrow L) \\
 \frac{}{A \rightarrow B, A, A, A \rightarrow (B \rightarrow C) \vdash C} \text{ } (\text{EL} \times 2) \\
 \frac{}{A, A, A \rightarrow B, A \rightarrow (B \rightarrow C) \vdash C} \text{ } (\text{CL}) \\
 \frac{}{A, A \rightarrow B, A \rightarrow (B \rightarrow C) \vdash C} \text{ } (\rightarrow R) \\
 \frac{}{A \rightarrow B, A \rightarrow (B \rightarrow C) \vdash A \rightarrow C} \text{ } (\rightarrow R) \\
 \frac{}{A \rightarrow (B \rightarrow C) \vdash (A \rightarrow B) \rightarrow (A \rightarrow C)} \text{ } (\rightarrow R) \\
 \frac{}{\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))} \text{ } (\rightarrow R)
 \end{array}$$

$$\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$$

$$\begin{array}{c}
 \begin{array}{c}
 \text{(I)} \frac{}{A \vdash A} \\
 \hline
 \end{array}
 \quad
 \begin{array}{c}
 \text{(I)} \frac{}{B \vdash B} \quad \frac{}{C \vdash C} \text{(I)} \\
 \hline
 B, B \rightarrow C \vdash C \quad (\rightarrow L)
 \end{array} \\
 \hline
 A \rightarrow B, A, B \rightarrow C \vdash C \quad (\rightarrow L) \\
 \hline
 A \rightarrow B, A, B \rightarrow C \vdash C \quad (\text{EL} \times 2) \\
 \hline
 A, B \rightarrow C, A \rightarrow B \vdash C \quad (\rightarrow R) \\
 \hline
 B \rightarrow C, A \rightarrow B \vdash A \rightarrow C \quad (\rightarrow R) \\
 \hline
 A \rightarrow B \vdash (B \rightarrow C) \rightarrow (A \rightarrow C) \quad (\rightarrow R) \\
 \hline
 \vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \quad (\rightarrow R)
 \end{array}$$



$$\vdash (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$$

$$\begin{array}{c}
 \begin{array}{c}
 \text{(I)} \frac{}{A \vdash A} \quad \frac{}{C \vdash C} \text{(I)} \\
 \hline
 A, A \rightarrow C \vdash C \quad (\rightarrow L)
 \end{array}
 \qquad
 \begin{array}{c}
 \text{(I)} \frac{}{B \vdash B} \quad \frac{}{C \vdash C} \text{(I)} \\
 \hline
 B, B \rightarrow C \vdash C \quad (\rightarrow L)
 \end{array} \\
 \hline
 A \vee B, A \rightarrow C, B \rightarrow C \vdash C, C \quad (\vee L) \\
 \hline
 A \vee B, A \rightarrow C, B \rightarrow C \vdash C, C \quad (\text{CR}) \\
 \hline
 A \vee B, A \rightarrow C, B \rightarrow C \vdash C \quad (\text{EL}) \\
 \hline
 A \vee B, B \rightarrow C, A \rightarrow C \vdash C \quad (\rightarrow R) \\
 \hline
 B \rightarrow C, A \rightarrow C \vdash A \vee B \rightarrow C \quad (\rightarrow R) \\
 \hline
 A \rightarrow C \vdash (B \rightarrow C) \rightarrow (A \vee B \rightarrow C) \quad (\rightarrow R) \\
 \hline
 \vdash (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C)) \quad (\rightarrow R)
 \end{array}$$

$$\vdash (A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$$

$$\begin{array}{c}
 \begin{array}{c}
 \text{(I)} \frac{}{A \vdash A} \\
 \hline
 \end{array}
 \quad
 \begin{array}{c}
 \text{(I)} \frac{}{A \vdash A} \quad \frac{}{B \vdash B} \text{(I)} \\
 \hline
 A, A \rightarrow B \vdash B \quad (\rightarrow L) \\
 \hline
 \neg B, A, A \rightarrow B \vdash \quad (\neg L)
 \end{array} \\
 \hline
 A \rightarrow \neg B, A, A, A \rightarrow B \vdash \quad (\rightarrow L) \\
 \hline
 A \rightarrow \neg B, A, A, A \rightarrow B \vdash \quad (\text{EL} \times 2) \\
 \hline
 A, A, A \rightarrow \neg B, A \rightarrow B \vdash \quad (\text{CL}) \\
 \hline
 A, A \rightarrow \neg B, A \rightarrow B \vdash \quad (\neg R) \\
 \hline
 A \rightarrow \neg B, A \rightarrow B \vdash \neg A \quad (\rightarrow R) \\
 \hline
 A \rightarrow B \vdash (A \rightarrow \neg B) \rightarrow \neg A \quad (\rightarrow R) \\
 \hline
 \vdash (A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A) \quad (\rightarrow R)
 \end{array}$$

$$\vdash (A \wedge B) \wedge C \rightarrow A \wedge (B \wedge C)$$

$$\begin{array}{c}
 \begin{array}{c}
 (I) \frac{}{A \vdash A} \\
 (\wedge L_1) \frac{}{A \wedge B \vdash A} \\
 (\wedge L_1) \frac{}{(A \wedge B) \wedge C \vdash A}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{B \vdash B} (I) \\
 \frac{}{A \wedge B \vdash B} (\wedge L_2) \\
 \frac{}{(A \wedge B) \wedge C \vdash B} (\wedge L_1) \\
 \frac{}{C \vdash C} (I) \\
 \frac{}{(A \wedge B) \wedge C \vdash C} (\wedge L_2) \\
 \frac{}{(A \wedge B) \wedge C, (A \wedge B) \wedge C \vdash B \wedge C} (\wedge R) \\
 \frac{}{(A \wedge B) \wedge C \vdash B \wedge C} (CL) \\
 \frac{}{(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)} (\wedge R) \\
 \frac{}{(A \wedge B) \wedge C, (A \wedge B) \wedge C \vdash A \wedge (B \wedge C)} (CL) \\
 \frac{}{(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)} (\wedge R) \\
 \frac{}{\vdash (A \wedge B) \wedge C \rightarrow A \wedge (B \wedge C)} (\rightarrow R)
 \end{array}
 \end{array}$$

$$\vdash (A \vee B) \vee C \rightarrow A \vee (B \vee C)$$

$$\begin{array}{c}
 \begin{array}{c}
 \text{(I)} \frac{}{A \vdash A} \\
 \text{(vR}_1) \frac{}{A \vdash A \vee (B \vee C)}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{B \vdash B} \text{(I)} \\
 \frac{}{B \vdash B \vee C} \text{(vR}_1) \\
 \frac{}{B \vdash A \vee (B \vee C)} \text{(vR}_2)
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{C \vdash C} \text{(I)} \\
 \frac{}{C \vdash B \vee C} \text{(vR}_2) \\
 \frac{}{C \vdash A \vee (B \vee C)} \text{(vR}_2)
 \end{array}$$


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$$\begin{array}{c}
 \frac{}{A \vee B \vdash A \vee (B \vee C), A \vee (B \vee C)} \text{(vL)} \\
 \frac{}{A \vee B \vdash A \vee (B \vee C)} \text{(CR)}
 \end{array}
 \quad
 \frac{}{C \vdash A \vee (B \vee C)} \text{(vL)}$$


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$$\frac{}{(A \vee B) \vee C \vdash A \vee (B \vee C), A \vee (B \vee C)} \text{(vL)} \\
 \frac{}{(A \vee B) \vee C \vdash A \vee (B \vee C)} \text{(CR)}$$


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$$\frac{}{\vdash (A \vee B) \vee C \rightarrow A \vee (B \vee C)} \text{(\rightarrow R)}$$

$$\vdash \neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$$

$$\begin{array}{c}
 \begin{array}{c}
 \text{(I) } \frac{}{A \vdash A} \\
 (\neg\text{R}) \frac{}{\vdash A, \neg A} \\
 (\vee\text{R}_1) \frac{}{\vdash A, \neg A \vee \neg B} \\
 (\text{ER}) \frac{}{\vdash \neg A \vee \neg B, A}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{}{B \vdash B} \text{(I)} \\
 (\neg\text{R}) \frac{}{\vdash B, \neg B} \\
 (\vee\text{R}_2) \frac{}{\vdash B, \neg A \vee \neg B} \\
 (\text{ER}) \frac{}{\vdash \neg A \vee \neg B, B}
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{\vdash \neg A \vee \neg B, A \wedge B} \text{(\wedge R)} \\
 \frac{}{\neg(A \wedge B) \vdash \neg A \vee \neg B, \neg A \vee \neg B} \text{(\neg L)} \\
 \frac{}{\neg(A \wedge B) \vdash \neg A \vee \neg B} \text{(CR)} \\
 \frac{}{\vdash \neg(A \wedge B) \rightarrow (\neg A \vee \neg B)} \text{(\rightarrow R)}
 \end{array}$$

$$\vdash (\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$$

$$\begin{array}{c}
 \frac{\frac{\frac{}{A \vdash A} \text{(I)}}{A \wedge B \vdash A} (\wedge L_1)}{\neg A, A \wedge B \vdash} (\neg L)}{A \wedge B, \neg A \vdash} \text{(EL)} \\
 \frac{}{\neg A \vdash \neg(A \wedge B)} (\neg R) \\
 \hline
 \frac{\frac{\frac{}{B \vdash B} \text{(I)}}{A \wedge B \vdash B} (\wedge L_2)}{\neg B, A \wedge B \vdash} (\neg L)}{A \wedge B, \neg B \vdash} \text{(EL)} \\
 \frac{}{\neg B \vdash \neg(A \wedge B)} (\neg R) \\
 \hline
 \frac{\neg A \vee \neg B \vdash \neg(A \wedge B), \neg(A \wedge B)}{\neg A \vee \neg B \vdash \neg(A \wedge B)} \text{(vL)} \\
 \frac{}{\neg A \vee \neg B \vdash \neg(A \wedge B)} \text{(CR)} \\
 \frac{}{\vdash (\neg A \vee \neg B) \rightarrow \neg(A \wedge B)} (\rightarrow R)
 \end{array}$$

$$\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$$

$$\begin{array}{c}
 \frac{}{A \vdash A} \text{ (I)} \quad \frac{}{B \vdash B} \text{ (I)} \\
 \hline
 A \rightarrow B, A \vdash B \quad \text{ (}\rightarrow\text{L)} \\
 \hline
 A, A \rightarrow B \vdash B \quad \text{ (EL)} \\
 \hline
 A \rightarrow B \vdash B, \neg A \quad \text{ (}\neg\text{R)} \\
 \hline
 A \rightarrow B \vdash \neg A, B \quad \text{ (ER)} \\
 \hline
 \neg B, A \rightarrow B \vdash \neg A \quad \text{ (}\neg\text{L)} \\
 \hline
 A \rightarrow B \vdash \neg B \rightarrow \neg A \quad \text{ (}\rightarrow\text{R)} \\
 \hline
 \vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A) \quad \text{ (}\rightarrow\text{R)}
 \end{array}$$

$$\vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$$

$$\begin{array}{c}
 \frac{}{B \vdash B} \text{ (I)} \qquad \frac{}{A \vdash A} \text{ (I)} \\
 \frac{}{\vdash B, \neg B} \text{ (\neg R)} \qquad \frac{}{\neg A, A \vdash} \text{ (\neg L)} \\
 \hline
 \frac{}{\neg B \rightarrow \neg A, A \vdash B} \text{ (\rightarrow L)} \\
 \frac{}{A, \neg B \rightarrow \neg A \vdash B} \text{ (EL)} \\
 \frac{}{\neg B \rightarrow \neg A \vdash A \rightarrow B} \text{ (\rightarrow R)} \\
 \hline
 \vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B) \text{ (\rightarrow R)}
 \end{array}$$